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New Fixed Point Approach For a Fully Nonlinear Fourth Order Boundary Value Problem*

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ABSTRACT: In this paper we propose a method for investigating the solvability and iterative solution of a nonlinear fully fourth order boundary value problem. Namely, by the reduction of the problem to an operator equation for the right-hand side function we establish the existence and uniqueness of a solution and the convergence of an iterative process. Our method completely differs from the methods of other authors and does not require the condition of boundedness or linear growth of the right-hand side function on infinity. Many examples, where exact solutions of the problems are known or not, demonstrate the effectiveness of the obtained theoretical results.

Key Words: Fully fourth order nonlinear boundary value problem, Existence and uniqueness of solution, Iterative method.

Contents

1. Introduction

In this paper we consider the boundary value problem

$$u^{(4)}(x) = f(x, u(x), u'(x), u''(x), u'''(x)), \quad 0 < x < 1, \tag{1.1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.2}$$

where $f : [0,1] \times \mathbb{R}^4$ is continuous.

This problem models the bending equilibrium of a beam on an elastic foundation, whose two ends are simply supported.

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The special case of equation (1.1), where f does not contain derivative terms u' and u''', i.e.

$$u^{(4)}(x) = f(x, u(x), u''(x)), \quad 0 < x < 1, \tag{1.3}$$

has studied by several authors. For example, in 1986, Aftabizadeh [1] showed the existence of a solution to the probem (1.3)-(1.2) under the restriction that f is a bounded function. In 1988 Yang [12] extended these results by letting f satisfy a growth condition of the form $|f(x, u, v)| \leq a|u| + b|v| + c$, where a, b, c are positive constants such that $a/\pi^4 + b/\pi^2 < 1$.

In 1997 Ma et al. [9] and in 2004 Bai et al. [4] by the monotone method in the presence of lower and upper solutions constructed two monotone sequences of functions converging to the extremal solutions of the problem under some monotone condition of f. The idea of Bai et al. is used in a recent work of Li [7]. Except for the existence Li successfully investigated the uniqueness of the problem. It should be emphasised that in the monotone method the assumption of the presence of lower and upper solutions is always needed and the finding of them is not easy.

Differently from the approaches to the problem (1.3)-(1.2) of the authors mentioned above and the approaches to other nonlinear fourth order differential equations with various boundary conditions including nonlocal equations and nonlocal boundary conditions, e.g., in [2,3,10,11], where the problem is led to integral operators for the unknown function $u(x)$, in [6] we reduce the original problem to an operator equation for the right-hand side function. This idea was used by ourselves first in a previous paper [5] when studying the Neumann problem for a biharmonic type equation.

For the fully fourth order nonlinear boundary value problem (1.1)-(1.2), in 2013, Li and Liang [8] established the existence of solution for the problem under the restriction of the linear growth of the function $f(x, u, y, v, z)$ in each variable on the infinity. In the present paper by a completely different approach, namely by the approach of [6] we free this restriction. Due to the reduction of the problem to an operator equation for the right hand side function, which will be proved to be contractive, we establish the existence and uniqueness of a solution and the convergence of an iterative method for finding the solution. Some examples demonstrate the applicability of our approach and the efficiency of the proposed iterative method.

2. The existence and uniqueness of a solution

For investigating the problem (1.1)-(1.2) we set

$$\varphi = f(x, u, u', u'', u'''), \quad v = u''. \tag{2.1}$$

Then the problem is reduced to the two second order problems

$$\begin{cases} v'' = \varphi, \quad 0 < x < 1, \\ v(0) = v(1) = 0, \end{cases} \tag{2.2}$$

$$\begin{cases} u'' = v, \quad 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{2.3}$$

Clearly, the solutions v and u of the above problems depend on φ, that is, $v = v_\varphi(x)$, $u = u_\varphi(x)$. Therefore, for φ we have the equation

$$\varphi = A\varphi, \tag{2.4}$$

where A is a nonlinear operator defined by

$$(A\varphi)(x) = f(x, u_\varphi(x), y_\varphi(x), v_\varphi(x), z_\varphi(x)), \tag{2.5}$$

with $y_\varphi(x) = u'_\varphi(x)$, $z_\varphi(x) = v'_\varphi(x)$.
We shall prove that under some conditions A is contractive operator.
For each number $M > 0$ denote

$$\mathcal{D}_M = \left\{ (x, u, y, v, z) |\ 0 \le x \le 1,\ |u| \le \frac{M}{64},\ |y| \le \frac{M}{16},\ |v| \le \frac{M}{8},\ |z| \le \frac{M}{2} \right\}, \tag{2.6}$$

and by $B[O, M]$ we denote the closed ball centered at O with the radius M in the space of continuous functions $C[0, 1]$ with the norm

$$\|\varphi\| = \max_{0 \le x \le 1} |\varphi(x)|.$$

Lemma 2.1. *Assume that there exist numbers $M, c_0, c_1, c_2, c_3 \ge 0$ such that*

$$|f(x, u, y, v, z)| \le M, \tag{2.7}$$

$$|f(x, u_2, y_2, v_2, z_2,) - f(x, u_1, y_1, v_1, z_1)| \le \\ c_0|u_2 - u_1| + c_1|y_2 - y_1| + c_2|v_2 - v_1| + c_3|z_2 - z_1| \tag{2.8}$$

for any $(x, u, y, v, z), (x, u_i, y_i, v_i, z_i) \in \mathcal{D}_M$ $(i = 1, 2)$.

Then, the operator A defined by (2.5), where v_φ, u_φ are the solutions of the problems (2.2),(2.3), maps the closed ball $B[O, M]$ into itself. Moreover, if

$$q := \frac{c_0}{64} + \frac{c_1}{16} + \frac{c_2}{8} + \frac{c_3}{2} < 1 \tag{2.9}$$

then A is contractive operator in $B[O, M]$.

Proof: Let φ be a function in $B[O, M]$. The solution of the problems (2.2) and (2.3) can be represented in the form

$$v(x) = -\int_0^1 G(x, t)\varphi(t)dt, \quad u(x) = -\int_0^1 G(x, t)v(t)dt$$

where $G(x, t)$ is the Green function for the differential operator $-u''$ with homogeneous Dirichlet boundary condition

$$G(x, t) = \begin{cases} x(1 - t), & 0 \le x \le t \le 1, \\ t(1 - x), & 0 \le t \le x \le 1. \end{cases}$$

Since

$$\int_0^1 |G(x,t)|dt \le \frac{1}{8}, \quad x \in [0,1] \tag{2.10}$$

$$\int_0^1 |G'_x(x,t)|dt \le \frac{1}{2}, \quad x \in [0,1] \tag{2.11}$$

we have

$$\|v\| \le \frac{1}{8}\|\varphi\| \le \frac{1}{8}M. \tag{2.12}$$

Therefore, for the solution of the problem (2.3) we have the estimate

$$\|u\| \le \frac{1}{8}\|v\| \le \frac{1}{64}M. \tag{2.13}$$

Hence

$$\begin{aligned} \|z\| &= \|v'\| \le \frac{1}{2}\|\varphi\| \le \frac{1}{2}M, \\ \|y\| &= \|u'\| \le \frac{1}{2}\|v\| \le \frac{1}{16}M. \end{aligned} \tag{2.14}$$

Therefore, taking into account (2.5) and the condition (2.7), we have $A\varphi \in B[0,M]$, i.e. the operator maps $B[0,M]$ into itself.
Now, let $\varphi_1, \varphi_2 \in B[O,M]$ and $v_1, v_2; u_1, u_2$ be the solutions of the problems (2.2), (2.3), i.e., for $i = 1,2$

$$\begin{cases} v_i'' = \varphi_i, \quad 0 < x < 1, \\ v_i(0) = v_i(1) = 0, \end{cases} \tag{2.15}$$

$$\begin{cases} u_i'' = v_i, \quad 0 < x < 1, \\ u_i(0) = u_i(1) = 0. \end{cases} \tag{2.16}$$

Using the representation of the solutions v_i and u_i via the Green function and the estimates (2.10) and (2.11) we obtain

$$\begin{aligned} &\|v_2 - v_1\| \le \frac{1}{8}\|\varphi_2 - \varphi_1\|, \|u_2 - u_1\| \le \frac{1}{64}\|\varphi_2 - \varphi_1\|, \\ &\|z_2 - z_1\| \le \frac{1}{2}\|\varphi_2 - \varphi_1\|, \|y_2 - y_1\| \le \frac{1}{2}\|v_2 - v_1\| \le \frac{1}{16}\|\varphi_2 - \varphi_1\|. \end{aligned} \tag{2.17}$$

Now from (2.5) and (2.8) it follows

$$\begin{aligned} |A\varphi_2 - A\varphi_1| &= |f(x, u_2, y_2, v_2, z_2) - f(x, u_1, y_1, v_1, z_1)| \\ &\le c_0|u_2 - u_1| + c_1|y_2 - y_1| + c_2|v_2 - v_1| + c_3|z_2 - z_1|. \end{aligned}$$

Using the estimate (2.17) we obtain

$$\|A\varphi_2 - A\varphi_1\| \le \left(\frac{c_0}{64} + \frac{c_1}{16} + \frac{c_2}{8} + \frac{c_3}{2}\right)\|\varphi_2 - \varphi_1\|$$

Therefore, A is an contractive operator in $B[O,M]$ provided the condition (2.9) is satisfied. $\square$

Theorem 2.1. *Under the assumptions of Lemma 2.1, the problem (1.1)-(1.2) has a unique solution u and there hold the estimates*

$$\|u\| \le \frac{M}{64}, \quad \|u'\| \le \frac{M}{16}, \quad \|u''\| \le \frac{M}{8}, \quad \|u'''\| \le \frac{M}{2}. \tag{2.18}$$

Proof: It is easy to see that the solution of the problem (1.1)-(1.2) is the function $u(x)$ obtained from the problems (2.2),(2.3), where φ is the unique fixed point of A. The estimates (2.18) indeed are the estimates (2.12)- (2.14). □

Now consider a particular case of Theorem 2.1. Let us denote

$$\mathcal{D}_M^+ = \left\{ (x,u,y,v,z) |\ 0 \le x \le 1;\ 0 \le u \le \frac{M}{64};\ |y| \le \frac{M}{16};\ \frac{-M}{8} \le v \le 0;\ |z| \le \frac{M}{2} \right\}, \tag{2.19}$$

and

$$S_M = \{\varphi \in C[0,1] \mid 0 \le \varphi(x) \le M\}.$$

□

Theorem 2.2. *(Positivity of solution) Suppose that in $\mathcal{D}_M^+$ the function f is such that*

$$0 \le f(x,u,y,v,z) \le M, \tag{2.20}$$

and satisfies the Lipschitz condition (2.8). Then, the operator A defined by (2.5), where v_φ, u_φ are the solutions of the problems (2.2),(2.3) maps the strip S_M into itself. Moreover, if (2.9) is satisfied then A is contractive operator in S_M. Therefore, the problem (1.1)-(1.2) has a unique nonnegative solution.

Proof: The proof of the theorem is similar to that of Lemma 2.1 and Theorem 2.1, where instead of the ball we consider the strip S_M. □

3. Iterative method

Consider the following iterative process:

1. Given

$$\varphi_0(x) = f(x,0,0,0,0). \tag{3.1}$$

2. Knowing φ_k $(k = 0,1,...)$ solve consecutively two problems

$$\begin{cases} v_k'' = \varphi_k(x), & 0 < x < 1, \\ v_k(0) = v_k(1) = 0, \end{cases} \tag{3.2}$$

$$\begin{cases} u_k'' = v_k(x), & 0 < x < 1, \\ u_k(0) = u_k(1) = 0. \end{cases} \tag{3.3}$$

3. Update

$$\varphi_{k+1} = f(x, u_k, u_k', v_k, v_k'). \tag{3.4}$$

Set $p_k = \dfrac{q^k}{1-q}\|\varphi_1 - \varphi_0\|$. We obtain the following result.

Theorem 3.1. *Under the assumptions of Lemma 2.1 the above iterative method converges with the rate of geometric progression and there hold the estimates*

$$\begin{aligned} &\|u_k - u\| \le \frac{p_k}{64}, \quad \|u_k' - u'\| \le \frac{p_k}{16}, \\ &\|u_k'' - u''\| \le \frac{p_k}{8}, \quad \|u_k''' - u'''\| \le \frac{p_k}{2}, \end{aligned} \tag{3.5}$$

where u is the exact solution of the problem (1.1)-(1.2).

Proof: Notice that the above iterative method is the successive iteration method for finding the fixed point of the operator A with the initial approximation (3.1) belonging to $B[O, M]$. Therefore, it converges with the rate of geometric progression and there is the estimate

$$\|\varphi_k - \varphi\| \le \frac{q^k}{1-q}\|\varphi_1 - \varphi_0\|. \tag{3.6}$$

Combining this estimate with those of the type (2.17) we obtain (3.5), and the theorem is proved. □

For numerical realization of the iterative method we use the difference schemes of fourth order accuracy for the Dirichlet problems (3.2), (3.3) on uniform grids $\overline{\omega}_h = \{x_i = ih,\ i = 0, 1, ..., N;\ h = 1/N\}$. Namely, for the typical second order problem

$$\begin{cases} v'' = \varphi(x), \quad 0 < x < 1, \\ v(0) = v(1) = 0. \end{cases}$$

we use the Numerov difference scheme

$$\begin{aligned} &\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} = \frac{\varphi_{i-1} + 10\varphi_i + \varphi_{i+1}}{12}, \ 1 \le i \le N-1, \\ &v_0 = v_N = 0 \end{aligned}$$

For grid functions on $\overline{\omega}_h$ we use the norm $\|u\|_{\overline{\omega}_h} = \max\limits_{0 \le i \le N} |u(x_i)|$, but in what follows for brevity we omit the subscript $\overline{\omega}_h$. The iterations are performed until $e_k = \|u_k - u_{k-1}\| \le 10^{-16}$. In the tables of results of computation n is the number of grid points, $error = \|u_k - u_d\|$, $error1 = \|u_k' - u_d'\|$, $error2 = \|u_k'' - u_d''\|$, $error3 = \|u_k''' - u_d'''\|$ where u_d is the exact solution.

4. Examples

In this section we consider some examples for demonstrating the applicability of the obtained theoretical results.

First, we consider an example for the case of known exact solution.

Example 4.1. *Consider the boundary value problem*

$$\begin{cases} u^{(4)}(x) = -\dfrac{u'''(x)}{3} + \cos\left(-\dfrac{\sin \pi x}{\pi^2} - u''(x)\right) - u'(x) - u^2(x) + \sin \pi x \\ \qquad + \dfrac{\cos \pi x}{\pi^3} + \dfrac{\sin^2 \pi x}{\pi^8} - \dfrac{\cos \pi x}{3\pi} - 1, \quad 0 < x < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

The exact solution of the problem is

$$u(x) = \frac{\sin(\pi x)}{\pi^4}.$$

In this example

$$f(x, u, y, v, z) = -\frac{z}{3} + \cos\left(-\frac{\sin \pi x}{\pi^2} - v\right) - y - u^2 + \sin \pi x$$
$$+ \frac{\cos \pi x}{\pi^3} + \frac{\sin^2 \pi x}{\pi^8} - \frac{\cos \pi x}{3\pi} - 1.$$

It is easy to see that the function $f(x, u, y, v, z)$ does not satisfy the conditions of [8, Theorem 1], so this theorem cannot guarantee the existence of a solution. Below, using the obtained theoretical results in Section 2 we show that the problem has a unique solution and the iterative method is very efficient for finding the solution.

First, choose M such that $|f(x, u, y, v, z)| \le M$. This number M may be defined from the inequality

$$|f(x, u, y, v, z)| \le \frac{M}{6} + 1 + \frac{M}{16} + \left(\frac{M}{64}\right)^2 + 1 + \frac{1}{\pi^3} + \frac{1}{\pi^8} + \frac{1}{3\pi} + 1 \le M$$

Clearly, $M = 5$ is a suitable choice. Then in the domain $\mathcal{D}_5$, since

$$f'_u = -2u, \quad f'_y = -1, \quad f'_v = \sin\left(-\frac{\sin \pi x}{\pi^2} - v\right), \quad f'_z = -\frac{1}{3},$$

$$|f'_u| \le 2\left(\frac{5}{64}\right) = \frac{5}{32}, \quad |f'_y| = 1, \quad |f'_v| \le 1, \quad |f'_z| = \frac{1}{3}$$

we can take $c_0 = \dfrac{5}{32}, c_1 = c_2 = 1, c_3 = \dfrac{1}{3}$. Then $q = \dfrac{c_0}{64} + \dfrac{c_1}{16} + \dfrac{c_2}{8} + \dfrac{c_3}{2} \approx 0.3566 < 1$. All the conditions of Theorem 2.1 are satisfied. Hence, the problem has a unique solution, and the iterative method converges.

The convergence of the iterative method for Example 4.1 is given in Table 1 and Figure 1.

Table 1: The convergence in Example 4.1

n	k	error	error1	error2	error3
30	11	1.0339e-8	7.3102e-7	5.2161e-8	7.3898e-6
50	11	1.3400e-9	9.5834e-8	6.7722e-9	9.7008e-7
100	11	8.3781e-11	6.0186e-9	4.2351e-10	6.0994e-8
1000	11	3.0210e-15	6.1669e-13	2.3273e-14	6.1727e-12

From Table 1 we observe that the convergence of the iterative method does not depend on the grid size.

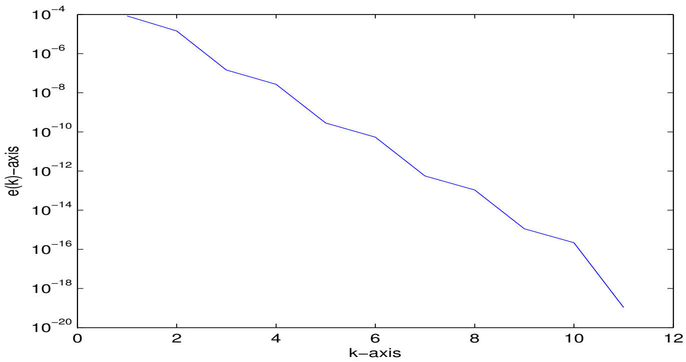

Figure 1: The graph of e_k in Example 4.1 for $n = 100$

In the next examples, the exact solution of problem (1.1)-(1.2) is not known.

Example 4.2. *Consider the boundary value problem*

$$\begin{cases} u^{(4)}(x) = -\dfrac{u'''(x)}{12} - u^2(x)\left(u''(x)\right)^2 - \left(u''(x)\right)^3 + \dfrac{u'(x)}{2} + \sin \pi x + 2, \\ \qquad 0 < x < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

In this example

$$f(x, u, y, v, z) = -\frac{z}{12} - u^2v^2 - v^3 + \frac{y}{2} + \sin \pi x + 2.$$

As in the previous example, obviously, that the function $f(x, u, y, v, z)$ does not satisfy the conditions of [8, Theorem 1], so this theorem cannot guarantee the existence of a solution.

Analogously as in Example 4.1 we can choose $M = 4$ and therefore, it is easy to verify that in the trip S_4 all the conditions of Theorem 2.2 are satisfied with $0 \leq f \leq 4$ and $c_0 = 0.03, c_1 = 0.5, c_2 = 0.75, c_3 = 0.083, q \approx 0.167 < 1$. Hence, the problem has a unique nonnegative solution, and the iterative method converges.

The numerical experiment for $n = 100$ shows that with the above stopping criterion after $k = 8$ iterations the iterative process stops and $e_8 = 6.5919e - 17$.

The convergence of the iterative method for Example 4.2 is given in Figure 2 and the graph of the approximate solution is depicted in Figure 3.

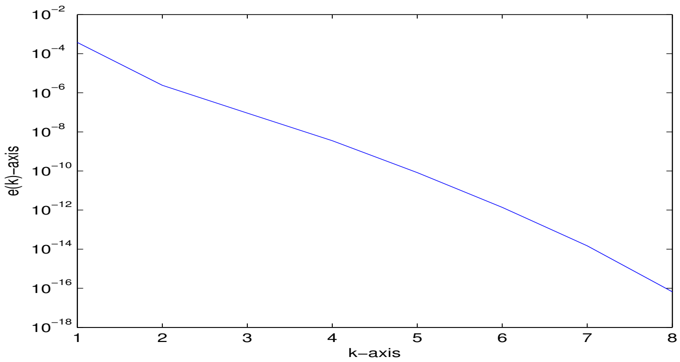

Figure 2: The graph of e_k in Example 4.2

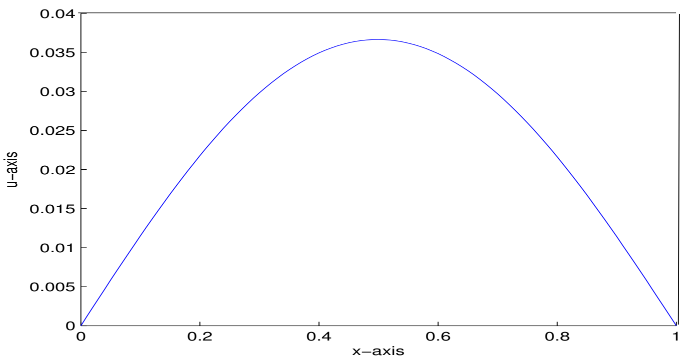

Figure 3: The graph of the approximate solution in Example 4.2

Example 4.3. *Consider the boundary value problem with exponential nonlinearity*

$$\begin{cases} u^{(4)}(x) = d_1 e^u + d_2 e^{u'} + d_3 e^{u''} + d_4 e^{u'''}, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

where $d_1, d_2, d_3, d_4 \in \mathbb{R}$.

In this example

$$f(x, u, y, v, z) = d_1 e^u + d_2 e^y + d_3 e^v + d_4 e^z.$$

Take, for example, $d_1 = 1, d_2 = 1, d_3 = 1, d_4 = 0$. We can choose $M = 4$. Then $c_0 = 1.0645, c_1 = 1.2840, c_2 = 1.6487, c_3 = 0$. Hence $q \approx 0.303$. Hence, the problem has a unique solution, and the iterative method converges.

The numerical experiment for $n = 100$ shows that with the above stopping criterion after $k = 13$ iterations the iterative process stops and $e_{13} = 6.9389e - 18$.

The convergence of the iterative method for Example 4.3 is given in Figure 4 and the graph of the approximate solution is depicted in Figure 5.

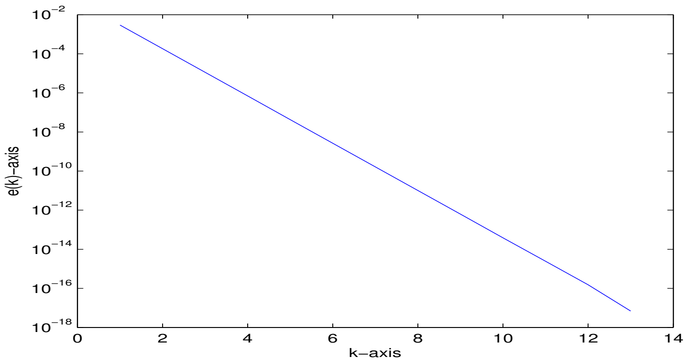

Figure 4: The graph of e_k in Example 4.3

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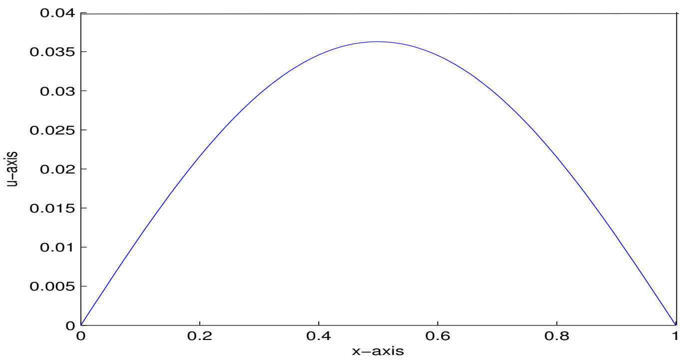

Figure 5: The graph of the approximate solution in Example 4.3

Moreover, below we show theoretically that this solution is nonnegative. Indeed, consider the domain

$$\mathcal{D}_4^+ = \left\{ (x, u, y, v, z) \mid 0 \le x \le 1; 0 \le u \le \frac{4}{64}; |y| \le \frac{4}{16}; \frac{-4}{8} \le v \le 0; |z| \le \frac{4}{2} \right\},$$

and the strip S_4

$$S_4 = \{ \varphi \in C[0,1] \mid 0 \le \varphi(x) \le 4 \}.$$

Therefore, it is easy to see that in $\mathcal{D}_4^+$ we have

$$0 \leq f(x, u.y, v, z) \leq 4$$

and all the conditions of Theorem 2.2 are satisfied. Hence, the problem has a unique nonnegative solution.

Example 4.4. *Consider the boundary value problem*

$$\begin{cases} u^{(4)}(x) = g_0(x)\left|u(x)\right|^{k_0} + g_1(x)\left|u'(x)\right|^{k_1} + g_2(x)\left|u''(x)\right|^{k_2} \\ \qquad\qquad + g_3(x)\left|u'''(x)\right|^{k_3}, \quad 0 < x < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

where $g_i \in C[0,1], k_i$ are integer constants, $k_i \geq 1, i = 0, 1, 2, 3.$
Let

$$\begin{aligned} f(x, u, y, v, z) &= g_0(x)|u|^{k_0} + g_1(x)|y|^{k_1} + g_2(x)|v|^{k_2} + g_3(x)|z|^{k_3}, \\ G_i &= \max_{x\in[0,1]} |g_i(x)|, i = 0, 1, 2, 3. \end{aligned} \tag{4.1}$$

Firstly, we prove that the $f(x, u, y, v, z)$ satisfies the Lipschitz condition with respect to u, y, v, z. For this reason we need the following claim.

Claim 1. *The function $t(x) = |x|^\alpha, \alpha \geq 1$ satisfies the Lipschitz condition in the interval $|x| \leq A$ (A is any positive constants), i.e. exists $L > 0$ such that*

$$|t(x) - t(y)| \leq L|x - y|, \quad \forall x, y \in \{z : |z| \leq A\}.$$

Proof: Since the role of x, y is the same it sufficies to consider the following cases of location of x, y.

Case 1. *If $0 \leq x \leq y \leq T$ then*

$$|t(x) - t(y)| = ||x|^\alpha - |y|^\alpha| = |x^\alpha - y^\alpha| = \alpha\xi^{\alpha-1}|x - y| \leq \alpha A^{\alpha-1}|x - y|,$$

where $\xi \in (x, y)$.

Case 2. *If $-A \leq y \leq x \leq 0$ then from Case 1 we have*

$$|t(x) - t(y)| = |t(-x) - t(-y)| \leq \alpha A^{\alpha-1}|x - y|.$$

Case 3. *If $-T \leq y \leq 0,\ 0 \leq x \leq A$ we have*

$$\begin{aligned} &|t(x) - t(y)| \leq |t(x) - t(0)| + |t(y) - t(0)| \leq \alpha A^{\alpha-1}x - \alpha A^{\alpha-1}y \\ &= \alpha A^{\alpha-1}(x - y) = \alpha A^{\alpha-1}|x - y|. \end{aligned}$$

Thus, in all cases we can choose $L = \alpha A^{\alpha-1}$, and Claim 1 is proved.

□

Now, return to the function f given by (4.1). Using Claim 1 we have

$$\begin{aligned}
&|f(x,u_1,y_1,v_1,z_1) - f(x,u_2,y_2,v_2,z_2)| \\
&\le G_0 \left||u_1|^{k_0} - |u_2|^{k_0}\right| + G_1 \left||y_1|^{k_1} - |y_2|^{k_1}\right| + G_2 \left||v_1|^{k_2} - |v_2|^{k_2}\right| \\
&+ G_3 \left||z_1|^{k_3} - |z_2|^{k_3}\right| \\
&\le G_0 k_0 \left(\frac{M}{64}\right)^{k_0-1} |u_1 - u_2| + G_1 k_1 \left(\frac{M}{16}\right)^{k_1-1} |y_1 - y_2| \\
&+ G_2 k_2 \left(\frac{M}{8}\right)^{k_2-1} |v_1 - v_2| + G_3 k_3 \left(\frac{M}{2}\right)^{k_3-1} |z_1 - z_2|.
\end{aligned}$$

Thus, the function f satisfies the Lipschitz condition (2.8) and the quantity q given by (2.9) has the value

$$q = \frac{k_0 G_0}{64}\left(\frac{M}{64}\right)^{k_0-1} + \frac{k_1 G_1}{16}\left(\frac{M}{16}\right)^{k_1-1} + \frac{k_2 G_2}{8}\left(\frac{M}{8}\right)^{k_2-1} + \frac{k_3 G_3}{2}\left(\frac{M}{2}\right)^{k_3-1}. \tag{4.2}$$

Assume that the constants G_0, G_1, G_2, G_3 satisfy the condition

$$\frac{G_0}{64} + \frac{G_1}{16} + \frac{G_2}{8} + \frac{G_3}{2} < 1. \tag{4.3}$$

Then, it is not difficult to choose $M > 0$, so that $q < 1$. Moreover, this M is such that $|f(x,u,y,v,z)| \le M$. Indeed, set $K = \min\{k_i,\ i = 0,1,2,3\} \ge 1$. Then for any $(x,u,y,v,z) \in \mathcal{D}_M$ we have

$$\begin{aligned}
&|f(x,u,y,v,z)| \le G_0\left(\frac{M}{64}\right)^{k_0} + G_1\left(\frac{M}{16}\right)^{k_1} + G_2\left(\frac{M}{8}\right)^{k_2} + G_3\left(\frac{M}{2}\right)^{k_3} \le \\
&\left(\frac{k_0 G_0}{64}\left(\frac{M}{64}\right)^{k_0-1} + \frac{k_1 G_1}{16}\left(\frac{M}{16}\right)^{k_1-1} + \frac{k_2 G_2}{8}\left(\frac{M}{8}\right)^{k_2-1} + \frac{k_3 G_3}{2}\left(\frac{M}{2}\right)^{k_3-1}\right)\frac{M}{K} \\
&= \frac{qM}{K}.
\end{aligned}$$

So, since $q < 1, K \ge 1$ we have $|f(x,u,y,v,z)| \le M$. Thus, all the conditions of Theorem 2.1 are satisfied. Hence, the problem has a unique solution.

Remark 4.1. *In all Examples 4.1-4.4 the right-hand side functions do not satisfy the condition of linear growth at infinity, therefore, [8, Theorem 1] cannot ensure the existence of a solution of the problems. But as seen above using the theory in Section 2 and 3 we have established the existence and uniqueness of a solution and the convergence of the iterative method. This convergence is also confirmed by numerical experiments.*

Remark 4.2. *Theorem 2.1 gives only sufficient conditions for the problem (1.1), (1.2) to have a unique solution and the iterative method (3.1)-(3.4) to be convergent. When these conditions are not met, in some cases the problem may have a solution and the iterative method may be convergent, too. Below, we show two examples, for one of them an exact solution is known, and for the other, Theorem 2.1 gives no information of its solution, but in the both examples the iterative method converges.*

Example 4.5. *Consider the problem*

$$\begin{cases} u^{(4)}(x) = \pi|u|^{\frac{1}{2}} + \pi|u'|^{\frac{1}{2}} - |u''|^{\frac{1}{2}} - |u'''|^{\frac{1}{2}} + \pi^4 \sin \pi x, \quad 0 < x < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

We can verify that the function $u(x) = \sin(\pi x)$ is an exact solution of the problem. It is interesting that for this example the starting approximation is $\varphi_0(x) = \pi^4 \sin \pi x$, and solving the problems at the $0-$iteration

$$\begin{cases} v_0'' = \varphi_0(x), \quad 0 < x < 1, \\ v_0(0) = v_0(1) = 0, \end{cases}$$

$$\begin{cases} u_0'' = v_0(x), \quad 0 < x < 1, \\ u_0(0) = u_0(1) = 0. \end{cases}$$

we obtain $u_0(x) = \sin \pi x$, which coincides with the exact solution.

Example 4.6. *Consider the boundary value problem*

$$\begin{cases} u^{(4)}(x) = \pi|u|^{\frac{1}{2}} + \pi|u'|^{\frac{1}{2}} + |u''|^{\frac{1}{2}} + |u'''|^{\frac{1}{2}} + 1, \quad 0 < x < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

In this example $f(x, u, y, v, z) = \pi|u|^{\frac{1}{2}} + \pi|y|^{\frac{1}{2}} + |v|^{\frac{1}{2}} + |z|^{\frac{1}{2}} + 1$. The convergence of the iterative method for Example 4.6 with the criterion for stopping iterations $e_k \leq 10^{-12}$ is given in Table 2 and the graph of the approximate solution is depicted in Figure 6.

Table 2: The convergence in Example 4.6

n	k	error
50	25	$5.7446e-13$
200	29	$9.2934e-13$
1000	27	$2.7030e-13$
10000	25	$9.1095e-13$
30000	26	$4.5841e-13$

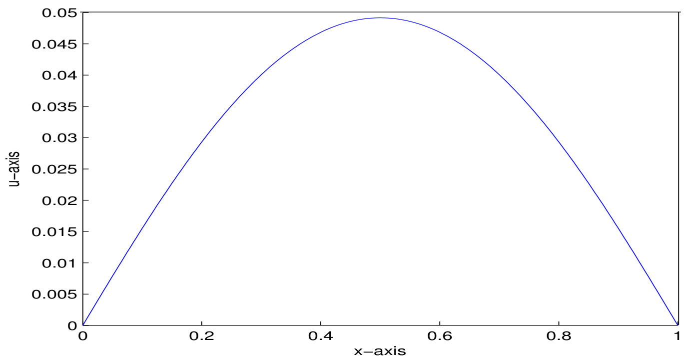

Figure 6: The graph of the approximate solution in Example 4.6 for $n = 200$

Notice that the existence of a solution in Examples 4.5 and 4.6 is guaranteed by [8].

5. Conclusion

In this paper we have established the existence and uniqueness of a solution of a fully fourth order nonlinear boundary value problem. Differently from the approaches of the other authors, we have reduced the problem to an operator equation for the right-hand side function. The investigation of the resulting operator equation does not require any condition for the right-hand side function with respect to all variables on infinity. The convergence of an iterative method has proved. Many examples have confirmed the validity of the obtained theoretical results and the wide applicability of the iterative method.

The proposed method can be used for some other nonlinear boundary value problems for ordinary and partial differential equations. This is the direction of our research in the future.

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