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Existence of Solutions For a Class of Strongly Coupled $p(x)$-laplacian System *

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abstract: The present work is concerned with the study of a strongly coupled nonlinear elliptic system on the whole space $\mathbb{R}^N$ involving the $p(x)$-laplacien operator. We employ variational methods and the theory of the variable exponent Sobolev spaces, in order to establish some sufficient conditions for the existence of non-trivial solutions.

Key Words: $p(x)$-Laplace operator, Generalized Lebesgue-Sobolev spaces, Strongly system, Variational methods.

Contents

1. Introduction

In this paper, we study the existence of nontrivial weak solutions for the following (p, q)-gradient elliptic system:

$$\begin{cases} -\Delta_{p(x)}u(x) + a(x)|u(x)|^{p(x)-2}u = f(x, u, v) \quad \text{in } \mathbb{R}^N, \\ \\ -\Delta_{q(x)}v(x) + b(x)|v(x)|^{q(x)-2}v = g(x, u, v) \quad \text{in } \mathbb{R}^N, \\ \\ (u, v) \in W^{1,p(x)}(\mathbb{R}^N) \times W^{1,q(x)}\left(\mathbb{R}^N\right). \end{cases} \tag{1.1}$$

Here $p, q : \Omega \to \mathbb{R}$ two functions of class $C(\overline{\Omega})$ such that $1 < p(x), q(x) < N$ $(N \geq 2)$ for all $x \in \mathbb{R}^N$ and the coefficients a, b, are variables. The real-valued functions f, g are given functions and $\Delta_{p(x)}u$ is the $p(x)$-Laplacian operator defined by $\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$.

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The operator $\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)-$Laplace where p is a continuous non-constant function. This differential operator is a natural generalization of the p-Laplace operator $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a real constant. However, the $p(x)-$Laplace operator possesses more complicated nonlinearity than p-Laplace operator, due to the fact that $\Delta_{p(x)}$ is not homogeneous. This fact implies some diffculties; for example, we can not use the Lagrange Multiplier Theorem in many problems involving this operator.

The study of differential and partial differential involving variable exponent conditions is a new and an interesting topic. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics, electrorheological fluids, image processing, flow in porous media, calculus of variations, nonlinear elasticity theory, heterogeneous porous media models (see Acerbi-Mingione [1] , Diening [5], Ružička [15], Zhikov [17]) etc.... These physical problems were facilitated by the development of Lebesgue and Sobolev spaces with variable exponent.

In literature, elliptic systems with standard and nonstandard growth conditions have been studied by many authors. Let us briefly recall the literature concerning related elliptic systems. In [3,2] the authors show the existence of nontrivial solutions for the following $p-$Laplacian problem:

$$\begin{cases} -\Delta_p u(x) = a(x)|u(x)|^{p-2}u + b(x)|u|^{\alpha}|v|^{\beta}v + f \quad \text{in } \mathbb{R}^N, \\ -\Delta_q v(x) = c(x)|u|^{\alpha}|v|^{\beta}v + d(x)|u(x)|^{q-2}u + g) \ \text{ in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = \lim\limits_{|x|\to\infty} v(x) = 0, \quad (u,v) > 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{1.2}$$

where the p-Laplace operator $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$, with $p > 1$, α, $\beta > 0$, $p, q > 1$, and is a real f, g are given functions. In [3], the author's obtain necessary and sufficient conditions on the coefficients for having a maximum principle for system (1.2). Then using the method of sup and super solutions, they prove the existence of positive solutions under some conditions on the functions f and g. In [2], the authors apply the theory of monotone operators to obtain the nontrivial solutions of the system (1.2).

In Khafagy-Serag [10] deal with the following problem:

$$\begin{gathered} -\Delta_{p,P}u = a(x)|u|^{p-2}u + b(x)|u|^{\alpha}|v|^{\beta}v + f \quad \text{in } \Omega, \\ -\Delta_{Q,q}v = c(x)|u|^{\alpha}|v|^{\beta}u + d(x)|v|^{q-2}v + g \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where the degenerate p-Laplacian defined as $\Delta_{p,P}u = \mathrm{div}[P(x)|\nabla u|^{p-2}\nabla u]$. Using an approximation method, they apply the Schauder's Fixed Point Theorem to get the nontrivial solutions of the system. Moreover, they gives necessary and sufficient conditions for having the maximum principle for this system.

In DJELLIT-YOUBI-TAS [6] show the existence of nontrivial solutions for the following $p(x)$-Laplacian system:

$$\begin{cases} -\Delta_{p(x)} u = \frac{\partial F}{\partial u}(x,u,v) & \text{in } \mathbb{R}^N \\ -\Delta_{q(x)} v = \frac{\partial F}{\partial v}(x,u,v) & \text{in } \mathbb{R}^N \end{cases} \tag{1.3}$$

Here $p, q : \Omega \to \mathbb{R}$ two functions of class $C(\overline{\Omega})$ such that $1 < p(x), q(x) < N$ $(N \geq 2)$ for all $x \in \mathbb{R}^N$. However, the function F belongs to $C(\mathbb{R}^N \times \mathbb{R}^2)$. Introducing some natural growth hypotheses on the right-hand side of the system which will ensure the semi-continuous and coercivity for the corresponding Euler-Lagrange functional of the system, the authors use critical point theory to obtain the existence of non-trivial weak solution of the system (1.3). In OGRAS-MASHIYEV-AVCI-YUCEDAG [13] using a weak version of the Palais-Smale condition, that is, Cerami condition, they apply the mountain pass theorem to get the nontrivial solutions of the system (1.3).

In XU-AN [16] study the following elliptic systems of gradient type with non-standard growth conditions

$$\begin{aligned} -\Delta_{p(x)} u + |u|^{p(x)-2} u &= \frac{\partial F}{\partial u}(x,u,v) \quad \text{in } \mathbb{R}^N \\ -\Delta_{p(x)} v + |v|^{q(x)-2} v &= \frac{\partial F}{\partial v}(x,u,v) \quad \text{in } \mathbb{R}^N. \end{aligned}$$

The potential function F needs to satisfy Caratheodory conditions. Using critical point theory, they establish existence and multiplicity of solutions in sub-linear and super-linear cases.

Inspired by the above-mentioned papers, we deal with the existence of nontrivial solutions for system (1.1). We know that in the study of $p(x)$-Laplace equations in $\mathbb{R}^N$, a main difficulty arises from the lack of compactness. In this paper we will overcome this difficulty by establishing some growth conditions and regularity on the nonlinearities f and g, which will ensure the mountain pass geometry and Cerami condition for the corresponding Euler-Lagrange functional. By the mountain pass theorem, the basic results on the existence of solutions of system (1.1) will be presented.

The outline of this paper is as follows. In section 2, we will recall some basic facts about the variable exponent Lebesgue and Sobolev spaces which we will use later. Our main results are stated in Section 3. Proofs of our results will be presented in section 4.

2. Preliminary Results

To deal with the $p(x)$-Laplacian problem, we need introduce some functional spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$, $W_0^{1,p(\cdot)}(\Omega)$ and properties of the $p(x)$-Laplacian which we will use later. Denote by $S(\Omega)$ be the set of all measurable real-valued functions defined in Ω. Note that two measurable functions are considered as the same element of $S(\Omega)$ when they are equal almost everywhere. Set

$$L_+^\infty(\Omega) = \{h; h \in L^\infty(\Omega), \text{ess inf } h(x) > 1 \text{ for all } x \in \bar{\Omega}\}$$

For any $h \in L_+^\infty(\Omega)$ we define

$$h^+ = ess \sup_{x\in\Omega} h(x) > 1 \quad \text{and} \quad h^- = ess \inf_{x\in\Omega} h(x) > 1.$$

Let

$$L^{p(\cdot)}(\Omega) = \big\{u \in S(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < \infty\big\},$$

with the norm

$$|u|_{p(\cdot)} = |u|_{L^{p(\cdot)}(\Omega)} = \inf\big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\big\}.$$

The space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ becomes a Banach space. We call it variable exponent Lebesgue space. Moreover, this space is a separable, reflexive and uniform convex Banach space; see [9, Theorems 1.6, 1.10, 1.14].

The variable exponent Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \big\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\big\},$$

can be equipped with the norm

$$\|u\| = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}, \quad \forall u \in W^{1,p(\cdot)}(\Omega).$$

Note that $W_0^{1,p(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ under the norm $\|u\| = |\nabla u|_{p(\cdot)}$. The spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable, reflexive and uniform convex Banach spaces (see [9, Theorem 2.1]). The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leq p_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

We denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, the Hölder type inequality

$$\Big|\int_\Omega u(x)v(x)dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)}, \tag{2.1}$$

holds true.

An important role in manipulating the generalized Lebesgue spaces is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx.$$

If $(u_n), u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$. Then the following relations hold true.

$$\|u\|_{L^{p(x)}} > 1 \Rightarrow \|u\|^{p^-}_{L^{p(x)}} \leq \rho_{p(x)}(u) \leq \|u\|^{p^+}_{L^{p(x)}}, \tag{2.2}$$

$$\|u\|_{L^{p(x)}} < 1 \Rightarrow \|u\|^{p^+}_{L^{p(x)}} \leq \rho_{p(x)}(u) \leq \|u\|^{p^-}_{L^{p(x)}}, \tag{2.3}$$

$$\|u_n - u\|_{L^{p(x)}} \to 0 \text{ if and if } \rho_{p(x)}(u_n - u) \to 0. \tag{2.4}$$

The following result generalizes the well-known Sobolev embedding theorem.

Theorem 2.1 ([8,11])**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary and assume that $p \in C(\bar{\Omega})$ with $p(x) > 1$ for each $x \in \bar{\Omega}$. If $r \in C(\bar{\Omega})$ and $p(x) \leq r(x) \leq p^*(x)$ for all $x \in \overline{\Omega}$, then there exists a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$. Also, the embedding is compact $r(x) < p^*(x)$ almost everywhere in $\overline{\Omega}$ where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

3. Main Results

Before stating our main results, we make the following assumptions throughout this paper:

(B1) $a(x), b(x) \in L^\infty_{loc}(\Omega)$ and there exist $a_0, b_0 > 0$ such that

$$a(x) \geq a_0,\ b(x) \geq b_0\ \forall\, x \in \mathbb{R}^N \text{ and } a(x) \to \infty,\quad b(x) \to \infty \quad as \quad |x| \to \infty.$$

(F1) $f(x,w),\ g(x,w) \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R}),\ f(x,0,0) = 0,\ g(x,0,0) = 0,\ \forall\, x \in \mathbb{R}^N$. Moreover, there exists a function $F(x,w) \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ such that

$$\frac{\partial F}{\partial u} = f(x,w), \frac{\partial F}{\partial v} = g(x,w), \forall x \in \mathbb{R}^N, w = (u,v) \in \mathbb{R}^2$$

(F2) There exist a constant $\mu > \max\,(p^+, q^+)$such that

$$0 < \mu F(x,w) \leq w.\nabla F(x,w)$$

(F3) For $p^* = \frac{Np^-}{N-p^-}, q^* = \frac{Nq^-}{N-q^-}$ and $p^+ < \frac{Np^-}{N-p^-}$, $q^+ < \frac{Nq^-}{N-q^-}$, there exist a_1, a_2, b_1, b_2 such that

$$\begin{aligned} |\nabla f(x,w)| &\leq a_1(x)\,|w|^{p_1-2} + a_2(x)\,|w|^{p_2-1} \\ |\nabla g(x,w)| &\leq b_1(x)\,|w|^{q_1-2} + b_2(x)\,|w|^{q_2-1} \end{aligned}$$

$$a_i(x) \in L^{\alpha_i}(\mathbb{R}^N) \cap L^{\beta_i}(\mathbb{R}^N), \quad b_i(x) \in L^{\gamma_i}(\mathbb{R}^N) \cap L^{\delta_i}(\mathbb{R}^N), \ i = 1, 2$$

$$\alpha_i = \frac{p^{*-}}{p^{*-} - (p_i - 1)}, \qquad \gamma_i = \frac{q^{*-}}{q^{*-} - (q_i - 1)},$$

$$\beta_i = \frac{p^{*-} q^{*-}}{p^{*-} q^{*-} - p^{*}\left(p_{i-2}\right) - q^{*}}, \qquad \delta_i = \frac{p^{*-} q^{*-}}{p^{*-} q^{*-} - q^{*-}\left(q_{1_i - 2}\right) - p^{*}}$$

$$2 < p_1, q_1 < \min(p^+ - 1, q^+ - 1), \ \max(p^+ - 1, q^+ - 1) < p_2,$$

$$q_2 < \min(p^{*+} - 1, q^{*+} - 1)$$

Now we denote by E the product space $D^{1,p(x)} \times D^{1,q(x)}$, defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|(u,v)\| = |\nabla u|_{p(x)} + |u|_{(a(x),p(x))} + |\nabla v|_{q(x)} + |v|_{(b(x),q(x))}$$

We remark that condition **(B1)** implies that $E \subset W^{1,p(x)}(\Omega) \times W^{1,q(x)}(\Omega)$.

Set

$$J(u,v) = \int \left(|\nabla u|^{p(x)} + a(x)\,|u|^{p(x)} \right) dx + \int \left(|\nabla v|^{q(x)} + b(x)\,|v|^{q(x)} \right) dx.$$

Then, for all $w \in E$, the following relations hold

$$\begin{aligned} \|(u,v)\| &> 1 \Rightarrow \|(u,v)\|^{\min(p^-,q^-)} \le J(u,v) \le \|(u,v)\|^{\max\left(p^+,q^+\right)} \\ \|(u,v)\| &< 1 \Rightarrow \|(u,v)\|^{\max\left(p^+,q^+\right)} \le J(u,v) \le \|(u,v)\|^{\min\left(p-,q^-\right)} \end{aligned}$$

We say that $(u,v) \in E$ is a weak solution of problem (1.1) if

$$\begin{aligned} |\nabla u|^{p(x)-2} \nabla u \nabla \Phi &+ |\nabla v|^{q(x)-2} \nabla v \nabla \Psi \\ &+ \int a(x)\,|u|^{p(x)-2} u\Phi + \int b(x)\,|v|^{q(x)-2} v\Psi \\ &= \int f(x,u,v)\Phi + \int g(x,u,v)\Psi, \end{aligned}$$

for all $(\Phi, \Psi) \in E$.

The main result of this paper is given by the following theorem:

Theorem 3.1. *Assume conditions* **(B1)** *and* **(F1)-(F3)** *are fulfilled. Then problem* (1.1) *has a non trivial weak solution.*

We point out the fact that the result of Theorem 3.1 extends the results from [12] , [14] where similar equations are studied in the case of $p-$laplacian operator.

4. Proof of Theorem 3.1

The energy functional corresponding to problem (1.1) is defined as $I : E \to R$,

$$I(u,v) = \int \frac{1}{p(x)} |\nabla u|^{p(x)} + a(x)\,|u|^{p(x)} \quad + \quad \int \frac{1}{q(x)} |\nabla v(x)|^{q(x)} + b(x)\,|v(x)|^{q(x)} \\ - \quad \int F(x,u,v)$$

Similar arguments as those used in [7] assure that $I \in C^1(E,R)$ *with*

$$\begin{aligned} I^{\backprime}(u,v)\,(\Phi,\Psi) &= \int |\nabla u|^{p(x)-2} \nabla u \nabla \Phi + \int |\nabla v|^{q(x)-2} \nabla v \nabla \Psi \\ &+ \int a(x)\,|u|^{p(x)-2} u(x)\Phi(x) + \int b(x)\,|v|^{q(x)-2} v\Psi \\ &- \int f(x,u,v)\Phi - \int g(x,u,v)\Psi, \end{aligned}$$

for all $(\Phi,\Psi) \in E$.

Thus, we observe that any critical points of the functional I are a weak solutions for problem (1.1).

Our idea is to prove Theorem 3.1 by applying the Mountain pass theorem (see [4]). With that end in view, we prove some auxiliary results which show that the functional $I(u,v)$ has a mountain pass geometry.

Lemma 4.1. *If* **(B1)** *and* **(F1)-(F3)** *holds, then there exist* $\tau > 0$ *and* $\delta > 0$ *such that for all* $(u,v) \in E$ *with* $\|(u,v)\| = \tau$

$$I(u,v) \geq \delta > 0$$

Proof: From **(F2)**, it is easy to see that

$$F(x,w) \geq \min_{|s|=1} F(x,s).\,|w|^{\mu}\,,\ \forall x \in \mathbb{R}^N \ \textit{and}\ |w| \geq 1, w \in \mathbb{R}^2 \tag{4.1}$$

and

$$0 < F(x,w) < \max_{|s|=1} F(x,s).\,|w|^{\mu}\,, \forall x \in \mathbb{R}^N \ \textit{and}\ 0 < |w| \leq 1 \tag{4.2}$$

Using **(F3)**,we have

$$\begin{aligned} F(x,u,v) &= \int_0^u \frac{\partial F}{\partial s}(x,s,v)\,ds + F(x,0,v) \\ &= \int_0^u \frac{\partial F}{\partial s}(x,s,v)\,ds + \int_0^v \frac{\partial F}{\partial s}(x,0,s)\,ds + F(x,0,0) \end{aligned}$$

and

$$F(x,u,v) \leq c_1[a_1(x)\left(|u|^{p_1}+|v|^{p_1-1}|u|\right)+a_2(x)\left(|u|^{p_2}+|v|^{p_2-1}|u|\right)$$
$$+\ b_1(x)|v|^{q_1}+b_2(x)|v|^{q_2}]$$

So $\max_{|w|=1} F(x,w)\leq C$ in view of **(F3)** and since

$$\max(p^+-1,q^+-1)<p_2,q_2<\min(p^{*+}-1,q^{*+}-1),$$

we have

$$\lim_{|w|\to\infty}\frac{F(x,w)}{|w|^{\frac{Np^{--}}{N-p^-}}}=0, \tag{4.3}$$
$$\lim_{|w|\to\infty}\frac{F(x,w)}{|w|^{\frac{Nq^-}{N-q^-}}}=0,$$

It follows, that

$$\lim_{|w|\to 0}\frac{F(x,w)}{|w|^{p+}}=0,\ \text{ uniformly for } x\in\mathbb{R}^N \tag{4.4}$$
$$\lim_{|w|\to 0}\frac{F(x,w)}{|w|^{q+}}=0, \text{ uniformly for } x\in\mathbb{R}^N$$

Thus, we obtain using condition **(F2)** and **(F3)**, that $\forall\varepsilon>0$, $\exists\, C_\epsilon>0$, such that

$$F(x,w)\leq\epsilon\,|w|^{\max(p^+,q^+)}+C_\epsilon\,|w|^{\max(\frac{Np^-}{N-p^-},\frac{Nq^-}{N-q^-})},\qquad \|w\|<1 \tag{4.5}$$

Using (4.5), we have

$$\begin{aligned}
I(w) &\geq \frac{1}{p^+}J_1(u)+\frac{1}{q^+}J_2(v)-\int F(x,W)\,dx\\
&\geq \frac{1}{p^+}J_1(u)+\frac{1}{q^+}J_2(v)-\epsilon\,|W|^{\max(p^+,q^+)}-C_\epsilon\,|W|^{\max(\frac{Np^{--}}{N-p^-},\frac{Nq^{--}}{N-q^-})}\\
&\geq \left(\frac{1}{\max(p^+,q^+)}\|W\|-\epsilon c_1\|W\|^{\max(p^+,q^+)}-C_\epsilon\|W\|^{\max(\frac{Np^{--}}{N-p^-},\frac{Nq^{--}}{N-q^-})}\right)\\
&\geq \delta>0,
\end{aligned}$$

for some fixed $\epsilon>0$, and δ, $\|W\|$ sufficiently small. The proof of the Lemma 4.1 is now completed. □

Lemma 4.2. *Assume conditions* **(B1)** *and* **(F1)-(F3)** *holds. Then there exists $e\in E$ with $\|e\|>\tau$ (τ is given in Lemma 4.1 such that $I(e)<0$).*

Proof: Denote

$$h(t) = \frac{F(x,tw)}{t^{\mu}}, \ \forall\ t > 0.$$

Then, using **(F3)**, we get

$$h'(t) = \frac{1}{t^{\mu+1}}[tu(f(x,w) + tvg(x,tw) - \mu F(x,tw)] \geq 0, \quad \forall\ t > 0$$

Thus, we deduce that for any $t \geq 1, F(x,tw) \geq t^{\mu} F(x,w)$
Choosing $w \in E$, with $\|w\| > 1$ and $\int F(x,w)dx > 0$ fixed and $t > 1$,we have

$$\begin{aligned}
I(tw) &= \int \frac{1}{p(x)}(|\nabla tu|^{p(x)} + a(x)\,|tu|^{p(x)}) \\
&+ \int \frac{1}{q(x)}(|\nabla tv(x)|^{q(x)} + b(x)\,|tv(x)|^{q(x)}) - \int F(x,tu,tv)dx \\
&= \int \frac{t^{p(x)}}{p(x)}(|\nabla u|^{p(x)} + a(x)\,|u|^{p(x)}) + \int \frac{t^{q(x)}}{q(x)}\,|\nabla v(x)|^{q(x)} \\
&+ b(x)\,|v(x)|^{q(x)} - t^{\mu} \int F(x,u,v)dx \\
&\leq t^{\max(p^+,q^+)}(\frac{\|u\|^{p^+}}{p^-} + \frac{\|v\|^{q^+}}{q^-}) - t^{\mu} \int F(x,u,v)dx
\end{aligned}$$

Since $\mu > \max(p^+, q^+)$, therefore $I(tw) \to -\infty$,when $t \to \infty$, which concludes our Lemma 4.2. □

Proof: [Proof of Theorem 3.1] We set

$$\Gamma := \{\gamma \in C([0,1], E); \gamma(0) = 0, \gamma(1) = e\}$$

where $e \in E$ is determined by Lemma 4.2 and

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$$

According to Lemma 4.2, we know that $\|e\| > \tau$, so every path $\gamma \in \Gamma$ intersects the sphere $\|w\| = \tau$. Then, Lemma 4.1 implies $c \geq \inf_{\|u\|=t} I(u) \geq \delta$ with constants $\delta > 0$. Thus $c > 0$. Hence, using the Mountain-pass theorem (see e.g., [4]) we obtain a sequence $(w_n)_n \subset E$ such that

$$I(w_n) \to c, \ I^{\backprime}(w_n) \to 0 \tag{4.6}$$

We claim that $(w_n)_n$ is bounded in E. Arguing by contradiction and passing to a subsequence, we have $\|w_n\| \to \infty$. Using (2.2), it follows that for n large enough, we have

$$c + 1 + \|w_n\| \geq I(w_n) - \frac{1}{\mu}\left\langle I^{\backprime}(w_n), w_n\right\rangle$$

Using the above inequality, we have

$$\begin{aligned} c+1+\|w_n\| &\geq \left(\frac{1}{p^+}-\frac{1}{\mu}\right)J_1(u_n)+\left(\frac{1}{q^+}-\frac{1}{\mu}\right)J_2(v_n) \\ &- \int F(x,w_n)-\frac{1}{\mu}[\int f(x,u_n,v_n)+\int g(x,u_n,v_n)]dx \end{aligned}$$

By (F3) we have

$$\int [F(x,w_n)-\frac{1}{\mu}(f(x,w_n)u_n+g(x,w_n)v_n)] \leq 0.$$

The above inequality combined with relations (2.2), (2.3) yields

$$\begin{aligned} c+1+\|w_n\| &\geq (\frac{1}{p^+}-\frac{1}{\mu})J_1(u_n)+(\frac{1}{q^+}-\frac{1}{\mu})J_2(u_n) \\ &\geq (\frac{1}{p^+}-\frac{1}{\mu})\|u_n\|^{p^-}+(\frac{1}{q^+}-\frac{1}{\mu})\|v_n\|^{q^-} \\ &\geq \min(\frac{1}{p^+}-\frac{1}{\mu},\frac{1}{q^+}-\frac{1}{\mu})(\|u_n\|^{p^-}+\|v_n\|^{q^-}) \end{aligned} \tag{4.7}$$

Now dividing by $\|u_n\|, \|v_n\|$ in (4.7) and passing to the limit as $n\to\infty$, we obtain a contradiction. So, up to a subsequence $(u_n,v_n)_n$ converges weakly in E to some $(u,v)\in E$. If Ω is a bounded domain, then there exists a compact embedding $E(\Omega)\hookrightarrow L^{\frac{Np}{N-p^-}}(\Omega)\times L^{\frac{Nq^-}{N-q^-}}(\Omega)$. Then $(u_n,v_n)\to(u,v)$ in $L^{\frac{Np}{N-p^-}}(\Omega)\times L^{\frac{Nq^-}{N-q^-}}(\Omega)$, for all Ω bounded domains in $\mathbb{R}^N$.

Claim :

$$\langle I'(u_n,v_n),(\Phi,\Psi)\rangle \to \langle I'(u,v),(\Phi,\Psi)\rangle \quad \forall\ (\Phi,\Psi)\in C_0^\infty(\mathbb{R}^N) \tag{4.8}$$

Assuming this Claim, using (4.6), (u,v) is a weak solution of the problem (1.1) since C_0^∞ is dense in E. Finally, let us prove the Claim. To do this, let $(\Phi,\Psi)\in C_0^\infty(\mathbb{R}^N)$ be fixed. Firstly, we prove that

$$\begin{aligned} \lim_{n\to\infty}[\int_{\mathbb{R}^N} f(x,u_n,v_n)\Phi &+ \int g(x,u_n,v_n)\Psi] = \int_{\mathbb{R}^N} f(x,u,v)\Phi dx \\ &+ \int_{\mathbb{R}^N} g(x,u,v)\Psi dx \end{aligned}$$

A simple calculation implies

$$\begin{aligned} &\left|\int (f(x,u_n,v_n)-f(x,u,v))\,\Phi dx\right| \leq \int |(f(x,u_n,v_n)-f(x,u,v))|\,.\,|\Phi|\,dx \\ &\leq \|\Phi\|_{L^\infty}\int\left|\frac{(f(x,u_n,v_n)-f(x,u,v))}{(u_n-u)}\right|\times|u_n-u|\,dx \\ &\leq \|\Phi\|_{L^\infty}\int f_u(x,u_n^*,v_n)\,|u_n-u|\,, \end{aligned}$$

where $u_n^* \in [u_n, u]$ or $[u, u_n]$. Similarly

$$\left|\int (g(x,u_n,v_n) - g(x,u,v))\, \Psi dx\right| \leq \|\Psi\|_{L^\infty} \int g_v(x,u_n,v_n^*)\, |v_n - v|$$

where $v_n^* \in [v_n, v]$ or $[v, v_n]$. Now using condition **(F2)**, we obtain

$$\left|\int (f(x,u_n,v_n) - f(x,u,v))\, \Phi dx\right| + \left|\int (g(x,u_n,v_n) - g(x,u,v))\, \Psi dx\right| \leq$$

$$\max\left(\|\Phi\|_\infty, \|\Psi\|_\infty\right) [\left(\|a_1(x)\|_{\alpha_1} \|u_n^*\|_{p*^-}^{p_1-1} \|a_2(x)\|_{\alpha_2} \|v_n\|_{q*^-}^{p_2-1}\right) \|u_n - u\|_{p*^-} +$$

$$\left(\|b_1(x)\|_{\beta_1} \|u_n\|_{p*^-}^{q_1-1} + \|b_2(x)\|_{\beta_2} \|v_n^*\|_{q*^-}^{q_2-1}\right) \|v_n - v\|_{q*^-}]$$

Taking into account that $u_n \to u$ in $L^{\frac{Np}{N-p^-}}(\Omega)$ and $v_n \to v$ in $L^{\frac{Nq^-}{N-q^-}}(\Omega)$, and for all $n \geq 1$, there exist $\lambda_n(x) \in [0,1]$ such that

$$u_n^* = \lambda_n(x) u_n(x) + [1 - \lambda_n(x)]\, u(x),$$

we deduce that

$$\int |u_n^* - u|^s\, dx = \int |\lambda_n(x)|^s\, |u_n - u|^s\, dx \to 0$$

as $n \to \infty$. It results that

$$\int |u_n^*|^s \to \int |u|^s\, dx \quad \text{as} \quad n \to \infty$$

Similarly

$$\int_\Omega |v_n^*|^s \to \int_\Omega |v|^s\, dx \quad \text{as} \quad n \to \infty$$

From the above considerations, we obtain

$$\left|\int (f(x,u_n,v_n) - f(x,u,v))\, \Phi dx\right| + \left|\int (g(x,u_n,v_n) - g(x,u,v))\, \Psi dx\right| \to 0,$$

as $n \to \infty$

Since $C_0^\infty\left(\mathbb{R}^N\right)$ is dense in E, the above relation implies

$$\lim_{n\to\infty} \left|\int (f(x,u_n,v_n) - f(x,u,v))\, \Phi dx\right| + \left|\int (g(x,u_n,v_n) - g(x,u,v))\, \Psi dx\right| = 0,$$

as $n \to \infty$

Next, since $(u_n, v_n) \rightharpoonup (u, v)$ *in* E, it follows that

$$\lim_{n\to\infty} \left| \int f(x,u,v)\,(u_n - u)\,dx \right| + \left| \int g(x,u,v)\,(v_n - v)\,dx \right| = 0$$

Thus, actually, we find

$$\lim_{n\to\infty} \left| \int f(x,u_n,v_n)\,(u_n - u)\,dx \right| + \left| \int (g(x,u_n,v_n)(v_n - v)\,dx \right| = 0 \tag{4.9}$$

On the other hand, we have

$$\lim_{n\to\infty} \langle I'(u_n,v_n), (u_n - u)\,(v_n - v)\rangle = 0 \tag{4.10}$$

Combining (4.9) with (4.10), to deduce that

$$\lim_{n\to\infty} \int |\nabla u_n|^{p(x)-2}\,\nabla u_n \nabla\,(u_n - u) + |\nabla v_n|^{q(x)-2}\,\nabla v_n \nabla\,(v_n - v)$$

$$+ \int a(x)\,|u_n|^{p(x)-2}\,u_n\,(x)\,(u_n\,(x) - u\,(x))$$

$$+ \int c(x)\,|v_n|^{q(x)-2}\,v_n\,(x)\,(v_n\,(x) - v\,(x)) = 0$$

Since relation (4.10) holds true and $(u_n, v_n) \rightharpoonup (u, v)$ in E. By [7, Lemma 3.1], we deduce that $(u_n, v_n) \to (u, v)$ in E.

Then since $I \in C^1(E, R)$,we conclude that

$$I'\,(u_n, v_n) \to I'\,(u, v)\,, \text{ as } \; n \to \infty \tag{4.11}$$

Relations (4.6) and (4.11) show that $I'\,(u, v) = 0$ and thus (u, v) is a weak solution for (P_λ), Moreover, by relation (10), it follows that $I\,(u, v) > 0$ and (u, v) is a nontrivial. □

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