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Some Results on Complex Valued Metric Spaces Employing contractive conditions with Complex Coefficients and its Applications

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abstract: In this paper, we establish coincidence point and common fixed point theorems involving two pairs of weakly compatible mappings satisfying contraction condition with complex coefficients are proved in complex valued metric spaces. The presented theorems generalize, extend and improve many existing results in the literature.

Key Words: Common fixed point; Contractive type mapping; Complex valued metric space; Complex Coefficient.

Contents

1. Introduction with Preliminaries

In 2011, Azam et al.(cf. [2]) and latter Fayyaz et al. (cf. [7]) studied complex valued metric spaces wherein some fixed point theorems for mappings satisfying a rational inequality were established. Naturally, this new idea can be utilized to define complex valued normed spaces and complex valued inner product spaces which ,in turn, offer a lot of scope for further investigation. Though complex valued metric spaces form a special class of cone metric spaces, yet this idea is intended to define rational expressions which are not meaningfull in cone metric spaces and thus many results of analysis cannot be generalized to cone metric spaces. Indeed the definition of a cone metric space banks on the underlying Banach space which is not a division Ring . However , in complex valued metric spaces, we can study improvements of a host of results of analysis involving divisions.
In this paper we prove coincidence point and common fixed point theorems involving two pairs of weakly compatible mappings satisfying complex inequality in complex valued metric space.

To begin with, we collect some definitions and basic facts on the complex valued metric space, which will be needed in the sequel.

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Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2).$$

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

In particular, we write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that $0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|$, and $z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Definition 1.1. (cf. [2]) Let X be a nonempty set whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies the following conditions:

(d_1). $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d_2). $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3). $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Example 1.1. *(cf. [7]) Let $X = \mathbb{C}$ be a set of complex number. Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, by*

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X, d) is a complex valued metric space.

Example 1.2. *(cf. [8]) Let $X = \mathbb{C}$ be a set of complex number. Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, by*

$$d(z_1, z_2) = e^{ik}|z_1 - z_2|$$

where $k \in \mathbb{R}$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X, d) is a complex valued metric space.

Definition 1.2. (cf. [2]) Let (X, d) be a complex valued metric space and $B \subseteq X$

(i) $b \in B$ is called an interior point of a set B whenever there is $0 \prec r \in \mathbb{C}$ such that

$$N(b, r) \subseteq B$$

where $N(b, r) = \{y \in X : d(b, y) \prec r\}$.

(ii) A point $x \in X$ is called a limit point of B whenever for every $0 \prec r \in \mathbb{C}$,

$$N(x, r) \cap (B \backslash X) \neq \emptyset.$$

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A. A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B.The family

$$F = \{N(x, r) : x \in X, 0 \prec r\}$$

is a sub-basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.

Definition 1.3. (cf. [2]) Let (X, d) be a complex valued metric space and $\{x_n\}_{n\geq 1}$ be a sequence in X and $x \in X$. We say that

(i) the sequence$\{x_n\}_{n\geq 1}$ converges to x if for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, . We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$,

(ii) the sequence$\{x_n\}_{n\geq 1}$ is Cauchy sequence if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$,

(iii) the metric space (X, d) is a complete complex valued metric space If every Cauchy sequence is convergent.

Definition 1.4. (cf. [5]) Two families of self-mappings $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ are said to be pairwise commuting if:
(i)$T_iT_j = T_jT_i$, $i, j \in \{1, 2, ...m\}$.
(ii)$S_iS_j = S_jS_i$, $i, j \in \{1, 2, ...n\}$.
(iii)$T_iS_j = S_jT_i$, $i \in \{1, 2, ...m\}$, $j \in \{1, 2, ...n\}$.

Definition 1.5. let $S : \mathbb{C} \to \mathbb{C}$ be a given mapping. we say that S is a non-decreasing mapping with respect $\precsim$ if for every $x, y \in \mathbb{C}, x \precsim y$ implies $Sx \precsim Sy$.

Definition 1.6. (cf. [1]) let S and T be two self-maps defined on set X. Then S and T are said to be weakly compatible if they commute at their coincidence points.

In [2], Azam et al. established the following two lemmas.

Lemma 1.3. *(cf. [2]) Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.4. *(cf. [2]) Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

2. Main Results

Theorem 2.1. *If S, T, I and J are self-mappings defined on complex valued metric space (X, d) satisfying $TX \subseteq IX, SX \subseteq JX$ and*

$$\lambda d(Sx, Ty) \precsim Ad(Ix, Jy) + Bd(Ix, Sx) + Cd(Jy, Ty) + Dd(Ix, Ty) + Ed(Sx, Jy), \tag{2.1}$$

for all $x, y \in X$, where $D, E \in \mathbb{R}^+$, $\lambda, A, B, C \in \mathbb{C}_+$ and $0 \prec A+B+C+D+E \prec \lambda$. If one of SX, TX, IX or JX is a complete subspace of X, then:
(a) $\{S, I\}$ and $\{T, J\}$ have a unique point of coincidence in X,
(b) if $\{S, I\}$ and $\{T, J\}$ are weakly compatible, then S, T, I and J have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $SX \subseteq JX$, we find a point x_1 in X such that $Sx_0 = Jx_1$. Also, since $TX \subseteq IX$, we choose a point x_2 with $Tx_1 = Ix_2$. Thus in general for the point x_{2n-2} one find a point x_{2n-1} such that $Sx_{2n-2} = Jx_{2n-1}$ and then a point x_{2n} with $Tx_{2n-1} = Ix_{2n}$ for $n = 1, 2, \cdots$. Repeating such arguments one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$y_{2n-1} = Sx_{2n-2} = Jx_{2n-1}, \;\; y_{2n} = Tx_{2n-1} = Ix_{2n}, \;\; n = 0, 1, 2, \cdots. \tag{2.2}$$

Using inequality 2.1, we have

$$\begin{aligned} \lambda d(Sx_{2n}, Tx_{2n+1}) \;\; \precsim \;\; & Ad(Ix_{2n}, Jx_{2n+1}) + Bd(Ix_{2n}, Sx_{2n}) \\ & + Cd(Jx_{2n+1}, Ty_{2n+1}) + Dd(Ix_{2n}, Tx_{2n+1}) \\ & + Ed(Sx_{2n}, Jx_{2n+1}), \end{aligned}$$

or

$$\begin{aligned} \lambda d(y_{2n+1}, y_{2n+2}) \;\; \precsim \;\; & Ad(y_{2n}, y_{2n+1}) + Bd(y_{2n}, y_{2n+1}) \\ & + Cd(y_{2n+1}, y_{2n+2}) + Dd(y_{2n}, y_{2n+2}) \\ & + Ed(y_{2n+1}, y_{2n+1}), \end{aligned}$$

and

$$\begin{aligned} \lambda d(y_{2n+1}, y_{2n+2}) \;\; \precsim \;\; & Ad(y_{2n}, y_{2n+1}) + Bd(y_{2n}, y_{2n+1}) \\ & + Cd(y_{2n+1}, y_{2n+2}) + Dd(y_{2n}, y_{2n+2}), \end{aligned}$$

since $D \in \mathbb{R}^+$,

$$(\lambda - C - D)(d(y_{2n+1}, y_{2n+2})) \precsim (A + B + D)(d(y_{2n}, y_{2n+1})),$$

therefore,

$$|\lambda - C - D||d(y_{2n+1}, y_{2n+2})| \leq |A + B + D||d(y_{2n}, y_{2n+1})|,$$

and

$$|d(y_{2n+1}, y_{2n+2})| \leq \frac{|A+B+D|}{|\lambda-(C+D)|}|d(y_{2n}, y_{2n+1})|,$$

$$|d(y_{2n+1}, y_{2n+2})| \leq h_1|d(y_{2n}, y_{2n+1})|, \tag{2.3}$$

where, $h_1 = |\frac{A+B+D}{\lambda-(C+D)}|$.
since $D, E \in \mathbb{R}^+, \lambda, A, B, C \in \mathbb{C}_+$ and $0 \prec A+B+C+D+E \prec \lambda$ then $h_1 = |\frac{A+B+D}{\lambda-(C+D)}| < 1$.
Again, using inequality 2.1,

$$\begin{aligned}\lambda d(Sx_{2n}, Tx_{2n-1}) \quad &\precsim \quad Ad(Ix_{2n}, Jx_{2n-1}) + Bd(Ix_{2n}, Sx_{2n}) \\ &+Cd(Jx_{2n-1}, Ty_{2n-1}) + Dd(Ix_{2n}, Tx_{2n-1}) \\ &+Ed(Sx_{2n}, Jx_{2n-1}),\end{aligned}$$

or

$$\begin{aligned}\lambda d(y_{2n+1}, y_{2n}) \quad &\precsim \quad Ad(y_{2n}, y_{2n-1}) + Bd(y_{2n}, y_{2n+1}) \\ &+Cd(y_{2n-1}, y_{2n}) + Dd(y_{2n}, y_{2n}) \\ &+Ed(y_{2n+1}, y_{2n-1}).\end{aligned}$$

Since $E \in \mathbb{R}^+$,

$$(\lambda - C - E)(d(y_{2n+1}, y_{2n})) \precsim (A+B+E)(d(y_{2n}, y_{2n-1})),$$

therefore,

$$|d(y_{2n+1}, y_{2n})| \leq \frac{|A+B+E|}{|\lambda-(C+E)|}|d(y_{2n}, y_{2n-1})|,$$

and

$$|d(y_{2n+1}, y_{2n})| \leq h_2|d(y_{2n}, y_{2n-1})|, \tag{2.4}$$

where, $h_2 = |\frac{A+B+E}{\lambda-(C+E)}|$.
since $E \in \mathbb{R}^+, \lambda, A, B, C \in \mathbb{C}_+$ and $0 \prec A+B+C+D+E \prec \lambda$ then $h_2 = |\frac{A+B+E}{\lambda-(C+E)}| < 1$.
Combining 2.3 and 2.4, we have

$$|d(y_{2n+1}, y_{2n+2})| \leq h|d(y_{2n}, y_{2n-1})|,$$

where $h = h_1 h_2$.
Continuing this process, we get

$$|d(y_{2n+1}, y_{2n+2})| \leq h^n|d(y_1, y_2)|. \tag{2.5}$$

By using inequality 2.1, we have

$$\begin{aligned}\lambda d(y_{2n+3}, y_{2n+2}) &= \lambda d(Sx_{2n+2}, Tx_{2n+1}) \\ &\precsim Ad(y_{2n+2}, y_{2n+1}) + Bd(y_{2n+2}, y_{2n+3}) \\ &\quad +Cd(y_{2n+1}, y_{2n+2}) + Dd(y_{2n+2}, y_{2n+2}) \\ &\quad +Ed(y_{2n+3}, y_{2n+1}) \\ &= Ad(y_{2n+2}, y_{2n+1}) + Bd(y_{2n+2}, y_{2n+3}) \\ &\quad +Cd(y_{2n+1}, y_{2n+2}) + Ed(y_{2n+3}, y_{2n+1}).\end{aligned}$$

Since $E \in \mathbb{R}^+$, we get,

$$\begin{aligned}\lambda d(y_{2n+3}, y_{2n+2}) &\precsim Ad(y_{2n+2}, y_{2n+1}) + Bd(y_{2n+2}, y_{2n+3}) \\ &\quad +Cd(y_{2n+1}, y_{2n+2}) + E(d(y_{2n+3}, y_{2n+2}) \\ &\quad +d(y_{2n+2}, y_{2n+1})).\end{aligned}$$

and,

$$(\lambda - E - B)d(y_{2n+3}, y_{2n+2}) \precsim (A + C + E)d(y_{2n+2}, y_{2n+1}),$$

therefore,

$$d(y_{2n+3}, y_{2n+2}) \leq \left|\frac{A + C + E}{\lambda - E - B}\right| d(y_{2n+2}, y_{2n+1}) = h_3 d(y_{2n+2}, y_{2n+1}), \tag{2.6}$$

where $h_3 = |\frac{A+C+E}{\lambda-E-B}|$. Combining 2.5 and 2.6, we have

$$|d(y_{2n+2}, y_{2n+3})| \leq h^n h_3 |d(y_1, y_2)|. \tag{2.7}$$

From 2.5 and 2.7, we get

$$|d(y_n, y_{n+1})| \leq \frac{\max\{1, h_3\}}{h}(\sqrt{h})^n |d(y_1, y_2)|, \text{for } \ n = 2, 3, \cdots .$$

Therefore, for any $m > n$, we have

$$\begin{aligned}|d(y_n, y_m)| &\leq |d(y_n, y_{n+1})| + |d(y_{n+1}, y_{n+2})| \\ &\quad +|d(y_{n+2}, y_{n+3})| + \cdots + |d(y_{m-1}, y_m)| \\ &\leq \frac{\max\{1, h_3\}}{h}[(\sqrt{h}^n + \sqrt{h}^{n+1} + \sqrt{h}^{n+2} + \cdots + \sqrt{h}^{m-1}]|d(y_1, y_2)| \\ &\leq [\frac{\sqrt{h}^n}{h(1 - \sqrt{h})}]\max\{1, h_3\}|d(y_1, y_2)|\end{aligned}$$

since $0 < h < 1$, so that

$$|d(y_n, y_m)| \leq [\frac{\sqrt{h}^n}{h(1 - \sqrt{h})}]\max\{1, h_3\}|d(y_1, y_2)| \to 0 \quad \text{as} \quad n \to \infty.$$

In view of Lemma 1.4, the sequence $\{y_n\}$ is Cauchy sequence in (X, d).Now suppose IX is complete subspace of X,then the subsequence $y_{2n} = Tx_{2n-1} = Ix_{2n}$ converges to some u in IX. That is,

$$y_{2n} = Ix_{2n} = Tx_{2n-1} \to u \quad \text{as} \quad n \to \infty. \tag{2.8}$$

As $\{y_n\}$ is a Cauchy sequence which contains a convergent subsequence $\{y_{2n}\}$, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of the subsequence $\{y_{2n-1}\}$ being a subsequence of convergent sequence $\{y_n\}$. Consequently, we can fined $v \in X$ such that

$$Iv = u. \tag{2.9}$$

We calim that $Sv = u$. Using inequality 2.1 and 2.9, we have

$$\begin{aligned}\lambda d(Sv, y_{2n}) = d(Sv, Tx_{2n-1}) &\precsim Ad(Iv, Jx_{2n-1}) + Bd(Iv, Sv)\\ &\quad +Cd(Jx_{2n-1}, Tx_{2n-1}) + Dd(Iv, Tx_{2n-1})\\ &\quad +Ed(Sv, Jx_{2n-1})\\ &= Ad(u, y_{2n-1}) + Bd(u, Sv) + Cd(y_{2n-1}, y_{2n})\\ &\quad +Dd(u, y_{2n}) + Ed(Sv, y_{2n-1}).\end{aligned}$$

Letting $n \to \infty$ in the above inequality, using 2.8, we have

$$\lambda d(Sv, u) \precsim (B + E)d(Sv, u).$$

since $0 \prec B + E \prec \lambda$, this implies that $d(Sv, u) = 0$, that is,

$$Sv = u. \tag{2.10}$$

Now, combining 2.9 and 2.10, we have

$$Iv = Sv = u,$$

that is, u is a point of coincidence of I and S.
Since $u = Sv \in SX \subseteq JX$, there exists $w \in X$ such that

$$u = Jw. \tag{2.11}$$

We claim that $Tw = u$. Using inequality 2.1, we have

$$\begin{aligned}\lambda d(u, Tw) = \lambda d(Sv, Tw) &\precsim Ad(Iv, Jw) + Bd(Iv, Sv)\\ &\quad +Cd(Jw, Tw) + Dd(Iv, Tw)\\ &\quad +Ed(Sv, Jw),\end{aligned}$$

or,

$$\lambda d(u, Tw) \precsim Cd(u, Tw) + Dd(u, Tw),$$

which by using, $0 \prec C + D \prec \lambda$, we have $d(u, Tw) = 0$, that is

$$u = Tw. \tag{2.12}$$

Combining 2.11 and 2.12, we have

$$u = Jw = Tw,$$

that is, u is a point of coincidence of J, T.
Now, suppose that u' is another point of coincidence of I and S, that is,

$$u' = Iv' = Sv',$$

for some $v' \in X$. Using inequality 2.1, we have

$$\begin{aligned}\lambda d(u', u) = \lambda d(Sv', Tw) &\precsim Ad(u', u) + Bd(u', u') \\ &\quad +Cd(u, u) + Dd(u', u) \\ &\quad +Ed(u', u) \\ &= Ad(u', u) + Dd(u', u) + Ed(u', u)\big),\end{aligned}$$

which implies (by using $0 \prec A + D + E \prec \lambda$) that $d(u', u) = 0$, that is, $u' = u$.
Now, suppose that $\overline{u}$ is another point of coincidence of J and T, that is ,

$$\overline{u} = Jw' = Tw',$$

for some $w' \in X$. Using inequality 2.1, we get

$$\lambda d(u, \overline{u}) = \lambda d(Sv, Tw') \precsim Ad(u, \overline{u}) + Dd(u, \overline{u}) + Ed(u, \overline{u})\big),$$

which implies (by using $0 \prec A + D + E \prec \lambda$) that $d(u, \overline{u}) = 0$, that is, $u = \overline{u}$.
Therefore, we proved that u is the unique point of coincidence of $\{I, S\}$ and $\{J, T\}$.
Now, we prove S, T, I and J, have a unique common fixed point.
Since $\{I, S\}$ and $\{J, T\}$ are weakly compatible, and $u = Iv = Sv = Jw = Tw$, we can write

$$Su = S(Iv) = I(Sv) = Iu = w_1 \text{ (say)}$$

and,

$$Tu = T(Jw) = J(Tw) = Ju = w_2 \text{ (say)}.$$

By using inequality 2.1, we get

$$\begin{aligned}\lambda d(w_1, w_2) = \lambda d(Su, Tu) &\precsim Ad(Iu, Ju) + Bd(Iu, Su) \\ &\quad +Cd(Ju, Tu) + Dd(Iu, Tu) \\ &\quad +Ed(Su, Ju) \\ &= Ad(w_1, w_2) + Dd(w_1, w_2) \\ &\quad +Ed(w_1, w_2),\end{aligned}$$

which implies(by using $0 \prec A + D + E \prec \lambda$) that $w_1 = w_2$, that is,

$$Su = Iu = Tu = Ju.$$

Now, setting $x = v$ and $y = u$ in 2.1, we have

$$\begin{aligned} \lambda d(Sv, Tu) &\precsim Ad(Iv, Ju) + Bd(Iv, Sv) + Cd(Ju, Tu) + Dd(Iv, Tu) \\ &\quad +Ed(Sv, Ju) = Ad(Sv, Tu) + Dd(Sv, Tu) + Ed(Sv, Tu), \end{aligned}$$

we deduce (by using $0 \prec A + D + E \prec \lambda$) that $Sv = Tu$, that is, $u = Tu$. This implies that

$$u = Su = Iu = Tu = Ju.$$

Then, u is the unique common fixed point of S, I, J and T. The proofs for the cases in which SX, JX, or TX is complete are similar, and are omitted. □

3. Application

As an application of Theorem 2.1, we prove the following theorem for four finite families of mappings.

Theorem 3.1. *If $\{T_i\}_1^m$,$\{J_i\}_1^p$,$\{S_i\}_1^l$ and $\{I_i\}_1^n$ are four finite pairwise commuting finite families of self-mapping defined on a complex valued metric space (X, d) such that the mappings S,T,I and J (with $T = T_1T_2...T_m$, $J = J_1J_2...J_p$, $I = I_1I_2...I_n$ and $S = S_1S_2...S_l$) satisfy $TX \subset IX$ and $SX \subset JX$ and the inequality 2.1.If one of TX, SX, IX or JX is complete subspace of X, then the component maps of the four families $\{T_i\}_1^m$,$\{J_i\}_1^p$, $\{S_i\}_1^l$ and $\{I_i\}_1^n$ have a unique common fixed point.*

Proof. Appealing to componentwise commutativity of various pairs, one immediately concludes that $SI = IS$ and $TJ = JT$ and hence, obviously both the pairs (S, I) and (T, J) are weak compatible. Note that all the conditions of Theorem 2.1 (for mappings S, T, I and J) are satisfied ensuring the existence of unique common fixed point u in X, i.e. $Su = Tu = Iu = Ju = u$. We are required to show that u is common fixed point of all the component maps of the families. For this consider

$$\begin{aligned} S(S_k u) &= ((S_1S_2...S_l)S_k)u = (S_1S_2...S_{l-1})((S_lS_k)u) \\ &= (S_1...S_{l-2})(S_{l-1}S_k(S_lu)) = (S_1...S_{l-2})(S_kS_{l-1}(S_lu)) = ... \\ &= S_1S_k(S_2S_3S_4...S_lu) = S_kS_1(S_2S_3S_4...S_lu) = S_k(Su) = S_ku \end{aligned}$$

Similarly one can show that

$$\begin{aligned} T_ku = T_kJu = JT_ku, T_ku = T_kTu = TT_ku \\ J_ku = TJ_ku = JJ_ku, S_ku = IS_ku = SS_ku \\ I_ku = II_ku = SI_ku, T_ku = TT_ku = JT_ku, \end{aligned}$$

which show that (for every k)S_ku, T_ku, I_ku and J_ku are other fixed points of S, T, I *and* J.

By using the uniqueness of common fixed point for S, T, I *and* J, we can write $S_k u = T_k u = I_k u = J_k u = u$ (for every k) which shows that u is a common fixed point of the family $\{T_i\}_1^m$, $\{S_i\}_1^l$,$\{I_i\}_1^p$ and $\{J_i\}_1^n$ (for every k). This completes the proof of the theorem. □

By setting $S_1 = S_2 = ... = S_l = G, T_1 = T_2 = ... = T_m = F, I_1 = I_2 = ... = I_n = Q$ and $J_1 = J_2 = ... = J_p = R$ in Theorem 3.1, we derive the following common fixed point theorem involving iterates of mappings.

Corollary 3.2. *If F, R,G and Q are four commuting self-mappings defined on a complex valued metric space (X, d) satisfying $F^m X \subseteq Q^n X, G^l X \subseteq R^p X$ and*

$$\begin{aligned}\lambda d(G^l x, F^m y) \precsim\ & Ad(Q^n x, R^p y) + Bd(Q^n x, G^l x)\\ & + Cd(R^p y, F^m y) + Dd(Q^n x, F^m y)\\ & + Ed(G^l x, R^p y)),\end{aligned}$$

for all $x, y \in X$. If one of $G^l X, F^m X, Q^n X$ or $R^p X$ is a complete subspace of X, then G, F, Q and R have a unique common fixed point in X.

Remark 3.3. *If $S = T$ and I and J are identity mappings, $\lambda = 1, A = B = 0$ and $C \neq 0$, in the particular case, when (X, d) is a metric space, we obtain Kannan fixed point theorem (cf. [6]).*

Remark 3.4. *If $S = T$ and I and J are identity mappings, $A = C = 0, B \in \mathbb{C}_+$ and $B \neq 0$, in the particular case, when (X, d) is a metric space, we obtain Chatterjia theorem (cf. [3]).*

Remark 3.5. *If $S = T$ and I and J are identity mappings, $A, B, C \in \mathbb{R}^+$ and $\lambda = 1$ in the particular case, when (X, d) is a metric space, we obtain Hardy and Rogers theorem (cf. [4]).*

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