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$(\psi-\alpha)$-Meir-Keeler-Khan Type Fixed Point Theorem in Partial Metric Spaces

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abstract: In this paper, we introduce a new concept of $(\psi-\alpha)$-Meir-Keeler-Khan type mappings in partial metric spaces. The presented theorems generalize and improve many existing results in the literature. Moreover, an examples is given to illustrate our results.

Key Words: $(\psi-\alpha)$-Meir-Keeler-Khan mappings, partial metric spaces, fixed point.

Contents

1. Introduction

In 1978, Fisher [1] proved the following revised version of result of Khan[2].

Theorem 1.1: ([1]) Let (X,d) be a metric space and f be a self map on X satisfying the following:

$d(fx,fy)\leq k\frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{d(x,fy)+d(y,fx)}, k\in[0,1),$

if

$d(x,fy)+d(y,fx)\neq 0,$

and

$d(fx,fy)=0 \quad if \quad d(x,fy)+d(y,fx)=0.$

Then f has a unique fixed point $t\in X$. Moreover, for every $t_0\in X$, the sequence $\{f^nt_0\}$ converges to t.

In the sequel, Ψ denotes the family of all (c)-comparison functions. A self map ψ on $[0,\infty)$ is said to be a (c)-comparison function, if $\sum_{n=1}^{\infty}\psi^n(t)<\infty$ for each $t>0$, where ψ^n is the nth iterate of ψ.Clearly, $\psi(0)=0$ and $\psi(t)<t$ for all $t>0$.

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Recently, Samet et al. [3] introduced the notion of α-admissible mappings as follows:

Definition 1.2:([3]) Let f be a self map on X and $\alpha : X^2 \to [0, \infty)$. If $\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1$, for all $x, y \in X$, then f is said to be α-admissible.

One can refer [4-5] for class of α-admissible mappings and more information on subject.

Matthews [6] introduced the notion of partial metric spaces as follows:

Let X be a nonempty set and $p : X^2 \to [0, \infty)$ satisfy the following:

(pm1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(pm2) $p(x, x) \leq p(x, y)$;
(pm3) $p(x, y) = p(y, x)$;
(pm4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$,

for all $x, y, z \in X$. Then p is called a partial metric and the pair (X, p) is called a partial metric space.

We note that the function $d_p : X \times X \to \mathbb{R}^+$ defined by

$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$

satisfies the conditions of a metric space X and hence it is a usual metric on X.

Definition 1.3: [6]
(i) A sequence $\{x_n\}$ in the PMS (X, p) converges to x if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.
(ii) A sequence $\{x_n\}$ in the PMS (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.
(iii) A PMS (X, p) is called complete, if every Cauchy sequence $\{x_n\}$ in X converges.

The following Lemma will be used in the sequel.
Lemma 1.4: [6]
1. A sequence $\{x_n\}$ is a Cauchy sequence in the PMS (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).
2. A partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete. Moreover
$\lim_{n\to\infty} d_p(x, x_n) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m)$.

In 1969, Meir and Keeler [7] proved an interesting fixed point theorem on a metric space (X, d). Further, Redjel et al. [8] introduced the concept of $(\alpha - \psi)$-Meir-

Keeler-Khan mappings in metric spaces.

2. Main Results

In this section, we introduce a new concept of $(\psi-\alpha)$-Meir-Keeler-Khan mappings in partial metric spaces and we establish a fixed point theorem via α-admissible mappings. In the sequel, we consider that if $T: X \to X$, then

for all $x, y \in X, x \neq y \Rightarrow p(x,Ty)+p(y,Tx) \neq 0$.

Definition 2.1.: Let (X,p) be a partial metric space, $T: X \to X$ and $\psi \in \Psi$. Then T is called a generalized Meir-Keeler-Khan type ψ-contraction whenever for each $\epsilon > 0$, there exists $\delta > 0$ such that

$\epsilon \leq \psi(\frac{p(x,Tx)p(x,Ty)+p(y,Ty)p(y,Tx)}{p(x,Ty)+p(y,Tx)}) < \epsilon + \delta(\epsilon) \Rightarrow p(Tx,Ty) < \epsilon$.

Definition 2.2.: Let (X,p) be a partial metric space, $T: X \to X$, $\psi \in \Psi$ and $\alpha: X^2 \to [0,\infty)$. Then T is called a generalized Meir-Keeler-Khan type $\psi-\alpha$-contraction if the following conditions are satisfied:
(i) T is α admissible;
(ii) for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} \epsilon &\leq \psi(\frac{p(x,Tx)p(x,Ty)+p(y,Ty)p(y,Tx)}{p(x,Ty)+p(y,Tx)}) \\ &< \epsilon + \delta(\epsilon) \\ &\Rightarrow \alpha(x,x)\alpha(y,y)p(Tx,Ty) < \epsilon. \end{aligned} \tag{2.1}$$

Remark 2.3.: It is clear that if $T: X \to X$ be an $\psi-\alpha$-Meir-Keeler-Khan type mapping then

$$\alpha(x,x)\alpha(y,y)p(Tx,Ty) \leq \psi(\frac{p(x,Tx)p(x,Ty)+p(y,Ty)p(y,Tx)}{p(x,Ty)+p(y,Tx)}), \tag{2.2}$$

for all $x, y \in X$.

Theorem 2.4.: Let (X,p) be a complete partial metric space and $\psi \in \Psi$. If $\alpha: X^2 \to \mathbb{R}^+$ satisfies the following conditions:
(i) there exists $x_0 \in X$ such that $\alpha(x_0,x_0) \geq 1$;
(ii) if $\alpha(x_k,x_k) \geq 1$ for all $k \in \mathbb{N}$, then $\lim_{k\to\infty} \alpha(x_k,x_k) \geq 1$;
(iii) $\alpha: X^2 \to \mathbb{R}^+$ is a continuous function in each coordinate.
Suppose that $T: X \to X$ is a generalized Meir-Keeler-Khan type $\psi-\alpha$-contraction. Then T has a fixed point in X.
Proof.: Let $x_0 \in X$ and $x_{k+1} = Tx_k = T^k x_0$, for $k = 0,1,2,3,...$. Since T is α-admissible and $\alpha(x_0,x_0) \geq 1$, we have

$\alpha(Tx_0, Tx_0) = \alpha(x_1, x_1) \geq 1.$

Proceeding in the same manner, we get

$$\alpha(x_k, x_k) \geq 1, \tag{2.3}$$

for all $k \in \mathbb{N} \cup \{0\}$.
If $x_{k_0+1} = x_{k_0}$ for some $k_0 \in \mathbb{N}$, then x_{k_0} is the fixed point of T.So, we suppose that $x_{k+1} \neq x_k$ for all $k \in \mathbb{N} \cup \{0\}$. Using the definition of ψ, we have

$\psi(\frac{p(x_k,Tx_k)p(x_k,Tx_{k+1})+p(x_{k+1},Tx_{k+1})p(x_{k+1},Tx_k)}{p(x_k,Tx_{k+1})+p(x_{k+1},Tx_k)}) > 0,$

for all $k \in \mathbb{N} \cup \{0\}$.
We shall assert that

$\lim_{k\to\infty} p(x_k, x_{k+1}) = 0, i.e., \lim_{k\to\infty} d_p(x_k, x_{k+1}) = 0.$

From (2) and (3), we have

$$\begin{aligned} p(x_{k+1}, x_{k+2}) &= p(Tx_k, Tx_{k+1}) \\ &\leq \alpha(x_k, x_k)\alpha(x_{k+1}, x_{k+1})p(Tx_k, Tx_{k+1}) \\ &< \psi(\frac{p(x_k, Tx_k)p(x_k, Tx_{k+1}) + p(x_{k+1}, Tx_{k+1})p(x_{k+1}, Tx_k)}{p(x_k, Tx_{k+1}) + p(x_{k+1}, Tx_k)}) \\ &= \psi(\frac{p(x_k, x_{k+1})p(x_k, x_{k+2}) + p(x_{k+1}, x_{k+2})p(x_{k+1}, x_{k+1})}{p(x_k, x_{k+2}) + p(x_{k+1}, x_{k+1})}) \end{aligned} \tag{2.4}$$

If $p(x_k, x_{k+1}) \leq p(x_{k+1}, x_{k+2})$, then

$$\begin{aligned} p(x_{k+1}, x_{k+2}) &= \psi(\frac{p(x_{k+1}, x_{k+2})p(x_k, x_{k+2}) + p(x_{k+1}, x_{k+2})p(x_{k+1}, x_{k+1})}{p(x_k, x_{k+2}) + p(x_{k+1}, x_{k+1})}) \\ &= \psi(p(x_{k+1}, x_{k+2})) \\ &< p(x_{k+1}, x_{k+2}), \end{aligned}$$

which is a contradiction, and hence $p(x_k, x_{k+1}) < p(x_{k-1}, x_k)$.

Using the same argument as above, we have for each $n \in \mathbb{N}$,

$$\begin{aligned} p(x_{k+1}, x_{k+2}) &= p(Tx_k, Tx_{k+1}) \\ &\leq p(x_k, x_{k+1}). \end{aligned} \tag{2.5}$$

Since the sequence $\{p(x_k, x_{k+1})\}$ is decreasing, it must converge to some $\epsilon \geq 0$, that is,

$$\lim_{k\to\infty} p(x_k, x_{k+1}) = \epsilon. \tag{2.6}$$

From (5) and (6), we have

$$\lim_{k\to\infty} \psi(p(x_k, x_{k+1})) = \epsilon. \tag{2.7}$$

Here $\epsilon = inf\{p(x_k, x_{k+1}) : k \in \mathbb{N}\}$.We assert that $\epsilon = 0$. On the contrary, suppose that, $\epsilon > 0$. Since T is a generalized Meir-Keeler-Khan type $\psi - \alpha$-contraction, corresponding to ϵ use, and using (7), there exists $\delta > 0$ and a natural number n such that

$$\epsilon \leq \psi(p(x_n, x_{n+1})) < \epsilon + \delta \Rightarrow \alpha(x_n, x_n)\alpha(x_{n+1}, x_{n+1})p(Tx_n, Tx_{n+1}) < \epsilon,$$

implies that,

$$p(x_{n+1}, x_{n+2}) = p(Tx_n, Tx_{n+1}) \leq \alpha(x_n, x_n)\alpha(x_{n+1}, x_{n+1})p(Tx_n, Tx_{n+1}) < \epsilon,$$

which is a contradiction, since

$$\epsilon = inf\{p(x_k, x_{k+1}) : k \in \mathbb{N}\}.$$

Thus, we have that

$$\lim_{k\to\infty} p(x_k, x_{k+1}) = 0. \tag{2.8}$$

Also, from (pm2), we have

$$\lim_{k\to\infty} p(x_k, x_k) = 0. \tag{2.9}$$

Since $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, for all $x, y \in X$, using (8) and (9), we get

$$\lim_{k\to\infty} d_p(x_k, x_{k+1}) = 0. \tag{2.10}$$

Now, we assert that $\{x_k\}$ is a Cauchy sequence in the partial metric space (X, p). To show, it is sufficient to that $\{x_k\}$ is a Cauchy sequence in the metric space

(X, d_p). On the contrary, let us suppose $\{x_k\}$ is not a Cauchy sequence. So, there exists $\eta > 0$ such that for any $c \in \mathbb{N}$, there are $n_c, m_c \in \mathbb{N}$ with $n_c > m_c \geq c$ satisfying

$$d_p(x_{m_c}, x_{n_c}) \geq \eta. \tag{2.11}$$

Also, for $m_c \geq c$, we can choose a smallest positive integer n_c such that $n_c > m_c \geq c$ and $d(x_{2m_c}, x_{2n_c}) \geq \eta$.
Therefore, we have

$$d_p(x_{m_c}, x_{n_c-2}) < \eta. \tag{2.12}$$

Now, we have that for all $c \in \mathbb{N}$,

$$\begin{aligned} \eta &\leq d_p(x_{m_c}, x_{n_c}) \\ &\leq d_p(x_{m_c}, x_{n_c-2}) + d_p(x_{n_c-2}, x_{n_c-1}) + d_p(x_{n_c-1}, x_{n_c}) \\ &< \eta + d_p(x_{n_c-2}, x_{n_c-1}) + d_p(x_{n_c-1}, x_{n_c}). \end{aligned} \tag{2.13}$$

Letting $c \to \infty$, we get

$$\lim_{c\to\infty} d_p(x_{m_c}, x_{n_c}) = \eta. \tag{2.14}$$

On the other hand, we have

$$\begin{aligned} \eta &\leq d_p(x_{m_c}, x_{n_c}) \\ &\leq d_p(x_{m_c}, x_{m_c+1}) + d_p(x_{m_c+1}, x_{n_c+1}) + d_p(x_{n_c+1}, x_{n_c}) \\ &\leq d_p(x_{m_c}, x_{m_c+1}) + d_p(x_{m_c+1}, x_{m_c}) + d_p(x_{m_c}, x_{n_c}) \\ &\quad + d_p(x_{n_c}, x_{n_c+1}) + d_p(x_{n_c+1}, x_{n_c}). \end{aligned}$$

Letting $c \to \infty$, we get

$$\lim_{c\to\infty} d_p(x_{m_c+1}, x_{n_c+1}) = \eta. \tag{2.15}$$

Since $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ and using (14) and (15), we have that

$$\lim_{c\to\infty} d_p(x_{m_c}, x_{n_c}) = \frac{\eta}{2}. \tag{2.16}$$

and

$$\lim_{c\to\infty} d_p(x_{m_c+1}, x_{n_c+1}) = \frac{\eta}{2}. \tag{2.17}$$

From (2), we have

$$\begin{aligned} p(x_{m_c+1}, x_{n_c+1}) &= p(Tx_{m_c}, Tx_{n_c}) \\ &\leq \alpha(x_{m_c}, x_{m_c})\alpha(x_{n_c}, x_{n_c})p(Tx_{m_c}, Tx_{n_c}) \\ &< \psi(\frac{p(x_{m_c}, Tx_{m_c})p(x_{m_c}, Tx_{n_c}) + p(x_{n_c}, Tx_{n_c})p(x_{n_c}, Tx_{m_c})}{p(x_{m_c}, Tx_{n_c}) + p(x_{n_c}, Tx_{m_c})}) \\ &< \psi(\frac{p(x_{m_c}, x_{m_c+1})p(x_{m_c}, x_{n_c+1}) + p(x_{n_c}, x_{n_c+1})p(x_{n_c}, x_{m_c+1})}{p(x_{m_c}, x_{n_c+1}) + p(x_{n_c}, x_{m_c+1})}) \end{aligned} \tag{2.18}$$

Since,

$$p(x_{m_c}, x_{n_c+1}) \leq p(x_{m_c}, x_{m_c+1}) + p(x_{m_c+1}, x_{n_c+1}) - p(x_{m_c+1}, x_{m_c+1}), \tag{2.19}$$

and

$$p(x_{n_c}, x_{m_c+1}) \leq p(x_{n_c}, x_{n_c+1}) + p(x_{n_c+1}, x_{m_c+1}) - p(x_{n_c+1}, x_{n_c+1}). \tag{2.20}$$

Using (9), (18), (19) and (20) and making $c \to \infty$, we have

$$\frac{\eta}{2} < \psi(\frac{\eta}{2}) \leq \frac{\eta}{2},$$

a contradiction.
Hence $\{x_k\}$ is a Cauchy sequence in the metric space (X, d_p).
Now, we assert that T has a fixed point z.
Since (X, p) is complete, so by Lemma 1.4, (X, d_p) is also complete. Thus, there exists $z \in X$ such that $\lim_{k\to\infty} d_p(x_k, z) = 0$. Moreover, from Lemma 1.4, we have

$$p(z, z) = \lim_{k\to\infty} p(x_k, z) = \lim_{k,l\to\infty} p(x_k, x_l). \tag{2.21}$$

Further, since $\{x_k\}$ is a Cauchy sequence in the metric space (X, d_p), so $\lim_{k\to\infty} d_p(x_k, x_l) = 0$.
Since, $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, we get

$$\lim_{k,l\to\infty} p(x_k, x_l) = 0. \tag{2.22}$$

From (21) and (22), we have

$p(z,z) = \lim_{k\to\infty} p(x_k, z) = \lim_{k\to\infty} p(x_{k_c}, z) = 0.$

Again, from (2), we get

$$\begin{aligned} p(x_{k+1}, Tz) &= p(Tx_k, Tz) \\ &\leq \alpha(x_k, x_k)\alpha(z,z)p(Tx_k, Tz) \\ &< \psi(\frac{p(x_k, Tx_k)p(x_k, Tz) + p(z, Tz)p(z, Tx_k)}{p(x_k, Tz) + p(z, Tx_k)}) \\ &= \psi(\frac{p(x_k, x_{k+1})p(x_k, Tz) + p(z, Tz)p(z, x_{k+1})}{p(x_k, Tz) + p(z, x_{k+1})}). \end{aligned} \tag{2.23}$$

Making $k \to \infty$, we get

$p(z, Tz) \leq \psi(0) = 0$, that is, $Tz = z$.

Corollary 2.5.: Let (X, p) be a partial metric space and $\psi \in \Psi$. Suppose that $T : X \to X$ is a generalized Meir-Keeler-Khan type ψ-contraction. Then T has a fixed point in X.
Proof.: By putting $\alpha(x, y) = 1$ in Theorem 2.4, we get the result.

Example 2.6.: Let $X = [0, 1]$ and $p(x, y) = max\{x, y\}$, then (X, p) is a partial metric space. Define $\alpha : [0, 1]^2 \to \mathbb{R}^+$ by $\alpha(x, y) = 1 + x + y$, and $T : X \to X$ by $Tx = \frac{x}{8}$. Also, let $\psi : [0,\infty) \to [0,\infty)$ be defined by $\psi(t) = \frac{t}{4}$. Clearly, T is α-admissible. Without loss of generality, assume that $x \geq y$. Then for all $x, y \in [0, 1]$, we have $\alpha(x, x)\alpha(y, y)p(Tx, Ty) \geq \frac{x}{8}$. Now, $p(x, x) = x, p(y, Ty) = y, p(x, Ty) = p(x, \frac{y}{8}) = x, p(y, Tx) = p(y, \frac{x}{8})$.
Case 1. If $p(y, \frac{x}{8}) = y$, then

$$\psi(\frac{p(x,Tx)p(x,Ty)+p(y,Ty)p(y,Tx)}{p(x,Ty)+p(y,Tx)}) = \psi(\frac{x.x+y.y}{x+y}) = \psi(\frac{x^2+y^2}{4(x+y)}) \leq \frac{x^2+y^2}{4} \leq \frac{2x^2}{4} = \frac{x^2}{2}.$$

Case 2. If $p(y, \frac{x}{8}) = \frac{x}{8}$, then

$$\psi(\frac{p(x,Tx)p(x,Ty)+p(y,Ty)p(y,Tx)}{p(x,Ty)+p(y,Tx)}) = \psi(\frac{x.x+y.\frac{x}{8}}{x+\frac{x}{8}}) = \psi(\frac{8x+y}{9}) \leq \frac{9x}{36} = \frac{x}{4}.$$

Hence all the conditions of Theorem 2.4 are satisfied and 0 is the fixed point of T.

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