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The Stable Subgroup Graph

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ABSTRACT: In this paper we introduce stable subgroup graph associated to the group G. It is a graph with vertex set all subgroups of G and two distinct subgroups H_1 and H_2 are adjacent if $St_G(H_1)\cap H_2 \neq 1$ or $St_G(H_2)\cap H_1 \neq 1$, where $St_G(H_i) = \{g \in G : H_i^g = H_i\},\ i = 1, 2$. The planarity of the stable subgroup graph of solvable groups has been discussed. Finally, the induced subgraph of stable subgroup graph with vertex set whole non-normal subgroups is considered and its planarity is verified for some certain groups.

Key Words: Stabilizer, finite group, planar graph.

Contents

1. Introduction

Algebraic graph theory is part of algebraic methods in applied problems about graphs. One of the main branches of algebraic graph theory, involving the use of group theory, and the study of graph invariants. A graph represents the information about the relations among nodes which is a very efficient way of describing a structure.

Recently, mathematicians constructed very interesting graphs which are assigned to an algebraic structure by different methods. In this paper we consider a graph with vertex set of all subgroups of a group so let us name some graphs which has a connection with the graph that is defined here, for instance subgroup graphs, subgroup lattice and intersection graphs. The subgroup graph of a group is the graph whose vertices are the subgroups of the group and two vertices, H_1 and H_2, are connected by an edge if and only if $H_1 \leq H_2$ and there is no subgroup K such that $H_1 < K < H_2$ (see [1]). Starr and Turner [11] were the first to study groups G with planar subgroup graph and classified all planar abelian groups. Also, Schmidt [9,10], Bohanon and Reid [1] simultaneously classified all finite planar groups.

The intersection graph of a group G is an undirected graph without loops and multiple edges defined as follows: the vertex set is the set of all proper non-trivial subgroups of G, and there is an edge between two distinct vertices H and K if and only if $H \cap K \neq \{1\}$ where 1 denotes the trivial element of the group G.

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Already, planarity of subgroup graphs, subgroup lattice, intersection graphs are studied [1,6,9,10,11].

In this paper, we introduced stable subgroup graph Γ_G with vertex set all subgroups of the finite group G and two vertices H_1 and H_2 join by an edge if $H_2^x = H_2$ or $H_1^y = H_1$, for some non-identity elements $x \in H_1$, $y \in H_2$. This graph has a narrow connection with subgroup graphs and intersection graphs. For instance subgroup graph and intersection graph are induced subgraph of the stable subgroup graph. The stable subgroup graph is Γ_G is connected graph. If the vertex H intersects $Z(G)$ non-trivially, then has a complete degree in the graph. We observe that subgroups in the same conjugacy classes have similar properties in the stable subgroup graph for instance their degrees are equal. It is not hard to deduce that stable subgroup graph is regular if and only it is a complete graph. Moreover, the stable subgroup graph of Dedekind groups is a complete graph. We verify the planarity of stable subgroup graph for abelian groups, p-groups, nilpotent groups and supersoluble groups directly with out using the planarity of subgroup graphs or intersection graphs. Moreover, we clarify all soluble groups whose stable subgroup graphs are planar by the fact that subgroup graph is its induced subgraph. It is clear that normal subgroups have complete degree by definition of the stable subgroup graph so we consider the induced subgraph of stable subgroup graph by omitting its normal subgroups. Let us denote it by $\Gamma_{G\setminus Ns}$. It is a connected graph whenever $Z(G) \neq 1$. If G is a p-group of order p^3, then $\Gamma_{G\setminus Ns}$ is planar. The stable non-normal subgroup graph of dihedral groups D_8, D_{12}, D_{16}, D_{18} and D_{2p} are planar, where p is an odd prime number.

Throughout the paper, graphs are simple and all the notations and terminologies about the graphs are found in [2,4].

2. Stable subgroup graph

Let us start with the definition of stable subgroup graph associated to the group G.

Definition 2.1. *The stable subgroup graph Γ_G is a graph with vertex set all the subgroups of the group G and two subgroups H_1 and H_2 are adjacent whenever $H_2^x = H_2$ or $H_1^y = H_1$, for some non-identity elements $x \in H_1$, $y \in H_2$.*

The adjacency condition of H_1 and H_2 can be presented by the following equivalent statement. One can use each of them depends on the situation in the argument.
(i) $St_G(H_1) \cap H_2 \neq 1$ or $St_G(H_2) \cap H_1 \neq 1$.

The following three items are equivalent and (i) is deduced from them but not the converse.
(ii) $h_2 \in C_G(H_1)$ or $h_1 \in C_G(H_2)$, for some $1 \neq h_i \in H_i$, $i = 1, 2$.
(iii) $H_1 \subseteq C_G(h_2)$ or $H_2 \subseteq C_G(h_1)$, for some $1 \neq h_i \in H_i$, $i = 1, 2$.
(iv) There exists at least a non-identity element in one of the subgroups such that it commutes with all the elements of the other one.

Thus a graph whose adjacency condition satisfies (ii),(iii) or (iv) is induced subgraph of Γ_G.

It is clear that G and its trivial subgroup join to all other subgroups. Therefore, if two non-adjacent subgroups exist, then they join via G or trivial subgroup. Thus $\mathrm{diam}(\Gamma_G) \leq 2$ and $\mathrm{girth}(\Gamma_G) = 3$. Moreover, G and its trivial subgroup have the maximum degrees which implies that Γ_G is a regular graph if and only if Γ_G is a complete graph.

Proposition 2.1. *If H is a subgroup of G such that intersects the center of G non-trivially, then H is adjacent to all other subgroups.*

Consider the subgroup H of G such that $H \leq Z(G)$ or $1 \neq Z(G) \leq H$, then H joins to all other subgroups. Assume G is an abelian group. Then Γ_G is a complete graph by Proposition 2.1.

If the intersection of two subgroups is non-trivial, then these two subgroups are adjacent. Thus intersection graph is induced subgraph of stable subgroup graph.

Let H, K be two subgroups of the G such that they are adjacent in subgroup graph of G. Then H and K are adjacent in stable subgroup graph of G. This fact implies that the subgroup graph is an induced subgraph of Γ_G.

We denote the conjugacy class subgroup of H in G by $\mathrm{cl}_G(H)$ which is the set of all subgroups of G conjugate to H. In the following lemma we observe that subgroups in the same conjugacy class share some similar properties.

Lemma 2.1. *If $H_t \in \mathrm{cl}_G(H)$ and $H \cap Z(G) = 1$, then*

(i) H_t *intersects* $Z(G)$ *trivially.*

(ii) $|St_G(H)| = |St_G(H_t)|$.

(iii) $\deg(H) = \deg(H_t)$.

Proof: (i) Assume $x \in H_t \cap Z(G)$. As $H_t \in \mathrm{cl}_G(H)$ we have $x \in H^g$ and $x = h^g$, for some $g \in G$. By hypothesis we conclude that

$$x = x^{g^{-1}} = h \in H.$$

Hence $x = 1$.
(ii) It is enough to consider the bijection $\varphi : St_G(H) \to St_G(H_t)$ by $\varphi(l) = l^g$. This map is well-defined because

$$H_t^{l^g} = H_t^{g^{-1}lg} = H^{gg^{-1}lg} = H^{lg} = H^g = H_t.$$

Moreover, if $H_t^k = H_t$, then $k^{g^{-1}}$ stabilize H.
(iii) Let H join K. Then by adjacency condition we have $H^x = H$ or $K^y = K$, for some $x \in K$, $y \in H$. If the first case happened, then $x^g \in K^g$ exists such that $H_t^{x^g} = H_t$. Suppose $K^y = K$, for some $y \in H$. Therefore, $y^g \in H_t$ and $(K^g)^{y^g} = K^g$. These argument imply that K^g and H_t are adjacent. Hence we can define a bijection between the set of the neighbors of H and H_t.

□

Lemma 2.2. *If A and B are two adjacent subgroups which belong to two different conjugacy classes, then for every $A_i \in \mathrm{cl}_G(A)$ there is $B_i \in \mathrm{cl}_G(B)$ which join.*

Proof: Suppose $A_i = A^{g_i}$, then A_i joins B_j, where $B_j = B^{g_i}$, $g_i \in G$. □

Proposition 2.2. *If H and K are two adjacent vertices, then K join to all conjugates of H by K. The converse holds if K join to at least a conjugate of H by K.*

Proof: Suppose H and K are two adjacent vertices, so by definition of adjacency in the graph we have $H^k = H$ or $K^h = K$ for some $k \in K,\ h \in H$. If the first case happened, then $k[k', k]^{-1}$ exists in K such that $H^{k'}$ joins K, for $k' \in K$. Assume $K^h = K$ for some $h \in H$. Therefore, $K^{h^{k'}} = K$ for $h^{k'} \in H^{k'}$, which implies K and $H^{k'}$ are adjacent. By similar computation the second assertion follows. □

Proposition 2.3. *All normal subgroups of G are of complete degree.*

Theorem 2.1. *If G is abelian or the direct product of a quaternion group of order 8, an elementary abelian 2-group and an abelian group with all its elements of odd order, then Γ_G is a complete graph.*

Proof: Suppose G is a group which is mentioned in the theorem, thus by Theorem 5.3.7 in [8] it is a Dedekind group which means all its subgroups are normal. □

If G is a group satisfies Theorem 2.1, then $\omega(\Gamma_G)$ and $\chi(\Gamma_G)$ are the number of subgroups of G.

An element $a \in G$ is said to be a persistence element if $a \neq 1$ and a is contained in all nontrivial subgroups of G. A group G is persistent if it has a persistence element.

Proposition 2.4. *If G is a persistent group, then Γ_G is a complete graph.*

If G is a finite persistent group, then it is either cyclic or isomorphic to a generalized quaternion group. An example of infinite persistent group is $\mathbb{Z}_{p^\infty}$.

Let us denote the number of conjugacy classes of non-normal subgroups of G by $\nu(G)$. Clearly, $\nu(G) = 0$ if and only if G a is Dedekind group. R. Brandl in [3] and H. Mousavi in [7] classified finite groups which have respectively just one or exactly two conjugacy classes of non-normal subgroups. By their main theorems groups whose associated graphs have at most three kind of degrees are specified.

The group with exactly one and two non-trivial subgroup are $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_{pq}$, $\mathbb{Z}_{p^3}$ respectively. Therefore one could deduce the following result.

Proposition 2.5. *Let G be a cyclic group. Then*

(i) If G is of prime order, then Γ_G is K_2.

(ii) If G is of order p^2, then Γ_G is K_3.

(iii) If G is of order pq or p^3, then Γ_G is K_4.

Proof: The result follows by the lattice of subgroups of the group G. $\square$

Proposition 2.6. *Let G be an abelian group. Then Γ_G is planar if and only if $G \cong \{1\},\ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_p,\ \mathbb{Z}_{p^2},\ \mathbb{Z}_{p^3}$ or $\mathbb{Z}_{pq}$, where p, q are prime numbers.*

Proof: If we consider other abelian groups, then they have more than three non-trivial subgroups so the graph associated to them is not planar. $\square$

Example 2.1. *In this example we verify the planarity of some non-abelian groups.*

(i) *Γ_{S_3} is a planar graph.*

(ii) *Γ_{S_n} and Γ_{A_n} are not planar graphs, where $n \geq 4$. It is clear that Γ_{A_4} is induced subgraph of Γ_{A_n} while it is induced subgraph of Γ_{S_n}. Consider the subgroups $H_1 = \langle (1\ 2)(3\ 4) \rangle$, $H_2 = \langle (1\ 3)(2\ 4) \rangle$, $H_3 = \langle (1\ 4)(2\ 3) \rangle$ and $H_4 = \langle (1\ 3)(2\ 4), (1\ 2)(3\ 4) \rangle$ of A_4 together with A_4 and trivial subgroup we have $K_{3,3}$. In Example 3.1 we will observe that after omitting normal subgroups from the vertices of stable subgroup graph of A_4 the new graph is planar. Moreover, the induced subgraph of stable graph of the simple group A_5, which is obtained after omitting A_5 and identity subgroup from the vertices of Γ_{A_5} is still non-planar.*

(iii) *Let $D_{2n} = \langle a, b : a^n = b^2 = 1,\ a^b = a^{-1} \rangle$ be the dihedral group of size $2n$, $n \geq 4$. $\Gamma_{D_{2n}}$ is not a planar graph. If n is an even number, then the subgroups $\langle a \rangle$, $\langle a^{n/2} \rangle$, $\langle a^{n/2}, b \rangle$ and $\langle a^{n/2}, ab \rangle$ together with D_{2n} and trivial subgroup form $K_{3,3}$. By similar argument we can find $K_{3,3}$ in $\Gamma_{D_{2n}}$, whenever n is an odd number.*

(iv) *The graph associated to dicyclic group $Q_{4n} = \langle a, x : a^{2n} = 1,\ a^n = x^2,\ a^x = a^{-1} \rangle$ is not planar, where $n \geq 2$. By the subgroups $\langle a \rangle$, $\langle ax \rangle$, $\langle a^2 x \rangle$, the center of the group $\langle a^n \rangle$, trivial subgroup and the group itself we can form $K_{3,3}$.*

Theorem 2.2. *Let G be a p-group, for a prime number p. Then Γ_G is planar if and only if $G \cong \{1\},\ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_p,\ \mathbb{Z}_{p^2},\ \mathbb{Z}_{p^3}$.*

Proof: Suppose G is a p-group of order p^n, and S the set of all subgroups of order p^m. By Sylow theorem, we see Card(S) congruent to 1 modulo p. There is at least one normal subgroup for every power of p up to the order of the group. Therefore a lower bound for the total number of normal subgroups of G is $n + 1$. Since normal subgroups join to all other subgroups in this graph, all these normal subgroups form a clique. Assume Γ_G is planar. Thus $n \leq 3$ and the planarity of the groups $\{1\},\ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_p,\ \mathbb{Z}_{p^2},\ \mathbb{Z}_{p^3},\ (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p,\ \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p,\ D_8$ and Q_8 should be checked. By Proposition 2.6 and Example 2.1, it is enough to verify the planarity of Γ_G, where $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p,\ \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ and p is odd prime number. If $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p = \langle a, b : a^{p^2} = b^p = 1, a^b = a^{1+p} \rangle$, then the subgroups $\{\langle a \rangle, \langle b \rangle, \langle ab \rangle, Z(G), G, \{1\}\}$ form $K_{3,3}$. For $G \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p = \langle a, b, c : a^p = b^p = c^p = 1,\ c = [a, b],\ ca = ac,\ cb = bc \rangle$, the subgroups $\{\langle a \rangle, \langle b \rangle, \langle ac \rangle, Z(G), G, \{1\}\}$ form $K_{3,3}$. $\square$

Suppose $\prod_{i=1}^{n} G_i$ is direct product of the groups G_i. Obviously, $\Gamma_{\prod_{i=1}^{m} G_i}$ is isomorphic to an induced subgraph of $\Gamma_{\prod_{i=1}^{n} G_i}$, where $m \leq n$.

Proposition 2.7. *If G is a finite nilpotent group, then Γ_G is planar if and only if $G \cong \{1\}$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_p$, $\mathbb{Z}_{p^2}$, $\mathbb{Z}_{p^3}$, and $\mathbb{Z}_{pq}$.*

Proof: By the above argument and the fact that every finite nilpotent group is the direct product of p-groups, the assertion is clear by Proposition 2.6 and Theorem 2.2. □

For finite nilpotent group G, $\omega(\Gamma_G) \geq n+2$, where n is the number of its Sylow p-subgroups of G.

Theorem 2.3. *Let G be a finite supersoluble group. Then Γ_G is planar if and only if $G \cong \{1\}$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_p$, $\mathbb{Z}_{p^2}$, $\mathbb{Z}_{p^3}$, $\mathbb{Z}_{pq}$, S_3, where p, q are prime numbers.*

Proof: It is clear that $\Gamma_{G/G'}$ is isomorphic to an induced subgraph of Γ_G. Now suppose Γ_G is planar, so its induced subgraph $\Gamma_{G/G'}$ is planar. Since G/G' is abelian so by Proposition 2.6 $G/G' \cong \{1\}$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_p$, $\mathbb{Z}_{p^2}$, $\mathbb{Z}_{p^3}$ or $\mathbb{Z}_{pq}$, where p, q are prime numbers. If G is supersoluble, then G' is nilpotent. By the fact that $\Gamma_{G'}$ is induced subgraph of Γ_G and Preposition 2.7, we deduce that $G' \cong \{1\}$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_p$, $\mathbb{Z}_{p^2}$, $\mathbb{Z}_{p^3}$, and $\mathbb{Z}_{pq}$. It is enough to consider the different cases of G' in G/G'. In fact G is extension of a certain group by G'. Let us discuss some of them. Assume $G/G' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. If

(i) $G' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, therefore G is a 2-group and Γ_G is not planar by Theorem 2.2.

(ii) $G' \cong \mathbb{Z}_p$, then G is a group of order $4p$. If $p = 2$, then Γ_G is not planar by Theorem 2.2. Consider p is a prime number greater than 2. If we denote the number of its Sylow 2-subgroups by N_2, then $N_2 = 1$ or p. For $N_2 = 1$, as intersection of distinct Sylow p-subgroups is trivial, we have $N_p = 4$, where N_p is the number of Sylow p-subgroups. Since the Sylow 2-subgroup is normal it is adjacent to 3 Sylow p-subgroups and we have $K_{3,3}$ by the vertices G and $\{1\}$. Moreover, if $N_2 = p$, then in the worst case $4p = |G| = p(4-1) + N_p p$ which implies that $N_p = 1$. By normality of the Sylow p-subgroup, Γ_G contains $K_{3,3}$. Moreover, if Sylow 2-subgroups have non-trivial intersection, then we can form K_5 by use of intersection of two Sylow 2-subgroups which has the maximal order.

(iii) Analogously, for $G' \cong \mathbb{Z}_{p^2}$, $\mathbb{Z}_{p^3}$ we obtain a non-planar Γ_G.

(iv) $G' \cong \mathbb{Z}_{pq}$, then $|G| = 4pq$. If p or q are equal to 2, then Γ_G is not planar. Suppose p, q are odd prime numbers. If G is a non-simple group, then it has at least one non-trivial normal subgroup N. Furthermore, there are $x_1, x_2, x_3 \in G$ of order $2, p$ and q, respectively. If the cyclic group generated by them is not equal to N, then $K_{3,3}$ can be form by the vertices $\{\{1\}, G, N, \langle x_1 \rangle, \langle x_2 \rangle, \langle x_3 \rangle\}$. Otherwise by the Sylow t-subgroup, Γ_G contains $K_{3,3}$, where $t = 2, p$ or q depends on the situation. If G is a simple group of order $4pq$, then $|G| = 60$ and every simple group of order 60 is isomorphic to A_5. By Example 2.1, Γ_{A_5} is not planar.

Now, if $G/G' \cong \mathbb{Z}_p, \mathbb{Z}_{p^2}, \mathbb{Z}_{p^3}$, $\mathbb{Z}_{pq}$, then planarity of Γ_G can be discussed similarly. One of the cases which its argument is to some extent different from the previous cases is when $G/G' \cong \mathbb{Z}_p$ and $G' \cong \mathbb{Z}_q$, where p and q are different prime numbers. If G is extension of a cyclic group of order p by a cyclic group of order q, then $G = \langle a, b | a^p = 1,\ b^q = a^t,\ bab^{-1} = a^r \rangle$, where $b^i \notin \langle a \rangle$ $0 < i < q$, $r^q \equiv 1$ and $rt \equiv t$ (mod p), such a group exists for every choice of integers r, t with these property (see [5, Theorem 12.9]). By argument about the number of Sylow p-subgroups and Sylow q-subgroups we deduce that p or q is less or equal than 2. Thus G is a group of order $2p$, which is cyclic or dihedral group of order $2p$. Hence, the only non-abelian group with this structure which its associated graph is planar is S_3. Hence the result is clear. □

Recall that a group is planar if its subgroup graph is planar [1].

Proposition 2.8. *[1] There are no soluble planar groups whose orders have more than three distinct prime factors.*

Proposition 2.9. *Let G be a finite soluble group. Then Γ_G is planar if and only if $G \cong \{1\}$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_p$, $\mathbb{Z}_{p^2}$, $\mathbb{Z}_{p^3}$, $\mathbb{Z}_{pq}$, S_3, where p, q are prime numbers.*

Proof: Since the subgroup graph is induced subgraph of Γ_G, Proposition 2.8 implies that the order of G is divisible by at most three distinct prime numbers. If G is a nilpotent group, then we discussed about the planarity of Γ_G in Proposition 2.7. Therefore, suppose G is non-nilpotent of order $p^\alpha q^\beta r^\gamma$, where p, q and r are prime numbers and $\alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}$. Since the product of Sylow subgroups is a Hall subgroup of G, we can find K_5 by vertices $\{\{1\}, G, S_p, S_q, S_pS_q\}$ as induced subgraph of Γ_G, where S_i is Sylow i-subgroup of G, $i = p, q, r$. Thus $|G| = p^\alpha q^\beta$. As Γ_{S_i} is induced subgraph of Γ_G, we conclude that $0 \leq \alpha, \beta \leq 3$ by Theorem 2.2. In the case α or β are greater than 2, Γ_G is not planar. Since K_5 is the induced subgraph of Γ_G by vertices $\{\{1\}, G, S_pS_q, S_i, H\}$, where S_i is Sylow i-subgroup of G of order i^γ and H is the cyclic subgroup of G generated by $x \in S_i$, $|x| = i$, $\gamma \geq 2$ and $i = p$ or q. Hence $\alpha, \beta = 1$ or 0 and the assertion follows. □

Since every supersoluble group is soluble group, Theorem 2.3 can be deduced from the Proposition 2.9.

3. Stable non-normal subgroup graph

In general let us denote the induced subgraph of stable subgroup graph of the group G which is obtained by omitting the set X from the the set of vertices by $\Gamma_{V(G)\setminus X}$. Let us denote the subgraph of stable subgroup subgraph with vertex set whole non-normal subgroups of G, by $\Gamma_{G\setminus Ns}$. In this section, we focus on some properties of $\Gamma_{G\setminus Ns}$.

Proposition 3.1. *If G is a group such that $Z(G) \neq 1$, then* $\mathrm{diam}(\Gamma_{G\setminus Ns}) = 2$ *and* $\mathrm{girth}(\Gamma_{G\setminus Ns}) = 3$*.*

Proof: Suppose H_1, H_2 are two non-normal subgroups of G which are non-adjacent in $\Gamma_{G\setminus Ns}$. We claim that $H_1Z(G)$ is a proper subgroup of G, since otherwise $St_G(H_1) = G$ and the adjacency of H_1 and H_2 are deduced, which is a contradiction. Therefore $H_1Z(G)$ is a vertex which is adjacent to both H_1 and H_2.

Thus $\operatorname{diam}(\Gamma_{G\setminus Ns}) = 2$. Now assume H_1 and H_2 are two adjacent vertices. If $H_iZ(G) \neq G$, then both H_1 and H_2 are adjacent to it, for $i = 1$ or 2. Let $H_iZ(G) = G$, for $i = 1$ and 2. Thus $H \cap St_G(H_i) = H$ for all $H < G$, $i = 1, 2$. This implies that H_1, H_2 join H and the assertion is clear. □

The direct result of the above proposition is that $\Gamma_{G\setminus Ns}$ is connected if $Z(G) \neq 1$.

Example 3.1. *In this example we observe that if $Z(G) = 1$, then $\Gamma_{G\setminus Ns}$ may be non-connected.*

(i) $\Gamma_{S_3\setminus Ns}$ is an empty graph with 3 vertices.

(ii) $\Gamma_{S_4\setminus Ns}$ is a connected graph which is not planar. The vertices

$$\{\langle(1\ 2)\rangle, \langle(2\ 4)\rangle, \langle(1\ 4)\rangle, \langle(1\ 4), (1\ 4\ 2)\rangle, S_{\{1,2,4\}}, \langle(1\ 3)(2\ 4), (1\ 4)(2\ 3), (1\ 2)\rangle\}$$

form $K_{3,3}$ as induced subgraph of $\Gamma_{S_4\setminus Ns}$. Therefore, $\Gamma_{S_n\setminus Ns}$ is not planar for $n \geq 4$.

(iii) $\Gamma_{A_4\setminus Ns}$ is planar. It is a triangle and four non-adjacent vertices. Actually A_4 has 2 conjugacy classes of non-normal subgroups, $\mathrm{cl}_{A_4}(\langle(1\ 2)(3\ 4)\rangle)$ and $\mathrm{cl}_{A_4}(\langle(2\ 4\ 3)\rangle)$. The first class contains three adjacent subgroups while the second one has 4 non-adjacent points.

(iv) $\Gamma_{A_5\setminus Ns} = \Gamma_{V(A_5)\setminus\{A_5,\{(1)\}\}}$ is not planar. Since subgroups $\{\langle(1\ 2)(3\ 4)\rangle, \langle(1\ 3)(2\ 4)\rangle, \langle(1\ 4)(2\ 3)\rangle, A_4, \langle(1\ 2)(3\ 4), (1\ 3)(2\ 4)\rangle\}$ form K_5 as its subgraph.

Theorem 3.1. *If G is a p-group of order p^3, then $\Gamma_{G\setminus Ns}$ is planar.*

Proof: By considering Theorems 2.2 and 2.1, $\Gamma_{G\setminus Ns}$ is planar for Dedekind groups $G \cong D_8, Q_8$ and abelian groups of order p^3. It is enough to discuss about the planarity of $\Gamma_{G\setminus Ns}$, where $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$, $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ which are extra special p-group. Every subgroup of index p is normal so non-normal subgroups of these two groups are of order p. The intersection of distinct non-normal subgroups are trivial, also intersection of them with the center of the group is trivial too. Thus two non-normal subgroups H_i, H_j are adjacent if $St_G(H_i) \cap H_j \neq 1$ or $St_G(H_j) \cap H_i \neq 1$. Therefore $|St_G(H_i) \cap H_j| = p$ or $|St_G(H_j) \cap H_i| = p$, which implies $St_G(H_i) = H_j$ or $St_G(H_j) = H_i$. This shows every non-normal subgroup is adjacent to its stabilizer. This fact implies that there is no enough distinct vertices to form $K_{3,3}$ or K_5. □

In the following results the structure of stable subgroup graph of dihedral group of order $2n$ is verified. If D_{2p} is a dihedral group of order $2p$, then $\Gamma_{D_{2p}\setminus Ns}$ is a graph with p isolated vertices.

Proposition 3.2. *Let $D_{2n} = \langle a, b | a^n = b^2 = 1, a^b = a^{-1}\rangle$ be dihedral group of order $2n$ and $n > 4$ is an odd non-prime integer. Then*

(i) The number of vertices of $\Gamma_{D_{2n}\setminus Ns}$ is $\sigma(n) - 1$, where $\sigma(n) = \prod_{i=1}^{r} \left(\frac{p_i^{k_i+1} - 1}{p_i - 1}\right)$, $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ and p_i are distinct prime numbers.

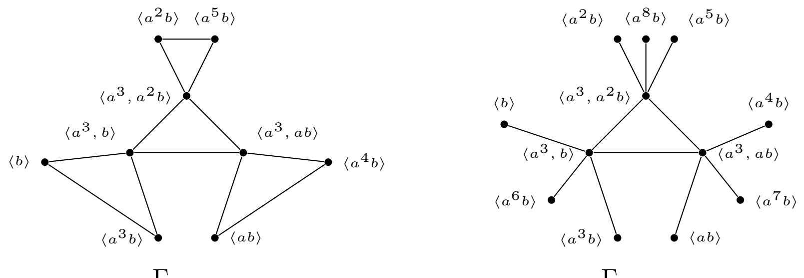
(ii) *The subgroups* $\langle a^i b\rangle$ *and* $\langle a^j b\rangle$ *are not adjacent, where* $0 \leq i, j \leq n-1$ *and* $i \neq j$*. Moreover,* $\langle a^d, a^i b\rangle$ *join to* $\langle a^i b\rangle$*,* $\langle a^{d+i} b\rangle$ *and* $\langle a^{\frac{n+d+2i}{2}} b\rangle$ *where* d *is a divisor of* n*.*

(iii) *The subgroups* $\langle a^d, a^i b\rangle$ *join to* $\langle a^d, a^j b\rangle$*, where* $0 \leq i, j, \leq d-1$*. Furthermore,* $\langle a^d, a^i b\rangle$ *join to* $\langle a^{d'}, a^i b\rangle$ *but* $\langle a^d, a^i b\rangle$ *and* $\langle a^{d'}, a^j b\rangle$ *are not adjacent, where* d, d' *are distinct divisor of* n*.*

(iv) *By third part* $\langle a^d, a^r b\rangle$ *form a component which may have a connection with the component* $\langle a^{d'}, a^{r'} b\rangle$ *by the possible vertices such that* $r = r'$*,* $0 \leq r, r' \leq d-1$*. The number of these components is* $\sigma_0(n) - 2$*, where* $\sigma_0(n)$ *is the number of the divisors of* n*.*

(v) $\Gamma_{D_{2n} \setminus Ns}$ *is planar if and only if* $n = 3^2$*.*

Proof: The subgroups of D_{2n} are $\langle a^d\rangle$ and $\langle a^d, a^r b\rangle$, where d is a divisor of n and $0 \leq r \leq d-1$. It is clear that $\langle a^d\rangle$ are normal subgroups so they are not vertices. The number of subgroups of the form $\langle a^d, a^r b\rangle$ is $\sigma(n)$ such that one of them is equal to D_{2n}. Hence (i) follows.
(ii) If $\langle a^i b\rangle$ and $\langle a^j b\rangle$ are adjacent, then $2j - i \equiv i \pmod{n}$. Since n is an odd integer, we have $2j - 2i = nt$, where t is an even number. Moreover $0 \leq j \leq n-1$ so the possible case for t is 2. Thus $j - i = n$ which is impossible. The adjacency of $\langle a^i b\rangle$ and $\langle a^d, a^i b\rangle$, follows by the fact that $\langle a^i b\rangle$ is subgroup of $\langle a^d, a^i b\rangle$. A computation imply the rest of the assertion. Third part follows similarly by computation.
(iv) Third part implies that for every divisor d of n we have subgroups $\langle a^d, a^r b\rangle$, where r move on $0 \leq r \leq d-1$. All such subgroups are adjacent. Therefore the number of these components is equal to the number of divisors of n, but when $d = 1$ or n such subgroups are D_{2n} or $\langle a^i b\rangle$, where $0 \leq i \leq n-1$.
(v) By fourth part we deduce that there are complete components that made of subgroups $\langle a^d, a^r b\rangle$, where $0 \leq r \leq d-1$. The component which has the largest number of vertices is coincides to the greatest $d \neq n$ and number of the vertices in that component is d. Thus if $\Gamma_{D_{2n}}$ is planar, then $d \leq 4$ and since n is an odd number $d = 3$. Hence the assertion is clear. $\square$



$\Gamma_{D_{12 \setminus Ns}}$ $\Gamma_{D_{18 \setminus Ns}}$

Figure 1:

The proof of the following proposition is very similar to the previous one so we omit it.

Proposition 3.3. *Let D_{2n} be dihedral group of order $2n$, $n \geq 4$ an even number and all the notions are the same as the Proposition 3.2. Then*

(i) $|V(\Gamma_{D_{2n}\setminus Ns})| = \sigma(n) - 3$.

(ii) The subgroups $\langle a^i b\rangle$ and $\langle a^j b\rangle$ are adjacent whenever $j = (n/2)+i$, $0 \leq i, j \leq n-1$ and $i \neq j$. Moreover, $\langle a^d, a^i b\rangle$ join to $\langle a^i b\rangle$ and $\langle a^{d+i} b\rangle$, where d is a divisor of n.

(iii) The subgroups $\langle a^d, a^i b\rangle$ join to $\langle a^d, a^j b\rangle$, where $0 \leq i, j, \leq d-1$. Furthermore, $\langle a^d, a^i b\rangle$ join to $\langle a^{d'}, a^i b\rangle$ but $\langle a^d, a^i b\rangle$ does not join $\langle a^{d'}, a^j b\rangle$, where d, d' are distinct divisor of n.

(iv) By third part $\langle a^d, a^r b\rangle$ form a component which may have a connection with the component $\langle a^{d'}, a^{r'} b\rangle$ by the possible vertices such that $r = r'$, $0 \leq r, r' \leq d-1$.

(v) $\Gamma_{D_{2n}\setminus Ns}$ is planar if and only if $n = 2^2, 6$ or 2^3.

By the above two proposition we deduce that the stable non-normal subgroup graph of $D_8, D_{12}, D_{16}, D_{18}$ and D_{2p} are planar, where p is an odd prime number. As we mentioned $\Gamma_{A_5\setminus Ns}$ is a connected graph which is not planar. Thus $\Gamma_{A_n\setminus Ns}$ is not planar for $n \geq 5$.

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