Bol. Soc. Paran. Mat. (3s.) **v. 36** 3 (2018): 207–217.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v36i3.31263

Rotational Hypersurfaces with L_r-Pointwise 1-Type Gauss Map

Akram Mohammadpouri

ABSTRACT: In this paper, we study hypersurfaces in $\mathbb{E}^{n+1}$ which Gauss map G satisfies the equation $L_rG = f(G + C)$ for a smooth function f and a constant vector C, where L_r is the linearized operator of the $(r + 1)$th mean curvature of the hypersurface, i.e., $L_r(f) = tr(P_r \circ \nabla^2 f)$ for $f \in \mathcal{C}^\infty(M)$, where P_r is the rth Newton transformation, $\nabla^2 f$ is the Hessian of f, $L_rG = (L_rG_1, \ldots, L_rG_{n+1}), G = (G_1, \ldots, G_{n+1})$. We show that a rational hypersurface of revolution in a Euclidean space $\mathbb{E}^{n+1}$ has L_r-pointwise 1-type Gauss map of the second kind if and only if it is a right n-cone.

Key Words: Linearized operators L_r, L_r-pointwise 1-type Gauss map, r-minimal, Rotational hypersurfaces.

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1. Introduction

An isometrically immersed submanifold $x : M^n \to \mathbb{E}^{n+k}$ is said to be of finite type if x has a finite decomposition as $x - x_0 = \sum_{i=1}^{p} x_i$, for some positive integer $p < +\infty$ such that $\Delta x_i = \lambda_i x_i, \lambda_i \in \mathbb{R}, 1 \le i \le p$, x_0 is constant, $x_i : M^n \to \mathbb{E}^{n+k}, 1 \le i \le p$ are non-constant smooth maps and Δ is the Laplace operator of M, (see the excellent survey of B. Y. Chen [7]). In [8], this definition was similarly extended to differentiable maps, in particular, to Gauss map of submanifolds. The notion of finite type Gauss map is especially a useful tool in the study of submanifolds (cf. [2,3,4,5,9,10,12,13]). If an oriented submanifold M of a Euclidean space has 1-type Gauss map G, then G satisfies $\Delta G = \lambda(G + C)$ for a constant $\lambda \in \mathbb{R}$ and a constant vector C. In [8], Chen and Piccinni made a general study on compact submanifolds of Euclidean spaces with finite type Gauss map, and for hypersurfaces they proved that a compact hypersurface M of $\mathbb{E}^{n+1}$ has 1-type Gauss map G if and only if M is a hypersphere in $\mathbb{E}^{n+1}$.

2010 *Mathematics Subject Classification:* 53D02, 53C40, 53C42.
Submitted March 10, 2016. Published October 29, 2016

As is well known, the Laplace operator of a hypersurface M immersed into $\mathbb{E}^{n+1}$ is an (intrinsic) second-order linear differential operator which arises naturally as the linearized operator of the first variation of the mean curvature for normal variations of the hypersurface. From this point of view, the Laplace operator Δ can be seen as the first one of a sequence of n operators $L_0 = \Delta, L_1, \ldots, L_{n-1}$, where L_r stands for the linearized operator of the first variation of the $(r+1)$th mean curvature arising from normal variations of the hypersurface (see [19]). These operators are given by $L_r(f) = tr(P_r \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where P_r denotes the rth Newton transformation associated to the second fundamental form of the hypersurface and $\nabla^2 f$ is the hessian of f (see the next section for details).

From this point of view, as an extension of finite type theory, S.M.B. Kashani ([11]) introduced the notion of L_r-finite type hypersurface in the Euclidean space, which has been followed in the author's doctoral thesis. One can see our results in the last section of the last chapter of B. Y. Chen's book ([6]), second edition.

Recently, in [15] the notion of pointwise 1-type Gauss map for the surfaces of Euclidean 3-space $\mathbb{E}^3$ was extended in a natural way in terms of the Chen-Yau operator $\Box$. Based on this definition rotational, helicoidal and canal surfaces in $\mathbb{E}^3$ with L_1-pointwise 1-type Gauss map were discussed in [16,18]. Motivated by such an idea, the following definition was given by the author in [17].

Definition 1.1. *An oriented hypersurface M of Euclidean space $\mathbb{E}^{n+1}$ is said to have L_r-pointwise 1-type Gauss map if its Gauss map satisfies*

$$L_r G = f(G + C) \tag{1.1}$$

for a smooth function $f \in C^\infty(M)$ and a constant vector $C \in \mathbb{E}^{n+1}$. An L_r-pointwise 1-type Gauss map is called proper if the function f is non-constant. More precisely, an L_r-pointwise 1-type Gauss map is said to be of the first kind if (1.1) is satisfied for $C = 0$; otherwise, it is said to be of the second kind. Moreover, if (1.1) is satisfied for a constant function f, then we say that M has-(global) 1-type Gauss map.

In the same paper, we focused on the hypersurfaces with constant $(r+1)$th mean curvature, constant mean curvature. We obtain some classification and characterization theorems for such hypersurfaces with L_r-pointwise 1-type Gauss map. Therefore, it seems natural and interesting to propose the following problem.

Open Problem. Classify hypersurfaces in $\mathbb{E}^{n+1}$ with L_r-1-type Gauss map.

On the other hand, rotational surfaces of Euclidean spaces and pseudo-Euclidean spaces with pointwise 1-type Gauss map have been studied in several papers [9, 12,14,20]. For example, in [9] the rotational surfaces of $\mathbb{E}^3$ with pointwise 1-type Gauss map have been studied by B.Y. Chen, M. Choi and Y.H. Kim. They proved that rotational surfaces of Euclidean spaces with pointwise 1-type Gauss map of the first kind coincide with rotational surfaces with constant mean curvature; and the right cones are the only rational surfaces of revolution with pointwise 1-type Gauss map of the second kind. Fortheremore, Dursun in [10] extentend the results given by B.Y. Chen, M. Choi and Y.H. Kim for surfaces of revolution with pointwise 1-type Gauss map in $\mathbb{E}^3$ ([9]) to the hypersurfaces of revolution with pointwise

1-type Gauss map in $\mathbb{E}^{n+1}$. He proved that a rational hypersurface of revolution of Euclidean space $\mathbb{E}^{n+1}$ has pointwise 1-type Gauss map if and only if it is an open portion of a hyperplane, a generalized cylinder, or a right n-cone.

In this paper, our aim is to study the hypersurfaces of revolution of a Euclidean space $\mathbb{E}^{n+1}$ in terms of L_r-pointwise 1-type Gauss map of the first and second kind. We first give some examples of hypersurfaces of revolution with proper L_r-pointwise 1-type Gauss map of the first kind and the second kind, respectively. Then, we classify rational rotational hypersurfaces of $\mathbb{E}^{n+1}$ with L_r-pointwise 1-type Gauss map which extend the results given in [10] on rational hypersurfaces of revolution with pointwise 1-type Gauss map to the rational hypersurfaces of revolution with L_r-pointwise 1-type Gauss map.

2. Preliminaries

In this section, we recall preliminary concepts from [1,17]. Let $x : M^n \to \mathbb{E}^{n+1}$ be an isometrically immersed hypersurface in the Euclidean space, with the Gauss map G. We denote by ∇^0 and ∇ the Levi-Civita connections on $\mathbb{E}^{n+1}$ and M^n, respectively. Then, the basic Gauss and Weingarten formulae of the hypersurface are written as $\nabla^0_X Y = \nabla_X Y + < SX, Y > G$ and $SX = -\nabla^0_X G$, for all tangent vector fields $X, Y \in \chi(M^n)$, where $S : \chi(M^n) \to \chi(M^n)$ is the shape operator (or Weingarten endomorphism) of M^n with respect to the Gauss map G.

As is well-known, for every point $p \in M^n$, S defines a linear self-adjoint endomorphism on the tangent space T_pM^n, and its eigenvalues $\lambda_1(p), \lambda_2(p), \dots, \lambda_{n-1}(p), \lambda_n(p)$ are the principal curvatures of the hypersurface. The characteristic polynomial $Q_S(t)$ of S is defined by

$$Q_S(t) = \det(tI - S) = (t - \lambda_1)(t - \lambda_2)\dots(t - \lambda_{n-1})(t - \lambda_n) = \sum_{k=0}^{n}(-1)^k a_k t^{n-k},$$

where a_k is given by

$$a_k = \sum_{\substack{i_1<\dots<i_k \\ i_j=1}}^{n} \lambda_{i_1}\dots\lambda_{i_k}, \text{ with } a_0 = 1. \tag{2.1}$$

The rth mean curvature H_r of M^n in $\mathbb{E}^{n+1}$ is defined by $\binom{n}{r}H_r = a_r,\ H_0 = 1$. If $H_{r+1} = 0$, then, we say that M^n is a r-minimal hypersurface. The r-th Newton transformation of M^n is the operator $P_r : \chi(M^n) \to \chi(M^n)$ defined by

$$P_r = \sum_{j=0}^{r}(-1)^j \binom{n}{r-j} H_{r-j} S^j = \sum_{j=0}^{r}(-1)^j a_{r-j} S^j.$$

Equivalently,

$$P_0 = I,\ P_r = \binom{n}{r} H_r I - S \circ P_{r-1}.$$

Associated to each Newton transformation P_r, we consider the second-order linear differential operator $L_r : C^\infty(M^n) \to C^\infty(M^n)$ given by $L_r(f) = tr(P_r \circ \nabla^2 f)$. Here, $\nabla^2 f : \chi(M^n) \to \chi(M^n)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and is given by $< \nabla^2 f(X), Y >=< \nabla_X(\nabla f), Y >$, $X, Y \in \chi(M^n)$.

Now we state the following lemma from [1], which is useful in the present paper..

Lemma 2.1. *Let $x : M^n \to \mathbb{E}^{n+1}$ be a connected orientable hypersurface immersed into the Euclidean space, with Gauss map G. Then, the Gauss map G of M satisfies*

$$L_r G = \binom{n}{r+1} \nabla H_{r+1} + \binom{n}{r+1} (nH_1 H_{r+1} - (n-r-1)H_{r+2})G. \tag{2.2}$$

Now, from the definition (1.1) and the lemma 2.1, we state the following theorem which characterize the hypersurfaces of Euclidean spaces with L_r-pointwise 1-type Gauss map of the first kind.

Theorem 2.1. *An oriented hypersurface M in $\mathbb{E}^{n+1}$ has proper L_r-pointwise 1-type Gauss map of the first kind if and only if H_{r+1} is constant and $nH_1H_{r+1} - (n-r-1)H_{r+2}$ is non-constant.*

We can have the following corollary on hypersurfaces with L_r-1-type Gauss map.

Corollary 2.1. *All oriented isoparametric hypersurfaces of a Euclidean space $\mathbb{E}^{n+1}$ has L_r 1-type Gauss map.*

So, hyperplanes, hyperspheres and the generalized cylinder $S^{n-k} \times \mathbb{E}^k$ of $\mathbb{E}^{n+1}$ have L_r-1-type Gauss map.

We can also state that:

Theorem 2.2. *If an oriented hypersurface M in $\mathbb{E}^{n+1}$ has proper L_r-pointwise 1-type Gauss map of the second kind, then the $(r+1)$th mean curvature of M is non-constant.*

3. Rotational hypersurfaces

Let $x_1 = \varphi(v)$, $x_{n+1} = \psi(v)$ be a curve in the x_1x_{n+1}-half plane lying in halfspace $x_1 = \phi(v) > 0$. Rotating this curve around the x_{n+1}-axis we obtain a rotational hypersurface M in $\mathbb{E}^{n+1}$. Let $\{\eta_1, \ldots, \eta_{n+1}\}$ be the standard orthonormal basis of $\mathbb{E}^{n+1}$ and $S^{n-1}(1)$ be the unit sphere in $\mathbb{E}^n$ spanned by $\{\eta_1, \ldots, \eta_n\}$. We can have an orthogonal parametrization of $S^{n-1}(1) \subset \mathbb{E}^n$ as

$$\begin{aligned} &Y_1 = \cos u_1,\ Y_2 = \sin u_1 \cos u_2, \ldots, \\ &Y_{n-1} = \sin u_1 \ldots \sin u_{n-2} \cos u_{n-1}, \\ &Y_n = \sin u_1 \ldots \sin u_{n-2} \sin u_{n-1}. \end{aligned} \tag{3.1}$$

It follows that

$$x(u_1, \ldots, u_{n-1}, v) = (\varphi(v)Y_1, \varphi(v)Y_2, \ldots, \varphi(v)Y_n, \psi(v)),\ Y_i = Y_i(u_1, \ldots, u_{n-1}),$$

is a parametrization of the rotational hypersurface M. Let us put

$$Y(u_1,\dots,u_{n-1}) = (Y_1(u_1,\dots,u_{n-1}),\dots,Y_n(u_1,\dots,u_{n-1}),0), \tag{3.2}$$

which is the position vector of the sphere $S^{n-1}(1)\subset\mathbb{E}^n$ in $\mathbb{E}^{n+1}$. Then, we can write

$$x(u_1,\dots,u_{n-1},v) = \varphi(v)Y(u_1,\dots,u_{n-1}) + \psi(v)\eta_{n+1}, \tag{3.3}$$

where $\eta_{n+1}=(0,0,\dots,0,1)$ is the axis of the rotation. Taking derivative we have the orthogonal coordinate vector fields on M as

$$x_{u_i} = \varphi(v)Y_{u_i},\ i=1,\dots,n-1,\ x_v = \varphi'(v)Y + \psi'(v)\eta_{n+1}. \tag{3.4}$$

Hence the Gauss map of the hypersurface of revolution is given by

$$G = \frac{1}{\sqrt{p}}(\psi' Y - \varphi'\eta_{n+1}),\ p = {\varphi'}^2 + {\psi'}^2. \tag{3.5}$$

By straightforward calculation we can have the Weingarten map as

$$S = \begin{pmatrix} -\frac{\psi'}{\varphi\sqrt{p}} I_{n-1} & 0 \\ 0 & \frac{\psi'\varphi''-\varphi'\psi''}{p\sqrt{p}} \end{pmatrix} \tag{3.6}$$

where I_{n-1} is the $(n-1)\times(n-1)$ identity map. Thus the $(r+1)$th mean curvature is

$$\begin{aligned}\binom{n}{r+1}H_{r+1} =& (-1)^{r+1}\binom{n-1}{r+1}\frac{(\psi')^{r+1}}{\varphi^{r+1}(\sqrt{p})^{r+1}} \\ &+ (-1)^r\binom{n-1}{r}\frac{(\psi')^{r+1}\varphi'' - (\psi')^r\psi''\varphi'}{\varphi^r p(\sqrt{p})^{r+1}}.\end{aligned} \tag{3.7}$$

Since the $(r+1)$th mean curvature H_{r+1} is the function of v, using (3.4) we can have the gradient of H_{r+1} as

$$\nabla H_{r+1} = \frac{H'_{r+1}}{p}(\varphi' Y + \psi'\eta_{n+1}). \tag{3.8}$$

3.1. Examples of rotational hypersurfaces with L_r-pointwise 1-type Gauss map

We can have the following examples of rotational hypersurfaces with proper L_r-pointwise 1-type Gauss map of the first kind and the second kind, respectively.

Example 3.1. *Let M be the rotational hypersurface in $\mathbb{E}^{n+1}$ parameterized by*

$$x(u_1,\dots,u_{n-1},v) = vY(u_1,\dots,u_{n-1}) + \int\frac{a\ dv}{\sqrt{v^q - a^2}}\eta_{n+1},\ v>0, \tag{3.9}$$

where a is nonzero constant, $q = \frac{2(n-r-1)}{r+1}$, $\eta_{n+1} = (0, 0, \dots, 0, 1) \in \mathbb{E}^{n+1}$ and $Y(u_1, \dots, u_{n-1})$ is defined in (3.1). M is the r-minimal hypersurface and the Gauss map G of M is given by

$$G = \frac{1}{v^{\frac{q}{2}}}(aY - \sqrt{v^q - a^2}\eta_{n+1}),$$

and hence by using (2.2), L_rG satisfies

$$L_{n-1}G = 0, \; L_rG = \frac{q}{2}\binom{n}{r+1}\frac{(-1)^r a^{r+2}}{v^{(r+2)(1+\frac{q}{2})}}G,$$

which implies that M has proper L_r-pointwise 1-type Gauss map of the first kind when $r \neq n-1$.

Note that, the author in [17] characterized hypersurfaces in $\mathbb{E}^{n+1}$ with at most 2 distinct principal curvatures that have L_{n-1}-(global) 1-type Gauss map as following.

Theorem 3.1. *An oriented hypersurface M in $\mathbb{E}^{n+1}$ with at most 2 distinct principal curvatures has L_{n-1}-(global) 1-type Gauss map of the first kind if and only if it is either an n-minimal hypersurface or an open part of a hypersphere, a hyperplane or a generalized cylinder.*

Example 3.2. *Consider the right n-cone C_a based on the sphere $S^{n-1}(1)$ which is parameterized by*

$$x(u_1, \dots, u_{n-1}, v) = vY(u_1, \dots, u_{n-1}) + av\eta_{n+1}, \; a \geq 0,$$

where $\eta_{n+1} = (0, 0, \dots, 0, 1) \in \mathbb{E}^{n+1}$ and $Y(u_1, \dots, u_{n-1})$ is defined in (3.1). Then, the Gauss map G of C_a is given by

$$G = \frac{1}{\sqrt{1+a^2}}(aY - \eta_{n+1}).$$

Hence by using (2.2), (3.7) and (3.8) for $\varphi(v) = v, \; v > 0$ and $\psi(v) = av$ we can have

$$L_rG = (-1)^r\binom{n-1}{r+1}\frac{r+1}{v^{(r+2)}(\sqrt{1+a^2})^r}(G + \frac{1}{\sqrt{1+a^2}}\eta_{n+1}), \; r \neq n-1,$$

which means that the right n-cone has proper L_r-pointwise 1-type Gauss map of the second kind.

3.2. Rotational hypersurfaces of rational kind with L_r-pointwise 1-type Gauss map

Let M be a rotational hypersurfaces in $\mathbb{E}^{n+1}$ parameterized by taking $\varphi(t) = t, \; t > 0$ and $\psi(t) = g(t)$ in (3.3)

$$x(u_1, \dots, u_{n-1}, t) = tY(u_1, \dots, u_{n-1}) + g(t)\eta_{n+1}, \tag{3.10}$$

where Y is given by (3.2). The Gauss map G of M parameterized by (3.10) is given by

$$G = \frac{1}{\sqrt{1+g'^2}}(g'Y - \eta_{n+1}). \tag{3.11}$$

When we consider (3.8) for the parametrization (3.10) we obtain from the equation (2.2)

$$L_r G = \big(tr\ (S^2 \circ P_r) + \binom{n}{r+1}\frac{H'_{r+1}}{\sqrt{1+g'^2}g'}\big)G + \binom{n}{r+1}\frac{H'_{r+1}}{g'}\eta_{n+1}, \tag{3.12}$$

where

$$\begin{aligned} tr\ (S^2 \circ P_r) = \frac{(-1)^r\binom{n-1}{r+1}g'^r}{t^r\big(\sqrt{1+g'^2}\big)^{r+2}}\Big[&\frac{(r+1)g'^2}{t^2} + \Big(\frac{r}{r+1} - \frac{nr+n-r-1}{n-r-1}\Big)\frac{g'g''}{t(1+g'^2)} \\ &+\Big(\frac{r+1}{n-r-1}\Big)\frac{g''^2}{(1+g'^2)^2}\Big] \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \binom{n}{r+1}H'_{r+1} =& \frac{(-1)^r\binom{n-1}{r+1}g'^r}{t^r\big(\sqrt{1+g'^2}\big)^{r+2}}\Big[\frac{(r+1)g'\sqrt{1+g'^2}}{t^2} - \frac{(r+1)g''\sqrt{1+g'^2}}{t} \\ &+ \frac{(r+1)g''g'^2}{\sqrt{1+g'^2}t} - \frac{r(r+1)g''^2g'^{-1}}{(n-r-1)\sqrt{1+g'^2}(1+g'^2)} \\ &- \frac{(r+1)g'''}{(n-r-1)\sqrt{1+g'^2}} - \frac{r(r+1)g''}{(n-r-1)t\sqrt{1+g'^2}} \\ &+ \frac{(r+1)^2g''^2g'}{(n-r-1)\sqrt{1+g'^2}(1+g'^2)} \\ &+ \frac{2(r+1)g''^2g'}{(n-r-1)\sqrt{1+g'^2}(1+g'^2)}\Big],\ (r \geq 1). \end{aligned} \tag{3.14}$$

Suppose that M has L_r-pointwise 1-type Gauss map of the second kind. Then (1.1) holds for some function f and some vector C. When the Gauss map is not L_r-harmonic (i.e. $L_rG = 0$), (1.1), (3.1), (3.11) and (3.12) imply that the first n components of C must be zero and

$$\begin{aligned} tr\ (S^2 \circ P_r) + \binom{n}{r+1}\frac{H'_{r+1}}{\sqrt{1+g'^2}g'} &= f \\ \binom{n}{r+1}\frac{H'_{r+1}}{g'} &= cf, \end{aligned} \tag{3.15}$$

where $C=(0,\dots,0,c)$ and f is independent of $u_1,\dots,u_{n-1}$.

Suppose that M is a hypersurface of revolution of polynomial kind, that is, $g(t)$ is a polynomial in t. Suppose that $deg\ g(t)=m$. Eliminating f in (3.15) and, using (3.13) and (3.14), we get following relation

$$A(t)=c\sqrt{1+g'^2}B(t), \tag{3.16}$$

where $A(t)$ and $B(t)$ are polynomials in t of degrees $(m-1)(r+9)$ and $(m-1)(r+7)$ respectively. So the polynomial $g(t)$ that satisfies (3.16) has degree $m=1$. So the parametrization of M reduces to

$$x(u_1,\dots,u_{n-1},t)=tY(u_1,\dots,u_{n-1})+(at+b)\eta_{n+1},\ \ a\neq 0,$$

which is the right n-cone. As a result we have the following.

Theorem 3.2. *A rotational hypersurface of polynomial kind in a Euclidean space $\mathbb{E}^{n+1}$ has L_r-pointwise 1-type Gauss map of the second kind if and only if it is a right n-cone.*

Let M be a rotational hypersurface of rational kind, that is, $g(t)$ is a rational function in t. In [9], it was proven that there is no rotational surface of rational kind, except polynomial kind, with pointwise 1-type Gauss map of the second kind. Following [9], one can see that the equation (3.15) does not have any rational solution, except polynomial. Therefore we can state the following.

Theorem 3.3. *There do not exist rational hypersurface of revolution, except polynomial kind, in a Euclidean space $\mathbb{E}^{n+1}$ with L_r-pointwise 1-type Gauss map of the second kind.*

Proof: We consider hypersurfaces of revolution of rational kind. In this case, the function $g(t)$ of (3.10) and $g'(t)$ are both rational functions in t. If $g'(t)$ is not a constant, we may put $g'(t)=\frac{s(t)}{q(t)}$, where $s(t)$ and $q(t)$ are relative prime polynomials, that is, $s(t)$ and $q(t)$ do not have a common factor of degree greater than or equal to one. If $g'(t)$ is non-constant, then there exists a polynomial $p(t)$ satisfying $q^2(t)+s^2(t)=p^2(t)$, where $q(t)$, $s(t)$ and $p(t)$ are relatively prime. Hence, $\sqrt{1+g'^2(t)}=\frac{p(t)}{q(t)}$. Since M has L_r-pointwise 1-type Gauss map of the second kind. Then (3.15) holds for some function f and some vector $C=(0,\dots,c)$. Eliminating f in (3.15) and using (3.13) and (3.14), we get following relation

$$C(t)=\frac{s^{r+3}(t)p^6(t)}{q(t)}, \tag{3.17}$$

where $C(t)$ is a polynomial in t. It follows that $\frac{s^{r+3}(t)p^6(t)}{q(t)}$ is a polynomial. This is a contradiction because $p(t),q(t)$ and $s(t)$ are relative prime. $\square$

We finally prove the following theorem:

Theorem 3.4. *A rotational hypersurface of rational kind of Euclidean space $\mathbb{E}^{n+1}$ has L_r-pointwise 1-type Gauss map if and only if it is an open portion of a hyperplane, a generalized cylinder, or a right n-cone.*

Proof: Let M be a rotational hypersurface parameterized by (3.3). If $\varphi = \varphi_0$ is constant, then the hypersurface is an open portion of the generalized cylinder $S^{n-1}(\varphi_0) \times \mathbb{R}$. When φ is not constant, we can consider the parametrization given by (3.10) for the rotational hypersurface. The rotational hypersurface has constant $(r+1)$th mean curvature if and only if $g = g(t)$ is a solution of the differential equation

$$\begin{aligned} &\binom{n}{r+1}\alpha t^{r+1}(\sqrt{1+g'^2})^{r+1}(1+g'^2) \\ &+(-1)^r\binom{n-1}{r+1}g'^{r+1}(1+g'^2)+(-1)^r\binom{n-1}{r}tg'^r g''=0. \end{aligned} \tag{3.18}$$

for some constant α. If we make the following change of variable: $g' = \sinh y$, then (3.18) becomes

$$\binom{n}{r+1}\alpha t^{r+1}+(-1)^r\binom{n-1}{r+1}\tanh^{r+1}y+t(-1)^r\binom{n-1}{r}\frac{\tanh^r y}{\cosh^2 y}y'=0, \tag{3.19}$$

After that we make another change of variable: $y = \tanh^{-1} w$, we get

$$y' = \frac{w'}{1-w^2}, \quad \sinh y = \frac{w}{\sqrt{1-w^2}}, \quad \cosh y = \frac{1}{\sqrt{1-w^2}}.$$

Thus, (3.19) becomes

$$\binom{n}{r+1}\alpha t^{r+1}+(-1)^r\binom{n-1}{r+1}w^{r+1}+t(-1)^r\binom{n-1}{r}w^r w'=0. \tag{3.20}$$

Solving (3.20) yields $w(t) = \left(\frac{(-1)^{r+1}\alpha t^n + a}{t^{n-r-1}}\right)^{\frac{1}{r+1}}$ for some constant a. Hence

$$\sinh\left(\tanh^{-1}\left(\frac{(-1)^{r+1}\alpha t^n + a}{t^{n-r-1}}\right)^{\frac{1}{r+1}}\right) = \frac{((-1)^{r+1}\alpha t^n + a)^{\frac{1}{r+1}}}{\sqrt{t^{\frac{2(n-r-1)}{r+1}} - ((-1)^{r+1}\alpha t^n + a)^{\frac{2}{r+1}}}}.$$

Therefore $g(t)$ is given by

$$g(t) = \int \frac{((-1)^{r+1}\alpha t^n + a)^{\frac{1}{r+1}}}{\sqrt{t^{\frac{2(n-r-1)}{r+1}} - ((-1)^{r+1}\alpha t^n + a)^{\frac{2}{r+1}}}}\, dt + c_1, \tag{3.21}$$

where a and c_1 are constant. If $a = \alpha = 0$, g is constant. Then, the hypersurface is an open portion of a hyperplane. If $\alpha = 0$ and $a \neq 0$, that is, M is an r-minimal hypersurface of revolution. When $n = r + 1$, then M is an open portion of a right n-cone. When $n \neq r+1$, then (3.21) implies that $g(t)$ can not be expressed in terms of rational functions. If $a = 0$ and $\alpha \neq 0$, then (3.21) gives $g(t) = (-1)^r \sqrt{\alpha^{\frac{-2}{r+1}} - (-1)^{r+1}t^2} + c_2$. In this case, the hypersurface M is an n-sphere which is not rational kind. If $a, \alpha \neq 0$, then (3.21) implies that $g(t)$ can be expressed in terms of elliptic functions and $g(t)$ is not a rational function of t.

If M is a rational hypersurface of revolution with L_r-pointwise 1-type Gauss map of the second kind, then M is an open portion of a right n-cone according to Theorem 3.2 and 3.3.

The converse is followed by Corollary 2.1 and Example 3.2 □

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Akram Mohammadpouri,
Department of Pure Mathematics,
Faculty of Mathematical Sciences,
University of Tabriz,
Tabriz,
Iran.
E-mail address: pouri@tabrizu.ac.ir