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# A Note On Bertrand Curves Of Constant Precession

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ABSTRACT: In this paper we define Bertrand curves of constant precession with a new approach and obtain their characterizations by using a moving alternative frame.

Key Words: Curves of constant precession, Frenet formula, Bertrand curve.

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## Introduction

Bertrand curve theory is widely studied by many mathematician since it is firstly introduced by Bertrand for the purpose of answering a question about the relationship between principal normals of two curves.

The theory is based on the question that for a principal normal of a curve, whether a second curve that has linear relationship with constant coefficient exist between curvature and torsion of the first one or not. The pair of this kind curves called Bertrand mates or conjugate Bertrand curves.

Going back to the drawing board, the Bertrand curve is firstly studied by Bertrand and from the beginning many mathematician extended the results to numerous spaces. Izumiya and Takeuchi[1] studied helices and Bertrand curves. In [2] the authors gave characterizations for Bertrand curves in n dimensional Lorentz space and obtained some results on this space. By a similar approach in [3] the authors gave some properties on non null curves for three dimensional Lorentzian space. Then in [4] the authors characterized Bertrand curves for null ones. Also we studied on Riemann-Otsuki space and obtained new properties of the curve[5]. In [6] the authors defined Bertrand curves for 3-dimensional Riemannian manifolds and found relationships between curvature and torsion of the curve. In addition the authors studied this type curves for Riemannian space forms with general helices [7]. This subject is also considered for non-flat spaces such as three dimensional sphere $S^3$ $[8-9]$. There are also many papers on this topic with different aspects $[10-15]$.

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On the other hand Scofield[16] studied on a curve having the property that is traversed with unit speed, its centrode (Darboux vector field)

$$W = \tau T + \kappa B \tag{1}$$

revolves about a fixed axis with constant angle and speed. The constant precession curve is characterized by

$$\kappa(s) = w \sin \mu(s), \tau(s) = w \cos \mu(s) \tag{2}$$

where $w > 0$ and $\mu$ are constants. Using Darboux vector field in terms alternative moving frame this vector provides the following conditions

$$D\Lambda N = N^{'}, D\Lambda C = C^{'}, D\Lambda W = W^{'} \tag{3}$$

Here

$$D = gN + fW. \tag{4}$$

Then we call it $C - cons\tan t$ precession curve [17]. In this paper, we focus on the Bertrand curves of C-constant precession with an alternative moving frame. We obtain some results and characterizations about curves of this type.

## 1. Preliminaries

Noting that each unit speed curve has at least four continuos derivatives in Euclidean 3- space, we can use well known orthogonal unit vector fields $T, N, B$. We may take an alternative moving frame along a curve $\alpha$ as follows:

Take the principal normal $N$ and derive it once, we may find a new vector that is orthogonal to the first one. Let us denote it by $C$ and by vector product we get

$$W = N\Lambda C. \tag{5}$$

Hence we compose the moving frame $\{N, C, \ N\Lambda C = W\}$. For the derivative of this alternative moving frame we have

$$\begin{bmatrix} N^{'}(s) \\ C^{'}(s) \\ W^{'}(s) \end{bmatrix} = \begin{bmatrix} 0 & f(s) & 0 \\ -f(s) & 0 & g(s) \\ 0 & -g(s) & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ C(s) \\ W(s) \end{bmatrix} \tag{6}$$

where $f = \kappa\sqrt{1 + H^2}$ and $g = \sigma f$. Here we denote $\sigma = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)^{'}$ which is also use for characterizing slant helices (See [18]) and $H$ is the harmonic curvature of the curve hence we may write $H = \frac{\tau}{\kappa}$ for the curve of this type.

One may also write Darboux vector field in terms of this moving frame as follows:

$$D = gN + fW.$$

Note that this vector has following properties

$$D\Lambda N = N' \; D\Lambda C = C' \; D\Lambda W = W'.$$

Let us consider a new vector field with fixed arbitrary constant $\delta > 0$, $\mu$. Then $\lambda = \sqrt{\delta^2 + \mu^2}$.

**Definition 1.1.** Let $\alpha : I \subseteq IR \to E^3$ be a unit speed curve in $E^3$ and $D$ be a Darboux vector in terms of the alternative moving frame $\{N, C, \; N\Lambda C = W\}$. If Darboux vector revolves about a fixed axis $d$ with a constant angle ($\langle d \,, D \rangle = const.$) and constant speed ( $\|D\| = const.$), then $\alpha$ is called a curve of $C-cons\tan t$ precession [17].

**Corollary 1.1.** Let $\alpha : I \subseteq IR \to E^3$ be a unit speed curve in $E^3$ in terms of the alternative moving frame $\{N, C, \; N\Lambda C = W\}$. Then $\alpha$ is a curve of $C - cons\tan t$ precession $\Leftrightarrow$ the curve $\alpha$ is a $C - slant$ helix. precession [17].

**Theorem 1.1.** Let $\alpha : I \subseteq IR \to E^3$ be a unit speed curve of $C - cons\tan t$ precession in $E^3$ in terms of the alternative moving frame $\{N, C, \; W, \; f, g\}$. So the curvature and the torsion of the curve

$$\kappa = (c_2 \cos\mu(s) - c_1 \sin\mu(s) \cos(\frac{c_1}{\mu} \sin\mu s - \frac{c_2}{\mu} \cos\mu s) \tag{7}$$

$$\tau = (c_2 \cos\mu(s) - c_1 \sin\mu(s) \sin(\frac{c_1}{\mu} \sin\mu s - \frac{c_2}{\mu} \cos\mu s), \tag{8}$$

respectively [17].

## 2. Bertrand Curves of Constant Precession and Their Characterizations

**Definition 2.1.** Let $X(s)$ and $X^*(s)$ be regular $C-constant$ precession curves in $E^3$ with alternative moving frames $\{N, C, \; W, \; f, \; g\}$ and $\{N^*, C^*, \; W^*, \; f^*, \; g^*\}$.

$X(s)$ and $X^*(s)$ are called C-Bertrand curves if $C(s)$ and $C^*(s)$ are linearly dependent. Also $X^*(s)$ is a Bertrand mate for $X(s)$. Thus $(X, X^*)$ is called C-Bertrand couple.

**Theorem 2.1.** Let $(X, X^*)$ be a C-Bertrand mate in $E^3$ and $X, X^*$ are given $(I, X)$ , $(I, X^*)$ coordinate neighbourhood respectively. Then $d(X(s), X^*(s)) = const.$

**Proof.** From Definition 2.1. we may write

$$X^*(s) = X(s) + \lambda(s) C(s) \tag{9}$$

Let us assume archlength parameter of $X$; s and archlength parameter of $X^*, s^*$ respectively. Then we obtain

$$C^*(s) = \frac{dX^*(s)}{ds^*} = X'(s) . \frac{ds}{ds^*} + \lambda'(s) C(s) + \lambda(s) (C(s))' \frac{ds}{ds^*} \tag{10}$$

$$\frac{ds^*}{ds}N^*(s) = N(s) + \lambda^{'}(s)\,C(s) + \lambda(s)\,(C(s))^{'}\,\frac{ds}{ds^*} \tag{11}$$

$$C^{'}(s) = -f(s)N(s) + g(s)\,W(s) \tag{12}$$

$$\frac{ds^*}{ds}N^*(s) = N(s) + \lambda^{'}(s)\,C(s) + \lambda(s)\,(-f(s)N(s) + g(s)\,W(s)) \tag{13}$$

$$\frac{ds^*}{ds}N^*(s) = N(s) + (-\lambda(s)\,f(s)\,N(s) + \lambda(s)\,g(s)\,W(s))\frac{ds}{ds^*} + \lambda^{'}(s)C(s) \tag{14}$$

Then using $C^*(s)$ and $C(s)$ are linearly dependent and from (14) we have

$$\begin{aligned} \langle N^*(s), C(s)\rangle &= \langle 1 - \lambda(s)\,f(s)\,N(s), C(s)\rangle \\ &\quad + \lambda^{'}(s)\,\langle C(s), C(s)\rangle \\ &\quad + \lambda(s)\,g(s)\,\langle W(s), C(s)\rangle \end{aligned} \tag{15}$$

then we get

$$\lambda^{'}(s) = 0 \tag{16}$$
$$\lambda(s) = const.$$

Then

$$d\,(X(s), X^*(s^*)) = \|X^*(s^*) - X(s)\| = \|\lambda\| = const. \tag{17}$$

This completes the proof.

**Theorem 2.2.** Let $X$ and $X^*$are given $(I, X)$ and $(I.X^*)$ coordinate neighbourhoods respectively. $X$ and $X^*$ are C-Bertrand curves if and only if there exist non-zero real numbers $\lambda, \mu$ such that

$$\mu\sigma f(s)\cot\theta + \lambda(s)\kappa(s)\sqrt{1 + H^2} = 1 \tag{18}$$

for any $s \epsilon I$.

**Proof.** Let us assume the angle between $N^*(s)$ and $C(s)$ is $\theta$ then we may write

$$N^*(s) = \cos\theta N(s) + \sin\theta C(s) \tag{19}$$

By differentiating both parts we obtain

$$\begin{aligned} N^{'*}(s)\,\frac{ds}{ds^*} &= \cos\theta N^{'}(s) + \sin\theta W^{'}(s) \\ &\quad + \frac{d}{ds}(\cos\theta)\,N(s) + \frac{d}{ds}(\cos\theta)\,W(s) \end{aligned} \tag{20}$$

Then using Frenet formula and using linear dependency of $\{C^*(s), C(s)\}$ we have $\theta = const.$

Calculating (20) we have

$$\begin{aligned} X^*(s) &= X(s) + \lambda C(s) \\ (X^*(s))^{'} &= X^*(s)\,\frac{ds}{ds^*} = N(s) - \lambda(s)\,(-f(s)\,N(s) + g(s)\,W(s) \end{aligned}$$

$$\begin{aligned} N^*(s)\frac{ds}{ds^*} &= N(s) - \lambda(s) P_i^j \kappa_1 g_j^i v_0^j(s) + \lambda(s) P_i^j \kappa_2 v_2^i(s) \\ &= [1-\lambda(s) f(s)] N(s) + \lambda(s) g(s) W(s) \end{aligned}$$

Then recalling (19) we may write

$$\begin{aligned} \frac{1-\lambda(s) f(s)}{\cos\theta} &= \frac{\lambda(s) g(s)}{\sin\theta} \\ \lambda(s) g(s)\cot\theta + \lambda(s) f(s) &= 1 \end{aligned}$$

Note that $f = \kappa\sqrt{1+H^2}$, $g = \sigma f$ we can write this equation as follows

$$\lambda(s)(\sigma f)(s)\cot\theta + \lambda(s)\kappa(s)\sqrt{1+H^2} = 1$$

If we write $\lambda\cot\theta = \mu$ we get

$$\mu\sigma f(s)\cot\theta + \lambda(s)\kappa(s)\sqrt{1+H^2} = 1$$

$\Leftarrow$) This part can be proved similarly.

**Theorem 2.3.** Let $(X, X^*)$ be a C-Bertrand mate in $E^3$. Then product of torsions $\kappa$ and $\kappa^*$ at the corresponding points of the C-Bertrand curves are constant where $\kappa$ and $\kappa^*$ are the torsions of the curves $X$ and $X^*$, respectively. There is a relation between torsions $\kappa$ and $\kappa^*$ as follows

$$\kappa\kappa^* = \frac{-\sin^2\theta}{\lambda^2(s)\sigma^2(s)\sqrt{(1+H^{*2})(1+H^2)}}$$

**Proof.** From Theorem 2.2 we may consider

$$N(s) = \cos\theta N^*(s) + \sin\theta W^*(s)$$

Arranging (20) for $N(s)$ we may write

$$1 + \lambda(s) f^*(s) = \cos\theta \tag{21}$$

$$-\lambda(s) g^*(s) = \sin\theta \tag{22}$$

Recall that $f = \kappa\sqrt{1+H^2}$, $g = \sigma f$ we arrange the equations above as follows

$$\begin{aligned} 1 + \lambda(s)\kappa^*(s)\sqrt{1+H^{*2}} &= \cos\theta \\ -\lambda(s)\sigma^*\kappa^*(s)\sqrt{1+H^{*2}} &= \sin\theta \end{aligned}$$

From (22)

$$\sin^2\theta = -\lambda^2(s)\left(\sigma^2\sqrt{(1+H^{*2})(1+H^2)}\right) \tag{23}$$

$$\kappa\kappa^* = \frac{-\sin^2\theta}{\lambda^2(s)\sigma^2(s)\sqrt{(1+H^{*2})(1+H^2)}}$$

**Conclusion 2.1.** In classical literature this relation known as Schell's theorem and

the relation given by (23) equals to constant. As seen above the Schell's theorem does not hold for this type of curves.

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