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Existence and multiplicity of a-harmonic solutions for a Steklov problem with variable exponents

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ABSTRACT: Using variational methods, we prove in a different cases the existence and multiplicity of a-harmonic solutions for the following elliptic problem:

$$\begin{aligned} div(a(x,\nabla u)) = 0 \quad &\text{in } \Omega, \\ a(x,\nabla u).\nu = f(x,u) \quad &\text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain of smooth boundary $\partial\Omega$ and ν is the outward unit normal vector on $\partial\Omega$. $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$, are fulfilling appropriate conditions.

Key Words: Variable exponents; Elliptic problem; Nonlinear boundary condition; a-harmonic solutions; Recceri's variational principle, mountain pass theorem.

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1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$ and consider the elliptic Steklov problem with variable exponents

$$\begin{aligned} div(a(x,\nabla u)) = 0 \quad &\text{in } \Omega, \\ a(x,\nabla u).\nu = f(x,u) \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where ν is the outward unit normal vector on $\partial\Omega$ and $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function which will be specified later.

Let $p \in C(\overline{\Omega})$ be a variable exponent. Throughout this paper, we denote

$$p^- = \min_{x\in\overline{\Omega}} p(x); \;\; p^+ = \max_{x\in\overline{\Omega}} p(x);$$

$$p^\partial(x) = \begin{cases} (N-1)p(x)/[N-p(x)] & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N, \end{cases}$$

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and

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : 1 < p^- < p^+ < \infty\}.$$

Our exponent p fulfills $p \in C_+(\overline{\Omega})$ and for this p we introduce a characterization of the Carathéodory function $a : \overline{\Omega} \times \mathbb{R}^N \mapsto \mathbb{R}^N$.

(H_0) $a(x, -s) = -a(x, s)$ for a.e. $x \in \overline{\Omega}$ and all $s \in \mathbb{R}^N$.

(H_1) There exists a Carathéodory function $A : \overline{\Omega} \times \mathbb{R}^N \mapsto \mathbb{R}$ continuously differentiable with respect to its second argument, such that $a(x, s) = \nabla_s A(x, s)$ all $s \in \mathbb{R}^N$ and a.e. $x \in \overline{\Omega}$.

(H_2) $A(x, 0) = 0$ for a.e. $x \in \overline{\Omega}$.

(H_3) There exists $c > 0$ such that a satisfies the growth condition $|a(x, s)| \leq c(1 + |s|^{p(x)-1})$ for a.e. $x \in \overline{\Omega}$ and all $s \in \mathbb{R}^N$, where $|.|$ denotes the Euclidean norm.

(H_4) The monotonicity condition $0 \leq [a(x, s_1) - a(x, s_2)](s_1 - s_2)$ holds for a.e. $x \in \overline{\Omega}$ and all s_1, $s_2 \in \mathbb{R}^N$, with equality if and only if $s_1 = s_2$.

(H_5) The inequalities $|s|^{p(x)} \leq a(x, s)s \leq p(x)A(x, s)$ hold for a.e. $x \in \overline{\Omega}$ and all $s \in \mathbb{R}^N$.

A first remark is that hypothesis (H_0) is only needed to obtain the multiplicity of solutions. As in [9], we have decided to use this kind of function a satisfying (H_0)-(H_5) because we want to assure a high degree of generality to our work. Here we invoke the fact that, with appropriate choices of a, we can obtain many types of operators. We give, in the following, two examples of well known operators which are present in lots of papers.

Examples:

1. If $a(x, s) = |s|^{p(x)-2}s$, we have $A(x, s) = \frac{1}{p(x)}|s|^{p(x)}$.
   (H_0) $-$ (H_5) are verified, and we arrive to the $p(x)$-Laplace operator
   $div(a(x, \nabla u)) = div(|\nabla u|^{p(x)-2}\nabla u) = \triangle_{p(x)}u$.
2. If $a(x, s) = (1 + |s|^2)^{(p(x)-2)/2}s$, we have $A(x, s) = \frac{1}{p(x)}[(1 + |s|^2)^{p(x)/2} - 1]$.
   (H_0) $-$ (H_5) are verified, and we find a generalized mean curvature operator
   $div(a(x, \nabla u)) = div((1 + |\nabla u|^2)^{(p(x)-2)/2}\nabla u)$.

The above operator appears in [16] and it is used in the study of two antiplane frictional contact problems of elastic cylinders. Functions fulfilling conditions related to (H_0)–(H_5) are used not only in the framework of the spaces with variable exponents [5], but also in the framework of the classical Lebesgue-Sobolev spaces [21] and the anisotropic spaces with variable exponents.

In the present paper, we study problem 1.1 in the particular case

$$f(x, t) = \lambda \left(|t|^{q(x)-2}t - |t|^{r(x)-2}t\right) - |t|^{p(x)-2}t,$$

where $\lambda \geq 0$ is a real number and $p, q, r \in C_+(\overline{\Omega})$. The energy functional corresponding to problem 1.1 is defined on $W^{1,p(x)}(\Omega)$ as

$$\Phi_\lambda(u) = \int_\Omega A(x, \nabla u)dx + \int_{\partial\Omega} \frac{|u|^{p(x)}}{p(x)} d\sigma - \lambda \int_{\partial\Omega} \left(\frac{|u|^{q(x)}}{q(x)} - \frac{|u|^{r(x)}}{r(x)} \right) d\sigma, \quad (1.2)$$

where $d\sigma$ is the $N-1$ dimensional Hausdorff measure. Let us recall that a weak solution of 1.1 is any $u \in W^{1,p(x)}(\Omega)$ such that

$$\int_\Omega a(x, \nabla u)\nabla v dx + \int_{\partial\Omega} |u|^{p(x)-2} uv d\sigma$$
$$= \lambda \int_{\partial\Omega} \left(|u|^{q(x)-2}uv - |u|^{r(x)-2}uv \right) d\sigma \quad \text{for all } v \in W^{1,p(x)}(\Omega).$$

The study of differential and partial differential equation with variable exponent has been received considerable attention in recent years. This importance reflects directly into a various range of applications. There are applications concerning elastic materials [22], image restoration [11], thermorheological and electrorheological fluids [4,19] and mathematical biology [13].

In the case when $p(x) = p$ is a constant and $a(x, s)) = |s|^{p-2}s$, the authors in [1] are considered the following Steklov problem

$$\begin{cases} \triangle_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda m |u|^{p-2} u & \text{on } \partial\Omega. \end{cases}$$

They are interested to the existence of p-harmonic solutions (when $\Delta_p u = 0$). Motivated by the recent works [5,6], we will study the existence and multiplicity of a-harmonic solutions (when $div(a(x, \nabla u)) = 0$) for the problem 1.1 with variable exponents. These solutions becomes $p(x)$-harmonic when $a(x, s) = |s|^{p(x)-2}s$. This is a generalization of the classical p-harmonic solutions obtained in the case when p is a positive constant.

Our main results in this paper are the proofs of the following theorems, which are based on the Ricceri Theorem and the Mountain Pass Theorem.

Theorem 1.1. *Assume (H_0)–(H_5) and let $p, q, r \in C_+(\bar{\Omega})$, such that $N < p^-$ and $1 \leq r^- \leq r^+ < q^- \leq q(x) \leq q^+ < p^-$, for all $x \in \overline{\Omega}$. Then there exist an open interval $\wedge \subset (0, \infty)$ and a positive constant $\rho > 0$ such that for any $\lambda \in \wedge$, problem 1.1 has at least three weak solutions whose norms are less than ρ.*

Theorem 1.2. *Assume (H_0)–(H_5) and let $p, q, r \in C_+(\bar{\Omega})$, such that $r^+ \leq p^+ < q^- \leq q^+ < p^\partial(x)$ for all $x \in \overline{\Omega}$, where $p^\partial(x)$ is defined above. Then for any $\lambda > 0$ problem 1.1 possesses a non trivial weak solutions.*

This present work extends some of the results known with Neuman or Dirichlet boundary conditions on bounded domain(see [16,18]), and generalize some results knouwn in the Steklov problems (see [2,3]).

This paper consists of four sections. Section 1 contains an introduction and the main results. In section 2, which has a preliminary character, we state some elementary properties concerning the generalized Lebesgue-Sobolev spaces and an embedding results. The proofs of our main theorems are given in Section 3 and Section 4.

2. Preliminaries

We first recall some basic facts about the variable exponent Lebesgue-Sobolev.
For $p \in C_+(\bar{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) := \left\{ u; u : \Omega \to \mathbb{R} \text{ is a measurable and } \int_\Omega |u|^{p(x)} dx < +\infty \right\},$$

endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega)} := \inf \left\{ \alpha > 0; \int_\Omega \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\},$$

which is separable and reflexive Banach space (see [15]).
Let us define the space

$$W^{1,p(x)}(\Omega) := \{ u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega) \},$$

equipped with the norm

$$\|u\|_\Omega = \inf \left\{ \alpha > 0; \int_\Omega \left| \frac{\nabla u(x)}{\alpha} \right|^{p(x)} dx + \int_\Omega \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\}; \ \forall u \in W^{1,p(x)}(\Omega).$$

Proposition 2.1. *[10] For any $u \in W^{1,p(x)}(\Omega)$.
Let $||u|| := |\nabla u|_{L^{p(x)}(\Omega)} + |u|_{L^{p(x)}(\partial\Omega)}$. Then the norm $||u||$ is a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to $||u||_\Omega$.*

Proposition 2.2. *[12,14]*

(1) $W^{1,p(x)}(\Omega)$ is separable reflexive Banach space;

(2) If $s \in C_+(\bar{\Omega})$ and $s(x) < p^\partial(x)$ for any $x \in \bar{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{s(x)}(\partial\Omega)$ is compact and continuous.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the mapping ρ defined by

$$\rho(u) := \int_\Omega |\nabla u|^{p(x)} dx + \int_{\partial\Omega} |u|^{p(x)} d\sigma, \ \forall u \in W^{1,p(x)}(\Omega).$$

Proposition 2.3. *[10] For $u, u_k \in W^{1,p(x)}(\Omega); k = 1, 2, ...,$ we have*

(1) $\|u\| \geq 1$ implies $\|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;

(2) $\|u\| \leq 1$ *implies* $\|u\|^{p^-} \geq \rho(u) \geq \|u\|^{p^+}$;

(3) $\|u_k\| \to 0$ *if and only if* $\rho(u_k) \to 0$;

(4) $\|u_k\| \to \infty$ *if and only if* $\rho(u_k) \to \infty$.

Remark 2.4. *If* $N < p^- \leq p(x)$ *for any* $x \in \overline{\Omega}$, *by Theorem 2.2 in [15] and remark 1 in [18], we have* $W^{1,p(x)}(\Omega)$ *is compactly embedded in* $C(\overline{\Omega})$. *Defining* $||u||_\infty = \sup_{x\in\overline{\Omega}} |u(x)|$, *we find that there exists a positive constant* $c > 0$ *such that* $||u||_\infty \leq c||u||$ *for all* $u \in W^{1,p(x)}(\Omega)$.

3. Proof of Theorem 1.1

The key argument in the proof of Theorem 1.1 is the following version of Ricceri theorem (see Theorem 1 in [8]).

Theorem 3.1. *[8] Let* X *be a separable and reflexive real Banach space;* $\Phi : X \to \mathbb{R}$ *a continuously Gâteaux differentiable and sequentially weakly lower. semicontinuous functional whose Gâteaux derivative admits a continuous inverse on* X^*; $\Psi : X \to \mathbb{R}$ *a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*
(i) $\lim_{||u||\to\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty$ *for all* $\lambda > 0$; *and that are* $r \in \mathbb{R}$ *and* $u_0, u_1 \in X$ *such that*
(ii) $\Phi(u_0) < r < \Phi(u_1)$;
(iii) $\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) > \frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}$.
Then there exist an open interval $\wedge \subset (0, +\infty)$ *and a positive real number* ρ_0 *such that for each* $\lambda \in \wedge$ *the equation* $\Phi'(u) + \lambda\Psi'(u) = 0$ *has at least three solutions in* X *whose norme are less than* ρ_0.

Let X denote the generalized Sobolev space $W^{1,p(x)}(\Omega)$.
In order to apply Ricceri's result we define the functionals $\Phi, \Psi : X \to \mathbb{R}$ by

$$\Phi(u) = \int_\Omega A(x, \nabla u)dx + \int_{\partial\Omega} \frac{1}{p(x)}|u|^{p(x)}d\sigma, \tag{3.1}$$

$$\Psi(u) = -\int_{\partial\Omega} \left(\frac{|u|^{q(x)}}{q(x)} - \frac{|u|^{r(x)}}{r(x)}\right) d\sigma, \tag{3.2}$$

Its clear that from (H_1), the Fréchet derivative of Φ is the operator $\Phi' : X \to X'$ defined as

$$\langle\Phi'(u), v\rangle = \int_\Omega a(x, \nabla u)\nabla v dx + \int_{\partial\Omega} |u|^{p(x)-2}uv d\sigma \text{ for any } u, v \in X.$$

On the other hand the Fréchet derivative of Ψ is Ψ' defined as

$$\langle\Psi'(u), v\rangle = -\int_{\partial\Omega} \left(|u|^{q(x)-2}uv - |u|^{r(x)-2}uv\right) d\sigma, \text{ for any } u, v \in X.$$

Thus we deduce that $u \in X$ is a weak solution of problem 1.1 if there exist $\lambda > 0$ such that u is a critical point of the operator $\Phi + \lambda\Psi$.

We start by proving some properties of the operator Φ'.

Theorem 3.2. *Suppose that the mapping a satisfies (H_0)-(H_5). Then the following statements holds.*

(1) Φ' is continuous, bounded and strictly monotone;

(2) Φ' is of (S_+) type;

(3) Φ' is an homeomorphism.

Proof. The same approach as in proof of Theorem 1.1 in [3], by taking $\lambda = 0$ and replacing the term $\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx$ by $\int_{\partial\Omega} \frac{1}{p(x)}|u|^{p(x)}d\sigma$ in the expression of energy functional $\phi_{\lambda,0}$ defined in [3]. □

Now we can give the proof of our main result.

Proof of Theorem 1.1. Set Φ and Ψ as 3.1, 3.2 . So, for each u, $v \in X$, one has

$$\langle \Phi'(u), v\rangle = \int_\Omega a(x, \nabla u)\nabla v dx + \int_{\partial\Omega} |u|^{p(x)-2}uv d\sigma,$$
$$\langle \Psi'(u), v\rangle = -\int_\Omega \left(|u|^{q(x)-2}u - |u|^{r(x)-2}u\right) v\, dx$$

From Theorem 3.2, the functional Φ is a continuous Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X', moreover, Ψ is continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Obviously, Φ is bounded on each bounded subset of X under our assumptions.

From (H_5) and using Proposition2.3, if $\|u\| \geq 1$ then

$$\begin{aligned}\Phi(u) &= \int_\Omega A(x, \nabla u)dx + \int_{\partial\Omega} \frac{1}{p(x)}|u|^{p(x)}d\sigma \\ &\geq \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega} \frac{1}{p(x)}|u|^{p(x)}d\sigma \\ &\geq \frac{1}{p^+}\rho(u) \\ &\geq \frac{1}{p^+}\|u\|^{p^-},\end{aligned}$$

Meanwhile, for each $\lambda \in \Lambda$,

$$\begin{aligned}\lambda\Psi(u) &= -\lambda \int_{\partial\Omega} \left(\frac{|u|^{q(x)}}{q(x)} - \frac{|u|^{r(x)}}{r(x)}\right) d\sigma \\ &\geq -\lambda(c_1||u||^{q^-} + c_2||u||^{q^+})\end{aligned}$$

for any $u \in X$, where c_1 and c_2 are positive constants. Combining the two inequalities above, we obtain

$$\Phi(u) + \lambda\Psi(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \lambda(c_1||u||^{q^-} + c_2||u||^{q^+}),$$

since $q^+ < p^-$, it follows that

$$\lim_{\|u\|\to+\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty \quad \forall u \in X, \quad \lambda \in [0, +\infty).$$

Then assumption (i) of Theorem 3.1 is satisfied.

Next, we will prove that assumption (ii) is also satisfied. In order to do that we define the function
$G : \overline{\Omega} \times [0, \infty[\to \mathbb{R}$ by $G(x,t) = \frac{t^{q(x)}}{q(x)} - \frac{t^{r(x)}}{r(x)}$, $\forall x \in \overline{\Omega}$ and $t \in (0, \infty)$.
It is clear that G is of class C^1 with respect to t, uniformly when $x \in \overline{\Omega}$. Define also the function $G_t(x,t) = t^{r(x)-1}(t^{q(x)-r(x)} - 1)$, $\forall x \in \overline{\Omega}$ and $t \in (0, \infty)$.
Thus $G_t(x,t) \geq 0$ for all $t \geq 1$ and all $x \in \overline{\Omega}$; $G_t(x,t) \leq 0$ for all $t \leq 1$ and all $x \in \overline{\Omega}$. Consequently $G(x,t)$ is increasing when $t \in (1, \infty)$ and decreasing when $t \in (0,1)$, uniformly with respect to x. Furthermore, $\lim\limits_{t\to+\infty} G(x,t) = +\infty$ uniformly which respect to $x \in \overline{\Omega}$. On the other hand $G(x,t) = 0$ imply that $t = t_0 = 0$ or $t = t_x = \left(\frac{q(x)}{r(x)}\right)^{\frac{1}{q(x)-r(x)}}$. So we have $G(x,t) \leq 0$ for all $0 \leq t \leq t_x$ and $G(x,t) > 0$ for all $t > t_x$ and all $x \in \overline{\Omega}$. Let a, b two real numbers such that $0 < a < \min(1, c)$, with c given in Remark 2.4 and $b > \max\left((\frac{q^+}{r^-})^{\frac{1}{q^- - r^+}}, (\frac{1}{|\partial\Omega|})^{\frac{1}{p^-}}\right)$.

Consider $u_0, u_1 \in X$, $u_0(x) = 0$, $u_1(x) = b$, for any $x \in \Omega$. Consequently by Remark 2.4 we have $u_0(x) = 0$ and $u_1(x) = b$, for any $x \in \overline{\Omega}$. Thus we have

$$\int_{\partial\Omega} \sup_{0\leq t\leq a} G(x,t)d\sigma \leq 0 < \int_{\partial\Omega} G(x,b)d\sigma.$$

We also define $r = \frac{1}{p^+}\left(\frac{a}{c}\right)^{p^+}$, we have $r \in (0,1)$ and $\Phi(u_0) = -\Psi(u_0) = 0$.
$\Phi(u_1) = \int_{\partial\Omega} \frac{1}{p(x)} b^{p(x)}dx \geq \frac{1}{p^+} b^{p^-}|\partial\Omega| > \frac{1}{p^+}.\left(\frac{a}{c}\right)^{p^+} = r$, $\Psi(u_1) = -\int_{\partial\Omega} G(x,b)d\sigma$.
Thus we deduce that $\Phi(u_0) < r < \Phi(u_1)$, so (ii) in Theorem 3.1 is verified.
On the other hand we have

$$-\frac{(\Phi(u_1) - r)\,\Psi(u_0) + (r - \Phi(u_0))\,\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)} = -r\frac{\Psi(u_1)}{\Phi(u_1)} = r\frac{\int_{\partial\Omega} G(x,b)d\sigma}{\int_{\partial\Omega} \frac{1}{p(x)} b^{p(x)}d\sigma} > 0.$$

Let $u \in X$ with $\Phi(u) \leq r < 1$. Then by Proposition 2.3, we have

$$\frac{1}{p^+}||u||^{p^+} \leq \frac{1}{p^+}\rho(u) \leq \Phi(u) \leq r = \frac{1}{p^+}\left(\frac{a}{c}\right)^{p^+} < 1.$$

Using Remark 2.4, we deduce that for any $u \in X$ with $\Phi(u) \leq r$, we have

$$|u(x)| \leq c.||u|| \leq c.(p^+.r)^{\frac{1}{p^+}} = a, \forall x \in \overline{\Omega}.$$

The above inequality shows that

$$-\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) = \sup_{u\in\Phi^{-1}((-\infty,r])} -\Psi(u) \leq \int_{\partial\Omega} \sup_{0\leq t\leq a} G(x,t)d\sigma \leq 0$$

Thus

$$-\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) < r\frac{\int_{\partial\Omega} G(x,b)d\sigma}{\int_{\partial\Omega} \frac{1}{p(x)} b^{p(x)} d\sigma},$$

i.e.

$$\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) > \frac{(\Phi(u_1) - r)\,\Psi(u_0) + (r - \Phi(u_0))\,\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)},$$

consequently the condition (iii) in Theorem 3.1 is verified. We proved that all assumptions of Theorem 1.2 are verified. We conclude that there exists an open interval $\wedge \subset (0, \infty)$ and a positive constant $\rho_0 > 0$ such that for any $\lambda \in \wedge$ the equation $\Phi'(u) + \lambda\Psi'(u) = 0$ has at least three solution in X whose norms are less than ρ_0. The proof of Theorem 1.1 is complete. □

4. Proof of Theorem 1.2

The proof of Theorem1.2 relies on the following version of the mountain pass theorem.

Theorem 4.1 ([20]). *Let X endowed with the norm $\|.\|_X$, be a Banach space. Assume that $\phi \in C^1(X;\mathbb{R})$ satisfies the Palais-Smale condition. Also, assume that ϕ has a mountain pass geometry, that is,*

(i) there exists two constants $\eta > 0$ and $\rho \in \mathbb{R}$ such that $\phi(u) \geq \rho$ if $\|u\|_X = \eta$;

(ii) $\phi(0) < \rho$ and there exists $e \in X$ such that $\|e\|_X > \eta$ and $\phi(e) < \rho$.

Then ϕ has a critical point $u_0 \in X$ such that $u_0 \neq 0$ and $u_0 \neq e$ with critical value

$$\phi(u_0) = \inf_{\gamma\in P} \sup_{u\in\gamma} \phi(u) \geq \rho > 0.$$

Where P denotes the class of the paths $\gamma \in C([0,1];X)$ joining 0 to e.

The energy functional corresponding to problem 1.1 is defined as

$$\Phi_\lambda(u) = \int_\Omega A(x,\nabla u)dx + \int_{\partial\Omega} \frac{|u|^{p(x)}}{p(x)} d\sigma - \lambda \int_{\partial\Omega} \left(\frac{|u|^{q(x)}}{q(x)} - \frac{|u|^{r(x)}}{r(x)} \right) d\sigma.$$

Where $d\sigma$ is the $N-1$ dimensional Hausdorff measure. Standard arguments imply that $\Phi_\lambda \in C^1(X;\mathbb{R})$

Lemma 4.2. *Assume* (H_0)–(H_5) *and let* $p, q, r \in C_+(\bar{\Omega})$, *such that* $r^+ \leq p^+ < q^- \leq q^+ < p^\partial(x)$ *for all* $x \in \overline{\Omega}$. *Then there exist* $\eta, b > 0$ *such that* $\Phi_\lambda(u) \geq b$ *for* $u \in W^{1,p(x)}(\Omega)$ *with* $||u|| = \eta$.

Proof. Since $q^+ < p^\partial(x)$ for all $x \in \bar{\Omega}$, by Proposition 2.2 and (H_5), we have the following inequality

$$\Phi_\lambda(u) \geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda}{q^-}\left(C_1\|u\|^{q^+} + C_2\|u\|^{q^-}\right) \text{ if } \|u\| \leq 1.$$

Thus

$$\Phi_\lambda(u) \geq \|u\|^{p^+}\left(\frac{1}{p^+} - \frac{\lambda}{q^-}\left(C_1\|u\|^{q^+-p^+} + C_2\|u\|^{q^--p^+}\right)\right) \text{ if } \|u\| \leq 1.$$

As $p^+ < q^- \leq q^+$, the functional $h : [0,1] \to \mathbb{R}$ defined by

$$h(t) = \frac{1}{p^+} - \frac{\lambda C_1}{q^-}t^{q^+-p^+} - \frac{\lambda C_2}{q^-}t^{q^--p^+}$$

is positive on neighborhood of the origin. So the Lemma 4.2 is proved. □

Lemma 4.3. *Assume* (H_0)–(H_5) *and let* $p, q, r \in C_+(\bar{\Omega})$, *such that* $r^+ \leq p^+ < q^- \leq q^+ < p^\partial(x)$ *for all* $x \in \overline{\Omega}$. *Then there exists* $e \in W^{1,p(x)}(\Omega)$ *with* $\|e\| > \eta$ *such that* $\Phi_\lambda(e) < 0$*; where* η *is given in Lemma 4.2.*

Proof. Choose $\varphi \in C_0^\infty(\bar{\Omega})$, $\varphi \geq 0$ and $\varphi \not\equiv 0$, on $\partial\Omega$. For $t > 1$, and using $(H_2), (H_3)$ we have

$$\Phi_\lambda(t\varphi) \leq t\bar{c}\int_\Omega |\nabla\varphi|dx + \frac{\overline{c_1}t^{p^+}}{p^-}\rho(\varphi) - \frac{\lambda t^{q^-}}{q^+}\int_{\partial\Omega}|\varphi|^{q(x)}d\sigma + \lambda\frac{t^{r^+}}{r^-}\int_{\partial\Omega}|\varphi|^{r(x)}d\sigma.$$

Since $r^+ \leq p^+ < q^-$, we deduce that $\lim\limits_{t\to+\infty}\Phi_\lambda(t\varphi) = -\infty$. Therefore for all $\varepsilon > 0$ there exists $\alpha > 0$ such that $|t| > \alpha$, $\Phi_\lambda(t\varphi) < -\varepsilon < 0$. This completes the proof. □

Lemma 4.4. *Assume* (H_0)–(H_5) *and let* $p, q, r \in C_+(\bar{\Omega})$, *such that* $r^+ \leq p^+ < q^- \leq q^+ < p^\partial(x)$ *for all* $x \in \overline{\Omega}$. *Then the functional* Φ_λ *satisfies the Palais-Smale (PS) condition.*

Proof. Let $(u_k) \subset W^{1,p(x)}(\Omega)$ be a sequence such that $C = \sup\limits_{k\in\mathbb{N}^*}\Phi_\lambda(u_k)$ and $\Phi'_\lambda(u_k) \to 0$. Suppose by contradiction that $\|u_k\| \to \infty$, there exists $k_0 \in \mathbb{N}^*$

such that $\|u_k\| > 1$ for any $k \geq k_0$, using (H_5) Then we have

$$
\begin{aligned}
& C+\|u_k\| \geq \Phi_\lambda(u_k)-\frac{1}{q^-}\langle \Phi'_\lambda(u_k), u_k\rangle \\
& \geq \int_\Omega A(x,\nabla u_k)dx+\int_{\partial\Omega}\frac{1}{p(x)}|u_k|^{p(x)}d\sigma-\lambda\int_{\partial\Omega}\left(\frac{1}{q(x)}|u_k|^{q(x)}-\frac{1}{r(x)}|u_k|^{r(x)}\right)d\sigma \\
& -\frac{1}{q^-}\left(\int_\Omega a(x,\nabla u_k)\nabla u_k dx+\int_{\partial\Omega}|u_k|^{p(x)}d\sigma\right)+\frac{\lambda}{q^-}\int_{\partial\Omega}\left(|u_k|^{q(x)}-|u_k|^{r(x)}\right)d\sigma \\
& \geq \left(\frac{1}{p^+}-\frac{1}{q^-}\right)\rho(u_k)+\lambda\int_{\partial\Omega}\left[\left(\frac{1}{q^-}-\frac{1}{q(x)}\right)|u_k|^{q(x)}+\left(\frac{1}{r(x)}-\frac{1}{q^-}\right)|u_k|^{r(x)}\right]d\sigma \\
& \geq \left(\frac{1}{p^+}-\frac{1}{q^-}\right)\rho(u_k) \\
& \geq \left(\frac{1}{p^+}-\frac{1}{q^-}\right)\|u_k\|^{p^-}.
\end{aligned}
$$

Since $p^+ < q^-$, this contradicts the fact that $p^- > 1$. So, the sequence (u_k) is bounded in $W^{1,p(x)}(\Omega)$. As $W^{1,p(x)}(\Omega)$ is reflexive (Proposition 2.2), for a subsequence still denoted (u_k), we have $u_k \rightharpoonup u$ in $W^{1,p(x)}(\Omega)$, $u_k \to u$ in $L^{p(x)}(\partial\Omega)$, $u_k \to u$ in $L^{q(x)}(\partial\Omega)$, $u_k \to u$ in $L^{r(x)}(\partial\Omega)$ (see Proposition 2.2). Therefore $\langle \Phi'_\lambda(u_k), u_k-u\rangle \to 0$, $\int_{\partial\Omega}|u_k|^{p(x)-2}u_k(u_k-u)d\sigma \to 0$, $\int_{\partial\Omega}|u_k|^{q(x)-2}u_k(u_k-u)d\sigma \to 0$ and $\int_{\partial\Omega}|u_k|^{r(x)-2}u_k(u_k-u)d\sigma \to 0$. Thus $\limsup_{k\to+\infty}\int_\Omega a(x,\nabla u_k)(\nabla u_k-\nabla u)dx \leq 0$. The following theorem assure that $u_k \to u$ strongly in $W^{1,p(x)}(\Omega)$ as $k \to +\infty$. □

Theorem 4.5. *([17] Theorem 4.1) The Carathéodory function $a : \overline{\Omega}\times\mathbb{R}^n \to \mathbb{R}^n$ described by (H_0)–(H_5) is an operator of type S_+ that is, if $u_n \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$ as $n \to +\infty$ and $\limsup_{n\to+\infty}\int_\Omega a(x,\nabla u_n)(\nabla u_n-\nabla u)dx \leq 0$, then $u_n \to u$ strongly in $W^{1,p(x)}(\Omega)$ as $n \to +\infty$.*

Proof of Theorem 1.2. Using the Lemmas 4.2 and 4.3, we obtain

$$\max(\Phi_\lambda(0), \Phi_\lambda(e)) = \Phi_\lambda(0) < \inf_{||u||=\mu}\Phi_\lambda(u) =: \beta.$$

By Lemma 4.4 and Theorem 4.1, we deduce the existence of critical points of Φ_λ associated of the critical value given by

$$\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}\Phi_\lambda(\gamma(t)) \geq \beta,$$

where

$$\Gamma = \{\gamma \in C([0,1], W^{1,p(x)}(\Omega)); \gamma(0) = 0 \text{ and } \gamma(1) = e\}.$$

This completes the proof. □

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