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Riesz Triple Probabilisitic of Almost Lacunary Cesàro C_{111} Statistical Convergence of χ^3 Defined by a Musielak Orlicz Function

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ABSTRACT: In this paper we study the concept of almost lacunary statistical Cesàro of χ^3 over probabilistic $p-$ metric spaces defined by Musielak Orlicz function. Since the study of convergence in PP-spaces is fundamental to probabilistic functional analysis, we feel that the concept of almost lacunary statistical Cesàro of χ^2 over probabilistic $p-$ metric spaces defined by Musielak in a PP-space would provide a more general framework for the subject.

Key Words: Analytic sequence, Orlicz function, chi sequence, Riesz space, statistical convergence, Cesàro $C_{1,1,1}$- statistical convergence.

Contents

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^3 for the set of all complex triple sequences (x_{mnk}), where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial work on double series is found in Apostol [1], Aotaibi et al. [2], Mursaleen et al. [19-22] and Mishra et al. [23-24] and double sequence spaces is found in Hardy [6], Deepmala et al. [7, 8] and many others. The initial work on triple sequence spaces is found in Sahiner et al. [11], Esi [3-4] and Esi et al. [5], Deepmala et al. [9], [10], Subramanian et al. [12], Shri Prakash et al. [13] and many others.

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Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. Then the triple series is said to be convergent if and only if the triple sequence (S_{mnk}) is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq}(m,n,k=1,2,3,...) .$$

A sequence $x = (x_{mnk})$is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequence is usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m,n,k \to \infty.$$

A sequence $x = (x_{mnk})$ is called triple gai sequence if $((m+n+k)! |x_{mnk}|)^{\frac{1}{m+n+k}} \to 0$ as $m,n,k \to \infty$. The triple gai sequences will be denoted by χ^3.

Consider a triple sequence $x = (x_{mnk})$. The $(m,n,k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq}\Im_{ijq}$ for all $m,n,k \in \mathbb{N}$,

$$\Im_{ijq} = \begin{bmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{bmatrix}$$

with 1 in the $(i,j,q)^{th}$ position and zero otherwise. The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [15] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m,n \in \mathbb{N}$.

Consider the triple difference sequence space is defined as
$\Delta_{mnk} = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1}$ and $\Delta^0 x_{mnk} = \langle x_{mnk}\rangle$.

2. Definitions and Preliminaries

Definition 2.1. An Orlicz function ([see [14]) is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function.

Lindenstrauss and Tzafriri ([17]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup \{|v| u - (f_{mnk})(u) : u \geq 0\}, m, n, k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, [see [16,18]] the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in w^3 : I_f(|x_{mnk}|)^{1/m+n+k} \to 0 \, as \, m, n, k \to \infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}\left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk}\right)$$

is an exteneded real number.

Definition 2.2. A triple sequence $x = (x_{mnk})$ of real numbers is called almost $P-$ convergent to a limit 0 if

$$lim_{p,q,u \to \infty} sup_{r,s,t \geq 0} \frac{1}{pqu} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} \sum_{k=t}^{t+u-1} ((m + n + k)! \, |x_{mnk}|)^{1/m+n+k} \xrightarrow{P-} 0.$$

that is, the average value of (x_{mnk}) taken over any rectangle
$\{(m, n, k) : r \leq m \leq r + p - 1, s \leq n \leq s + q - 1, t \leq k \leq t + u - 1\}$ tends to 0 as both p, q and u to ∞, and this $P-$ convergence is uniform in i, ℓ and j. Let denote the set of sequences with this property as $\left[\widehat{\chi^3}\right]$.

Definition 2.3. Let $(Q_r), (\overline{Q_s}), \left(\overline{\overline{Q_t}}\right)$ be sequences of positive numbers and

$$Q_r = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1s} & 0\ldots \\ q_{21} & q_{22} & \cdots & q_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ q_{r1} & q_{r2} & \cdots & q_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix} = q_{11} + q_{12} + \ldots + q_{rs} \neq 0,$$

$$\overline{Q}_s = \begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & ... & \overline{q}_{1s} & 0... \\ \overline{q}_{21} & \overline{q}_{22} & ... & \overline{q}_{2s} & 0... \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{q}_{r1} & \overline{q}_{r2} & ... & \overline{q}_{rs} & 0... \\ 0 & 0 & ...0 & 0 & 0... \end{bmatrix} = \overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \neq 0,$$

$$\overline{\overline{Q}}_t = \begin{bmatrix} \overline{\overline{q}}_{11} & \overline{\overline{q}}_{12} & ... & \overline{\overline{q}}_{1s} & 0... \\ \overline{\overline{q}}_{21} & \overline{\overline{q}}_{22} & ... & \overline{\overline{q}}_{2s} & 0... \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{\overline{q}}_{r1} & \overline{\overline{q}}_{r2} & ... & \overline{\overline{q}}_{rs} & 0... \\ 0 & 0 & ...0 & 0 & 0... \end{bmatrix} = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \ldots + \overline{\overline{q}}_{rs} \neq 0$$

and is given by
$T_{rst} = \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left((m+n+k)! \left|x_{mnk}\right|\right)^{1/m+n+k}$ is called the Riesz mean of triple sequence $x = (x_{mnk})$. If $P - lim_{rst} T_{rst}(x) = 0, 0 \in \mathbb{R}$, then the sequence $x = (x_{mnk})$ is said to be Riesz convergent to 0. If $x = (x_{mnk})$ is Riesz convergent to 0, then we write $P_R - limx = 0$.

Definition 2.4. The four dimensional matrix A is said to be RH-regular if it maps every bounded $P-$ convergent sequence into a $P-$ convergent sequence with the same $P-$ limit.

Definition 2.5. The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$m_0 = 0, h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty \text{ and}$$
$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty.$$
$$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ is determine by
$I_{i,\ell,j} = \{(m,n,k) : m_{i-1} < m < m_i \, and \, n_{\ell-1} < n \leq n_\ell \, and \, k_{j-1} < k \leq k_j\}$,
$q_k = \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}$.
Using the notations of lacunary Fuzzy sequence and Riesz mean for triple sequences. $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ be a triple lacunary sequence and $q_m \overline{q}_n \overline{\overline{q}}_k$ be sequences of positive real numbers such that $Q_{m_i} = \sum_{m \in (0,m_i]} p_{m_i}, Q_{n_\ell} = \sum_{n \in (0,n_\ell]} p_{n_\ell}, Q_{n_j} = \sum_{k \in (0,k_j]} p_{k_j}$ and $H_i = \sum_{m \in (0,m_i]} p_{m_i}, \overline{H} = \sum_{n \in (0,n_\ell]} p_{n_\ell}, \overline{\overline{H}} = \sum_{k \in (0,k_j]} p_{k_j}$. Clearly, $H_i = Q_{m_i} - Q_{m_{i-1}}, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}}, \overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}}$. If the Riesz transformation of triple sequences is RH-regular, and $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty, \overline{H} = \sum_{n \in (0,n_\ell]} p_{n_\ell} \to \infty$ as $\ell \to \infty, \overline{\overline{H}} = \sum_{k \in (0,k_j]} p_{k_j} \to \infty$ as $j \to \infty$, then $\theta'_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} = \left\{\left(Q_{m_i} Q_{n_j} Q_{k_k}\right)\right\}$ is a triple lacunary sequence. If the assumptions $Q_r \to \infty$ as $r \to \infty$, $\overline{Q}_s \to \infty$ as $s \to \infty$ and $\overline{\overline{Q}}_t \to \infty$ as $t \to \infty$ may be not enough to obtain the conditions $H_i \to \infty$ as $i \to \infty, \overline{H}_\ell \to \infty$ as $\ell \to \infty$

and $\overline{\overline{H}}_j \to \infty$ as $j \to \infty$ respectively. For any lacunary sequences $(m_i), (n_\ell)$ and (k_j) are integers.
Throughout the paper, we assume that $Q_r = q_{11}+q_{12}+\ldots+q_{rs} \to \infty\,(r \to \infty), \overline{Q}_s = \overline{q}_{11}+\overline{q}_{12}+\ldots+\overline{q}_{rs} \to \infty\,(s \to \infty), \overline{\overline{Q}}_t = \overline{\overline{q}}_{11}+\overline{\overline{q}}_{12}+\ldots+\overline{\overline{q}}_{rs} \to \infty\,(t \to \infty)$, such that $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}} \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}} \to \infty$ as $j \to \infty$.
Let $Q_{m_i,n_\ell,k_j} = Q_{m_i}\overline{Q}_{n_\ell}\overline{\overline{Q}}_{k_j}, H_{i\ell j} = H_i\overline{H}_\ell\overline{\overline{H}}_j$,
$I'_{i\ell j} = \left\{(m,n,k) : Q_{m_{i-1}} < m < Q_{m_i}, \overline{Q}_{n_{\ell-1}} < n < Q_{n_\ell} \; and \; \overline{Q}_{k_{j-1}} < k < \overline{Q}_{k_j}\right\}$,
$V_i = \frac{Q_{m_i}}{Q_{m_{i-1}}}, \overline{V}_\ell = \frac{Q_{n_\ell}}{Q_{n_{\ell-1}}}$ and $\overline{\overline{V}}_j = \frac{Q_{k_j}}{Q_{k_{j-1}}}$. and $V_{i\ell j} = V_i\overline{V}_\ell\overline{\overline{V}}_j$.
If we take $q_m = 1, \overline{q}_n = 1 \; and \; \overline{\overline{q}}_k = 1$ for all m, n and k then $H_{i\ell j}, Q_{i\ell j}, V_{i\ell j}$ and $I'_{i\ell j}$ reduce to $h_{i\ell j}, q_{i\ell j}, v_{i\ell j}$ and $I_{i\ell j}$.
Let $n \in \mathbb{N}$ and X be a real vector space of dimension m, where $n \leq m$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1), \ldots, d_n(x_n))\|_p$ on X satisfying the following four conditions:
(i) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p = 0$ if and and only if $d_1(x_1), \ldots, d_n(x_n)$ are linearly dependent,
(ii) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1), \ldots, \alpha d_n(x_n))\|_p = |\alpha| \|(d_1(x_1), \ldots, d_n(x_n))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p((x_1,y_1),(x_2,y_2)\cdots(x_n,y_n)) = (d_X(x_1,x_2,\cdots x_n)^p + d_Y(y_1,y_2,\cdots y_n)^p)^{1/p}$ for $1 \leq p < \infty$; is called the p product metric.

3. Almost Lacunary Cesàro C_{111}-statistical convergence of PP-triple sequence spaces

Let $A = [a^{pqr}_{mnk}]^\infty_{m,n,k=0}$ be a triple infinite matrix of real number for $p, q, r = 1, 2, \cdots$ forming the sum

$$\mu_{pqr}(X) = \sum_{m=0}^\infty \sum_{n=0}^\infty \sum_{k=0}^\infty a^{pqr}_{mnk} \left(\left((m+n+k)! \left(\frac{X_{mnk}}{Y_{mnk}}\right)\right)^{1/m+n+k}, \bar{0}\right) \tag{3.1}$$

is called a triple sequence space of summable to the limit 0, i.e.,

$$lim_{uvw\to\infty} \sum_m^u \sum_n^v \sum_k^w a^{pqr}_{mnk} \left((m+n+k)! \left(\frac{X_{mnk}}{Y_{mnk}}\right)\right)^{1/m+n+k} = \mu_{pqr}$$

and

$$\lim_{pqr\to\infty} \mu_{pqr} = 0$$

Define the means

$$\sigma^X_{pqr} = \frac{1}{pqr} \sum_{m=0}^p \sum_{n=0}^q \sum_{k=0}^r \left((m+n+k)! \left(\frac{X_{mnk}}{Y_{mnk}}\right)\right)^{1/m+n+k}$$

and

$$A\sigma^X_{pqr} = \frac{1}{pqr} \sum_{m=0}^p \sum_{n=0}^q \sum_{k=0}^r a^{pqr}_{mnk} \left(\left((m+n+k)! \left(\frac{X_{mnk}}{Y_{mnk}}\right)\right)^{1/m+n+k}, \bar{0}\right).$$

We say that $\left(\frac{X_{mnk}}{Y_{mnk}}\right)$ is statistically lacunary equivalent summable $(C,1,1,1)$ to 0, if the sequence $\sigma = \left(\sigma^X_{mnk}\right)$ is statistically convergent to $\bar{0}$, that is, $st_3 - lim_{pqr}\sigma^X_{pqr} = 0$. It is denoted by $C_{111}\left(st_3\right)$.
Let $q_m, \overline{q}_n$ and $\overline{\overline{q}}_k$ be sequences of positive numbers and $Q_r = q_{11} + \cdots + q_{rs}$, $\overline{Q}_s = \overline{q}_{11} + \cdots + \overline{q}_{rs}$ and $\overline{\overline{Q}}_t = \overline{\overline{q}}_{11} + \cdots + \overline{\overline{q}}_{rs}$.

Definition 3.1. A triple $(X, P, *)$ be a $PP-$ space. Then a triple sequence $X = (X_{mnk})$ is said to statistically convergent to $\bar{0}$ with respect to the probabilistic $p-$ metric $P-$ provided that for every $\epsilon > 0$ and $\gamma \in (0,1)$

$$\delta\Bigg(\Bigg\{m,n,k \in \mathbb{N} : P - lim_{r,s,t\to\infty}\frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k$$
$$\left[f\left(A\sigma^X_{pqr}\right)\left(\epsilon\right)\right] \leq 1-\gamma\Bigg\}\Bigg) = 0$$

or equivalently

$$P - lim_{r,s,t\to\infty}\frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t}\overset{lim_{k\ell v}\frac{1}{k\ell v}m \leq k, n\leq \ell, k \leq v\,:}{\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t}} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left(A\sigma^X_{pqr}\right)(\epsilon)\right] \leq 1-\gamma = 0$$

In this case we write $St_{PP} - lim_X = \bar{0}$.

Definition 3.2. A triple $(X, P, *)$ be a $PP-$ space. The two non-negative sequences $X = (X_{mnk})$ and $Y = (Y_{mnk})$ are said to be almost asymptotically statistical equivalent of multiple $\bar{0}$ in $PP-$ space X if for every $\epsilon > 0$ and $\gamma \in (0,1)$.

$$\delta\Bigg(\Bigg\{m,n,k \in \mathbb{N} : P - lim_{r,s,t\to\infty}\frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t}$$
$$\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left(A\sigma^X_{pqr}\right)\left(\epsilon\right), \bar{0}\right] \leq 1-\gamma\Bigg\}\Bigg) = 0$$

or equivalently

$$lim_{k\ell v}\frac{1}{k\ell}\left|\left\{m \leq k, n \leq \ell, k \leq v : P_{\left((m+n+k)!\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k} - \bar{0}}(\epsilon) \leq 1-\gamma\right\}\right| = 0.$$

In this case we write $X \overset{\widehat{S}(PP)}{\equiv} Y$.

Definition 3.3. A triple $(X, P, *)$ be a $PP-$ space and $\theta = (m_r n_s k_t)$ be a lacunary sequence. The two non-negative sequences $X = (X_{mnk})$ and $Y = (Y_{mnk})$ are said to be a almost asymptotically lacunary statistical equivalent of multiple $\bar{0}$ in $PP-$ space X if for every $\epsilon > 0$ and $\gamma \in (0,1)$

$$\delta_\theta\left(\left\{m,n,k \in I_{r,s,t} : P_{\left((m+n+k)!\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k} - \bar{0}}(\epsilon) \leq 1-\gamma\right\}\right) = 0 \qquad (3.2)$$

or equivalently

$$lim_{rst}\frac{1}{h_{rst}}\left|\left\{m,n\in I_{rst}:P_{\left((m+n+k)!\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\epsilon)\leq 1-\gamma\right\}\right|=0.$$

In this case we write $X \overset{\widehat{S_\theta}(PP)}{\equiv} Y.$

Lemma 3.4. *A triple* $(X,P,*)$ *be a* $PP-$ *space. Then for every* $\epsilon>0$ *and* $\gamma\in(0,1)$, *the following statements are equivalent:*

(1) $lim_{rst}\frac{1}{h_{rst}}\left|\left\{m,n,k\in I_{rst}:P_{\left((m+n+k)!\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\epsilon)\leq 1-\gamma\right\}\right|=0,$

(2) $\delta_\theta\left(\left\{m,n,k\in I_{r,s,t}:P_{\left((m+n+k)!\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\epsilon)\leq 1-\gamma\right\}\right)=0,$

(3) $\delta_\theta\left(\left\{m,n,k\in I_{r,s,t}:P_{\left((m+n+k)!\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\epsilon)\leq 1-\gamma\right\}\right)=1,$

(4) $lim_{rst}\frac{1}{h_{rst}}\left|\left\{m,n,k\in I_{rst}:P_{\left((m+n+k)!\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\epsilon)\leq 1-\gamma\right\}\right|=1.$

4. Main Results

Theorem 4.1. *Let* f *be a Musielak Orlicz function and a triple* $(X,P,*)$ *be a* $PP-$ *space. If two triple sequences* $X=(X_{mnk})$ *and* $Y=(Y_{mnk})$ *are almost asympototically lacunary statistical equivalent of multiple* $\bar{0}$ *with respect to the probabilistic* $p-$ *metric* P, *then* $\bar{0}$ *is unique sequence.*

Proof: Assume that $X \overset{\widehat{S_\theta^{\bar{0}}}(PP)}{\equiv} Y$. For a given $\lambda>0$ choose $\gamma\in(0,1)$ such that $(1-\gamma)>1-\lambda$. Then, for any $\epsilon>0$, define the following set:

$$K=\left\{m,n,k\in I_{r,s,t}:P_{\left((m+n+k)!\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\epsilon)\leq 1-\gamma\right\}$$

Then, clearly

$$lim_{rst}\frac{K\bigcap\bar{0}}{h_{rst}}=1,$$

so K is non-empty set,since $x \overset{\widehat{S_\theta^{\bar{0}}}(PP)}{\equiv} y, \delta_\theta(K)=0$ for all $\epsilon>0$, which implies $\delta_\theta(\mathbb{N}-K)=1$. If $m,n,k\in\mathbb{N}-K$, then we have

$$P_{\bar{0}}(\epsilon)=P_{\left((m+n+k)!\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\epsilon)>(1-\gamma)\geq 1-\lambda$$

since λ is arbitrary, we get $P_{\bar{0}}(\epsilon)=1$.
This completes the proof.

Theorem 4.2. *Let* f *be a Musielak Orlicz function and a triple* $(X,P,*)$ *be a* $PP-$ *space. For any lacunary sequence* $\theta=(m_rn_sk_t), \widehat{S_\theta}(PP)\subset\widehat{S}(PP)$ *if* $limsup_{rst}q_{rst}<\infty$.

Proof: If $limsup_{rst}q_{rst}<\infty$. then there exists a $B>0$ such that $q_{rst}<B$ for all $r,s,t\geq 1$. Let $X \overset{\widehat{S_\theta}(PP)}{\equiv} Y$ and $\epsilon>0$. Now we have to prove $\widehat{S}(PP)$. Set

$$K_{rst} = \left| \left\{ m, n, k \in I_{r,s,t} : P_{\left((m+n+k)! \left| \frac{X_{mnk}}{Y_{mnk}} \right| \right)^{1/m+n+k} - \bar{0}} (\epsilon) > 1 - \gamma \right\} \right|.$$

Then by definition, for given $\epsilon > 0$, there exists $r_0 s_0 t_0 \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that

$$\frac{K_{rst}}{h_{rst}} < \frac{\epsilon}{2B} \text{ for all } r > r_0, s > s_0 \text{ and } t > t_0.$$

Let $M = max \{K_{rst} : 1 \le r \le r_0, 1 \le s \le s_0, 1 \le t \le t_0\}$ and let uvw be any positive integer with $m_{r-1} < u \le m_r, n_{s-1} < v \le n_s$ and $k_{t-1} < w \le k_t$. Then
$\frac{1}{uvw} \left| \left\{ m \le u, n \le v, k \le w : P_{\left((m+n+k)! \left| \frac{X_{mnk}}{Y_{mnk}} \right| \right)^{1/m+n+k} - \bar{0}} (\epsilon) > 1 - \gamma \right\} \right| \le$
$\frac{1}{m_{r-1} n_{s-1} k_{t-1}} \left| \left\{ m \le m_r, n \le n_s, k \le k_t : P_{\left((m+n+k)! \left| \frac{X_{mnk}}{Y_{mnk}} \right| \right)^{1/m+n+k} - \bar{0}} (\epsilon) > 1 - \gamma \right\} \right| =$
$\frac{1}{m_{r-1} n_{s-1} k_{t-1}} \{K_{111} + \cdots + K_{rst}\}$
$\le \frac{M}{m_{r-1} n_{s-1} k_{t-1}} r_0 s_0 t_0 + \frac{\epsilon}{2B} q_{rst} \le \frac{M}{m_{r-1} n_{s-1} k_{t-1}} r_0 s_0 t_0 + \frac{\epsilon}{2}.$
This completes the proof.

Theorem 4.3. *Let f be a Musielak Orlicz function and a triple $(X, P, *)$ be a $PP-$ space. For any lacunary sequence $\theta = (m_r n_s k_t)$, $\widehat{S} (PP) \subset \widehat{S_\theta} (PP)$ if $liminf_{rst} q_{rst} > 1$.*

Proof: If $liminf_{rst} q_{rst} > 1$, then there exists a $\beta > 0$ such that $q_{rst} > 1 + \beta$ for sufficiently large rst, which implies

$$\frac{h_{rst}}{K_{rst}} \ge \frac{\beta}{1+\beta}.$$

Let $X \overset{\widehat{S^{\bar{0}}}(pp)}{\equiv} Y$, then for every $\epsilon > 0$ and for sufficiently large r, s, t we have
$\frac{1}{m_r n_s k_t} \left| \left\{ m \le m_r, n \le n_s, k \le k_t : P_{\left((m+n+k)! \left| \frac{X_{mnk}}{Y_{mnk}} \right| \right)^{1/m+n+k} - \bar{0}} (\epsilon) > 1 - \gamma \right\} \right| \ge$
$\frac{1}{m_r n_s k_t} \left| \left\{ m, n, k \in I_{rst} : P_{\left((m+n+k)! \left| \frac{X_{mnk}}{Y_{mnk}} \right| \right)^{1/m+n+k} - \bar{0}} (\epsilon) > 1 - \gamma \right\} \right| \ge$
$\frac{\beta}{1+\beta} \frac{1}{h_{rst}} \left| \left\{ m, n, k \in I_{rst} : P_{\left((m+n+k)! \left| \frac{X_{mnk}}{Y_{mnk}} \right| \right)^{1/m+n+k} - \bar{0}} (\epsilon) > 1 - \gamma \right\} \right|.$ Therefore
$X \overset{\widehat{S^{\bar{0}}_\theta}(pp)}{\equiv} Y.$
This completes the proof.

Corollary 4.4. *Let f be a Musielak Orlicz function and a triple $(X, P, *)$ be a $PP-$ space. For any lacunary sequence $\theta = (m_r n_s)$, with $1 < liminf_{rs} q_{rs} \le limsup_{rst} q_{rst} < \infty$, then $\widehat{S} (PP) = \widehat{S_\theta} (PP)$.*

Proof: The result clearly follows from Theorem 4.2 and Theorem 4.3.

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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