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p-$\mathcal{I}$-Generator And p_1-$\mathcal{I}$-Generator In Bitopology

S. Acharjee and B. C. Tripathy

ABSTRACT: In this article, we investigate several relations between p-$\mathcal{I}$-generator, p_1-$\mathcal{I}$-generator with p-Lindelöf and p_1-Lindelöf spaces by using τ_i-codense, (i, j)-meager set, (i, j)-nowhere dense set and perfect mapping of bitopological space. Various relations between p-compactness, p-Lindelöfness, p_1-Lindelöfness, topological ideal, (i, j)-meager, (i, j)-Baire space in bitopological space are investigated. Some properties are studied by using perfect mapping in a product bitopological space. It is found that bitopological space has many applications in real life.

Key Words: opological ideal, p-Lindelöf, p_1-Lindelöf, pairwise weakly Lindelöf, pairwise almost Lindelöf.

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1. Introduction, motivation and scopes of bitopological space in other areas of mathematics and natural science.

Kelly [1] introduced bitopological space via quasi-pseudo metric and systematically investigated its various important properties. It has drawn direct and indirect attentions of many point set topologists, fuzzy topologists, engineers, researchers of medical science, computer scientists, economists etc. for its applications in their respective areas.

Definition of topological ideal is well known. Topological ideal $\mathcal{I}$ and σ-ideal can be found in Dontchev et al.[2]. Ideal of all nowhere dense sets and ideal of all

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meager sets of an ideal topological space $(X, \tau, \mathcal{I})$ are denoted by $\mathcal{N}$ and $\mathcal{M}$; respectively. Throughout this paper, no separation axiom is considered unless otherwise stated.

Kuratowski [3] introduced the notion of local function of $A \subseteq X$ in (X, τ) with respect to $\mathcal{I}$ and τ (briefly A^*). $A^*(\mathcal{I})$ or $A^* = \{x \in X | U \cap A \notin \mathcal{I}, x \in U$ for all $U \in \tau\}$.

It is well known that $cl^*(A) = A^* \cup A$; defines a Kuratowski closure operator for a topology $\tau^*(\mathcal{I})$ finer than τ.

Throughout this paper, the word "bitopological space" will be denoted by BS.

A cover $\mathcal{U}$ of a BS (X, τ_1, τ_2) is called $\tau_1\tau_2$-open (Swart [4], Definition 4.1) if $\mathcal{U} \subseteq \tau_1 \cup \tau_2$. If in addition, $\mathcal{U}$ contains atleast one non-empty member of τ_1 and atleast one nonempty member of τ_2; then it is called pairwise open (see for instance Fletcher et al. [5]). Pairwise compactness was defined by Fletcher et al. [5]. p-compact, p_1- compact, p-Lindelöf and p_1-Lindelöf were defined by Kilićkman and Salleh [6]. According to Reilly [7]; (X, τ_1, τ_2) is pairwise Lindelöf (resp. pairwise compact) if each pairwise open cover has a countable (resp. finite) subcover. Cooke and Reilly [8] investigated relation between semi-compactness and pairwise compactness in bitopological space.

Kilićkman and Salleh [9-11] also studied various properties of pairwise Lindelöfness. Cocompactness, cotopology, (i, j)-Baire space etc. were studied by Dvalishvili [12].

Frolik [13] introduced weakly Lindelöf space, Willard and Dissanayake [14] introduced almost Lindelöf space in a topological space and their bitopological versions were studied by Kilićkman and Salleh [9]. In the last two decades, various developments have been observed in bitopological space. Still a little progress has been observed in case of generalized closed sets of bitopological space and related areas. Fuzzy versions of some generalized closed sets and related structures from both topology and bitopology have been studied (one may refer to [15-17]). Fuzzy version of topological ideal was introduced by Sarkar [18].

Bitopological space and its properties have many useful applications in real world. In 2010, Salama [19] used lower and upper approximations of Pawlak's rough sets; by using a class of generalized closed set of bitopological space for data reduction of rheumatic fever data sets. Fuzzy topology integrated support vector machine (FTSVM)-classification method for remotely sensed images based on standard support vector machine (SVM) was introduced by Zhang et al. [20]. For some recent indirect applications of topology or bitopology as fuzzy versions, one may refer to [19-21]. Ideal topological space has many applications. Recently Tripathy and Acharjee [22] introduced a class of generalized closed set in bitopological space using topological ideal, two expansion operators and local functions. The application of this set can be found in market price equilibrium [23]. There are maximum nine out of eleven strategies; under which expected price of a daily useful commodity, which is decided by a consumer and price; which is decided by government; are equal. Other two strategies are special cases. These are useful from the view point that; no one will have to face poverty in year 2017 if she has price lists of these commodities for 2016 and 2015. She has freedom to choose her daily useful commodities; according to her preferences.

One may refer to [41]; for interrelated research works on topology, orderings and utility theory of mathematical economics. In this paper one may find; how concepts of countability, compactness, normality, Lindelöfness etc. of general topology and order (i.e. LOTS etc.) have been used for countable representation of utility function. One may refer to Bosi and Mehta [43]; for their interlink of bitopology and choice via utility function.

Hence, there is a need to study different types of pairwise compactness, pairwise Lindelöfness from the point of view of topological ideal, (i, j)-meager set and (i, j)-Baire space.

In this paper, we try to give some possible answers of the following questions.

(i) Is there any relation between different forms of pairwise Lindelöfness, (i, j)-meager set and pairwise Baire space in a bitopological space?

(ii) Is there any relation between different forms of pairwise Lindelöfness and topological ideal in a bitopological space?

(iii) What are the results related to pairwise Lindelöfness in product bitopology using Datta's perfect mapping?

In this paper, we consider two types of pairwise Lindelöfnss. They are p-Lindelöf due to Kilićkman and Salleh [2] and p_1-Lindelöf due to Birsan [24] (as defined by Kilićkman and Salleh [2]). Dvalishvili [25] defined (i, j)-nowhere dense set. Dontchev et al. [26] studied ideal irresoluteness in topology. Datta [27] defined perfect map from the view point of bitopology. Researchers have studied Khalimsky digital line by considering generalized closed sets in topological space ([28-30]). Many topologists are now focusing on ideal and its various consequences. Systematic study on pairwise Lindelöfness can be found in Salleh and Kilićkman [31]. Throughout this paper; we will consider $i, j \in \{1, 2\}, i \neq j$

From the above; it is clear that bitopological space has drawn attentions as an applied branch for research. Many researchers have used bitopological properties as their tools to solve problems of mechanical engineering, medicine, economics etc. Hence, the above questions may play significant roles in near future. Often it is easy to assume results of bitopology as extensions of results of general topology; which is it not true in general. This can be understood from the fact that bitopology has many definitions of Lindelöfness using only pairwise open sets etc.

Variations of i and j between 1 and 2 often signify different properties in a bitopological space; which general topology never follows. In [44], Acharjee et al. answered some open questions and one suitable counterexample.

Lemma 1.1. ([6], Lemma 1) Every pairwise closed subset of a p-Lindelöf bitopological space is p-Lindelöf.

Lemma 1.2. ([6], Lemma 4) Every pairwise closed subset of a p_1-Lindelöf bitopological space is p_1-Lindelöf.

Lemma 1.3. ([44], Theorem 3.1.) Let (X, τ_1, τ_2) be a contra second countable bitopological space, then it is p_1-Lindelöf.

Lemma 1.4. ([44], Corollary 3.1.) Every pairwise closed subset of a contra second countable bitopological space is p_1-Lindelöf.

2. Some preliminary definitions

Definition 2.1. ([9], Definition 2.7) A BS (X, τ_1, τ_2) is said to be (i, j)-nearly Lindelöf (resp. (i, j)-almost Lindelöf, (i, j)-weakly Lindelöf), if every τ_i-open cover $\{U_\alpha | \alpha \in \Delta\}$ of X, there exists a countable subcollection $\{U_{\alpha_n} | n \in N\}$; such that $X = \bigcup_{n\in N} \tau_i int \tau_j cl(U_{\alpha_n})$ (resp. $X = \bigcup_{n\in N} \tau_j cl(U_{\alpha_n})$, $X = \tau_j cl(\bigcup_{n\in N} U_{\alpha_n})$).

(X, τ_1, τ_2) is said to be pairwise nearly Lindelöf, if it is both (i, j)-nearly Lindelöf and (j, i)-nearly Lindelöf. Similarly, one can define pairwise almost Lindelöf, pairwise weakly Lindelöf.

Definition 2.2. ([25], Definition 1.1) A subset A of a BS (X, τ_1, τ_2) is termed as (i, j)-nowhere dense, if $\tau_i int \tau_j cl(A) = \emptyset$. The family of all (i, j)-nowhere dense subsets of X is denoted by (i, j)-$\mathcal{ND}(X)$.

Let $\mathcal{I}$ be a topological ideal, then $\mathcal{I} \neq \emptyset$ and $\mathcal{I}$ is said to be codense [2] for a topological space (X, τ); if and only if $\mathcal{I} \cap \tau = \{\emptyset\}$. Similarly, one can define τ_i-codense; $i \in \{1, 2\}$ for a BS (X, τ_1, τ_2). An ideal $\mathcal{I}$ is said to be pairwise codense, if it is both τ_1-codense and τ_2-codense. We denote ideal of (i, j)-nowhere dense subsets of BS (X, τ_1, τ_2) by $\mathcal{I}_i\ \mathcal{N}_j(X)$

Definition 2.3. ([12], Definition 1.6) A subset A of a BS (X, τ_1, τ_2) is termed as (i, j)-first category (or (i, j)-meager), if $A = \bigcup_{n=1}^{\infty} A_n$; where $A_n \in (i, j)$-$\mathcal{ND}(X)$; for every $n \in N$ and A is of (i,j)-second category (or (i,j)-non meager), if it is not of (i,j)-first category. The family of all sets of (i,j)-first category (or (i,j)-second categories) in X is denoted by (i,j)-$Catg_I(X)((i,j)$-$Catg_{II}(X))$.

If $X \in (i,j)$-$Catg_I(X)$ (resp. $X \in (i,j)$-$Catg_{II}(X)$), then it is abbreviated as X is of (i,j)-$Catg_I$ (resp. (i,j)-$Catg_{II}$).

We denote σ-ideal [2] of (i,j)-meager subsets of a BS (X,τ_1,τ_2) by $\sigma_i\mathcal{M}_j(X)$.

Now, we define the following definitions.

Definition 2.4. A BS (X,τ_1,τ_2) is said to be (i,j)-non-nearly Lindelöf (resp. (i,j)-non-almost Lindelöf, (i,j)-non-weakly Lindelöf), if for every τ_i-open cover $\{U_\alpha | \alpha \in \Delta\}$ of X; there exists a τ_j-open countable sub-collection $\{U_{\alpha_n} | n \in N\}$ such that $X = \bigcup_{n\in N} \tau_j int \tau_i cl(U_{\alpha_n})$ (resp. $X = \bigcup_{n\in N} \tau_i cl(U_{\alpha_n})$, $X = \tau_i cl(\bigcup_{n\in N} U_{\alpha_n})$).

(X,τ_1,τ_2) is said to be pairwise non-nearly Lindelöf, if it is both (i,j)-non-nearly Lindelöf and (j,i)-non-nearly Lindelöf. Similarly, we have pairwise non-almost Lindelöf, pairwise non-weakly Lindelöf.

Kilićkman and Salleh defined p-Lindelöf ([6] Definition 6) and Birsan defined p_1-Lindelöf (One may refer to Definition 1 of [6]).

Definition 2.5. ([44], Definition 3.1) Let (X,τ_1,τ_2) be a bitopological space, then:

(i) (X,τ_1,τ_2) is said to be an (i,j)-second countable bitopological space, if (X,τ_i) is second countable with respect to τ_j.

(ii) (X,τ_1,τ_2) is said to be a contra second countable bitopological space, if it is both $(1,2)$-second countable bitopological space and $(2,1)$-second countable bitopological space.

We state the following results those will be used in this paper.

Lemma 2.1. ([6], Theorem 6) If (X,τ_1,τ_2) is second countable space, then (X,τ_1,τ_2) is p-Lindelöf.

Definition 2.6. [7] A bitopological space (X,τ_1,τ_2) is pairwise compact (resp. pairwise Lindelöf), if each pairwise open cover of (X,τ_1,τ_2) has a finite (resp. countable) subcover.

Definition 2.7. [46] (X,τ_1,τ_2) is said to be pairwise countably compact, if every countable pairwise open cover of (X,τ_1,τ_2) has a finite subcover.

Proposition 2.1. [7] In a pairwise Lindelöf space; pairwise countable compactness is equivalent to pairwise compactness.

Proposition 2.2. [7] Any second countable bitopological space is pairwise Lindelöf.

Proposition 2.3. [7] If (X, τ_1, τ_2) is pairwise Lindelöf and A is a proper subset of X which is τ_1-closed, then A is pairwise Lindelöf and τ_2-Lindelöf.

Proposition 2.4. [7] If (X, τ_1, τ_2) is pairwise Lindelöf and pairwise regular; then it is pairwise normal.

3. Main results

In this section, we define two new classes in a bitopological space, which generate p-Lindelöf space and p_1-Lindelöf space respectively.

Definition 3.1. A BS $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be τ_i-$\mathcal{I}$-generator (resp. τ_i^c-$\mathcal{I}$-generator); if for every τ_i-open cover $\{U_\alpha | \alpha \in \Delta\}$ of X, there exists a (resp. τ_j-open) countable sub-collection $\{U_{\alpha_n} | n \in N\}$; such that $X \setminus \bigcup_{n \in N} U_{\alpha_n} \in \mathcal{I}$.

$(X, \tau_1, \tau_2, \mathcal{I})$ is said to be p-$\mathcal{I}$-generator (resp. p_1-$\mathcal{I}$-generator), if it is both τ_i-$\mathcal{I}$-generator (resp. τ_i^c-$\mathcal{I}$-generator) and τ_j-$\mathcal{I}$-generator (resp. τ_j^c-$\mathcal{I}$-generator).

Remark 3.1. From definition of ideal, it is clear that $\mathcal{I} \neq \emptyset$. If $\mathcal{I} = \{\emptyset\}$, then Definition 3.1 reduces to p-Lindelöf (resp. p_1-Lindelöf) i.e. p-$\{\emptyset\}$-generator$\Leftrightarrow$ p-Lindelöf and p_1-$\{\emptyset\}$-generator$\Leftrightarrow$ p_1-Lindelöf.

From ([2],[26]), we know that a subset S of $(X, \tau, \mathcal{I})$ is a topological subspace with ideal $\mathcal{I}_S = \{I \cap S : I \in \mathcal{I}\}$.

A subset A of X of (X, τ_1, τ_2) is said to be pairwise clopen, if it is both τ_1-clopen and τ_2-clopen.

Theorem 3.1. (i) Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a p-$\mathcal{I}$-generator. If A is a pairwise closed subset of X, then $(A, \tau_1|_A, \tau_2|_A, \mathcal{I}_A)$ is also p-$\mathcal{I}_A$-generator.

(ii) Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a p_1-$\mathcal{I}$-generator. If A is a pairwise clopen subset of X, then $(A, \tau_1|_A, \tau_2|_A, \mathcal{I}_A)$ is also p_1-$\mathcal{I}_A$-generator.

Proof.(i) Let $\mathcal{U}_A = \{U_\alpha \cap A : U_\alpha \in \tau_i, \alpha \in \Delta\}$ be a $\tau_i|_A$-open cover of A. Thus, $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\} \cup \{(X \setminus A)\}$ is τ_i open cover of X. Thus, X has a countable sub-collection $\mathcal{V} = \{U_{\alpha_n} : U_{\alpha_n} \in \tau_i, n \in N\} \cup \{(X \setminus A)\}$ such that $X \setminus \{\bigcup_{n \in N} U_{\alpha_n} \cup (X \setminus A)\} = R(\text{say}) \in \mathcal{I}$.

Then, $A \subseteq \bigcup_{n\in N}\{U_{\alpha_n} : n \in N\} \cup R$. Thus, $A = \bigcup_{n\in N}(U_{\alpha_n} \cap A) \cup (R \cap A)$. So, we have $A \setminus \bigcup_{n\in N}\{(U_{\alpha_n} \cap A)\} \subseteq (R \cap A) \in \mathcal{I}_A$. Hence, $\mathcal{V}_A = \{U_{\alpha_n} \cap A : n \in N\}$ is satisfying the required condition for p-$\mathcal{I}_A$-generator. Hence the proof.

(ii) It can be established following the technique used in establishment of (i).

Remark 3.2. If $\mathcal{I} = \{\emptyset\}$, then $\mathcal{I}_A = \{\emptyset\}$. Then by Theorem 3.1, A is p-$\{\emptyset\}$-generator and it implies Lemma 1.1 of and vice-versa. Similarly, if A is p_1-$\{\emptyset\}$-generator, then it implies Lemma 1.2.

In view of Lemma 2.1 and Remark 3.1, we have the following result.

Corollary 3.1. Every second countable space is p-$\{\emptyset\}$-generator.

Theorem 3.2. (i) Let (X, τ_1, τ_2) be a BS, then X is pairwise weakly Lindelöf if and only if X is both τ_i-$\mathcal{I}_j\mathcal{N}_i$-generator and τ_j-$\mathcal{I}_i\mathcal{N}_j$-generator.

(ii) Let (X, τ_1, τ_2) be a BS, then X is pairwise non-weakly Lindelöf if and only if X is both τ_i-$\mathcal{I}_i\mathcal{N}_j$-generator and τ_j-$\mathcal{I}_j\mathcal{N}_i$-generator.

Proof. (i) Necessity.
We have only to show, if X is (i, j)-weakly Lindelöf; then it is τ_i-$\mathcal{I}_j\mathcal{N}_i$-generator.

Let us assume, X be (i, j)-weakly Lindelöf and let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be a τ_i-open cover of X. Then by Definition 2.1, there exists a countable sub-collection $\{U_{\alpha_n} | n \in N\}$ such that $X = \tau_j cl(\bigcup_{n\in N} U_{\alpha_n})$. Then, $X \setminus \bigcup_{n\in N} U_{\alpha_n} \in \mathcal{I}_j\mathcal{N}_i(X)$. Similarly, it can be established for (j, i)-weakly Lindelöf case.

Sufficiency.

We will only prove that if X is τ_i-$\mathcal{I}_j\mathcal{N}_i$-generator, then X is (i, j)-weakly Lindelöf.

Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be a τ_i-open cover of X, then by Definition 3.1; there exists a countable sub-collection $\{U_{\alpha_n} | n \in N\}$ such that $X \setminus \bigcup_{n\in N} U_{\alpha_n} \in \mathcal{I}_j\mathcal{N}_i(X)$. Then, $X = \tau_j cl(\bigcup_{n\in N} U_{\alpha_n})$. Thus, X is (i, j)-weakly Lindelöf. Similarly, we can prove for τ_j-$\mathcal{I}_i\mathcal{N}_j$-generator case.

(ii) It can be established by following the technique of proof of (i).

Theorem 3.3. (i) A BS (X, τ_1, τ_2) is pairwise weakly Lindelöf; if and only if it is both τ_i-R-generator and τ_j-S-generator for some τ_j-codense ideal R and τ_i-codense ideal S.

(ii) A BS (X, τ_1, τ_2) is pairwise non-weakly Lindelöf; if and only if it is both τ_i-R-generator and τ_j-S-generator for some τ_i-codense ideal R and τ_j-codense ideal S.

Proof. (i) Necessity.

If (X, τ_1, τ_2) is pairwise weakly Lindelöf, then by Theorem 3.2(i), X is both τ_i-$\mathcal{I}_j\mathcal{N}_i$-generator and τ_j-$\mathcal{I}_i\mathcal{N}_j$-generator. It is easy to verify $\mathcal{I}_j\mathcal{N}_i(X) \cap \tau_j = \{\emptyset\}$. So, $\mathcal{I}_j\mathcal{N}_i(X)$ is τ_j-codense. Similarly, we can establish the other case.

Sufficiency.

Let R be any τ_j-codense ideal and X is τ_i-R-generator. Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be any τ_i-open cover of X. Then, there is a countable subcover $\{U_{\alpha_n} | n \in N\}$ such that $X \setminus \bigcup_{n \in N} U_{\alpha_n} \in R$. Hence, $X = \tau_j cl(\bigcup_{n \in N} U_{\alpha_n})$. Thus, X is (i,j)-weakly Lindelöf. Similarly, we can prove for the other case. Thus, X is pairwise weakly Lindelöf. Hence the proof.

Dvalishvili ([12],[25]) defined (i,j)-Baire space and pairwise Baire space.

In next theorem, we establish the relation between pairwise weakly Lindelöf space and pairwise σ-ideal generator under certain condition.

Theorem 3.4. Let (X, τ_1, τ_2) is a pairwise Baire space. Then,

(i) (X, τ_1, τ_2) is pairwise weakly Lindelöf; if and only if (X, τ_1, τ_2) is both τ_i-$\sigma_j\mathcal{M}_i$-generator and τ_j-$\sigma_i\mathcal{M}_j$-generator.

(ii) (X, τ_1, τ_2) is pairwise non-weakly Lindelöf; if and only if (X, τ_1, τ_2) is both τ_i-$\sigma_i\mathcal{M}_j$-generator and τ_j-$\sigma_j\mathcal{M}_i$-generator.

Proof. (i) (X, τ_1, τ_2) is (i,j)-Baire space and (j,i)-Baire space $\Rightarrow$ X is (i,j)-$Catg_{II}$ and (j,i)-$Catg_{II}$.

(X, τ_1, τ_2) is (i,j)-Baire space and (j,i)-Baire space $\Leftrightarrow$ $\sigma_i\mathcal{M}_j(X)$ is τ_i-codense and $\sigma_j\mathcal{M}_i(X)$ is τ_j-codense. Then, the proof follows from Theorem 3.3(i). Hence the proof.

A BS (X, τ_1, τ_2) is said to have property *; if $\tau_i cl(\tau_j cl(U)) = \tau_j cl(U)$, whenever $U \subseteq X$ and $i, j \in \{1,2\}, i \neq j$.

We state the following result without proof.

Theorem 3.5.(i) If (X, τ_1, τ_2) is pairwise almost Lindelöf with property * ; then it is both τ_i-$\sigma_j\mathcal{M}_i$-generator and τ_j-$\sigma_i\mathcal{M}_j$-generator.

(ii) If (X, τ_1, τ_2) is pairwise non-almost Lindelöf with property * ; then it is both τ_i-$\sigma_i\mathcal{M}_j$-generator and τ_j-$\sigma_j\mathcal{M}_i$-generator.

In view of Theorem 3.4 and Theorem 3.5, we state the following result.

Corollary 3.2. (i) If a BS (X, τ_1, τ_2) is pairwise almost Lindelöf with property * and pairwise Baire space, then it is pairwise weakly Lindelöf.

(ii) If a BS (X, τ_1, τ_2) is pairwise non-almost Lindelöf with property * and pairwise Baire space, then it is pairwise non-weakly Lindelöf.

The following result is a consequence of Theorem 3.1 and Theorem 3.2.

Corollary 3.3. (i) If A be a pairwise clopen subset of a pairwise weakly Lindelöf space (X, τ_1, τ_2), then $(A, \tau_1|_A, \tau_2|_A)$ is pairwise weakly Lindelöf.

(ii) If A be a pairwise clopen subset of a pairwise non-weakly Lindelöf space (X, τ_1, τ_2), then $(A, \tau_1|_A, \tau_2|_A)$ is pairwise non-weakly Lindelöf.

During the preparation of this paper with refer to Kilièman and Salleh [6], some open questions were raised. Some answers of these questions are affirmative and one counter example is proved by Acharjee et al. in [44]; using interlocking and nest in a bitopological space. Notions of interlocking and nest can be found in [45]. The two main questions are stated below.

(i) What type of a countable space in a bitopological space is a p_1-Lindelöf space?

(ii) Does every p_1-Lindelöf space imply countable space of (i)?.

Theorem 3.6. Let (X, τ_1, τ_2) be a contra second countable bitopological space, then it is p_1-$\{\emptyset\}$-generator.

Proof. The proof follows from Remark 3.1. and Lemma 1.1.

Theorem 3.7. Every pairwise closed subset of a contra second countable bitopological space is p_1-$\{\emptyset\}$-generator.

Proof. It can be proved by Lemma 1.4 and Remark 3.1.

4. Relations of p-$\mathcal{J}$-generator and p_1-$\mathcal{J}$-generator with perfect mapping

The following definition of perfect mapping is due to Datta [27].

Definition 4.1. ([27], Definition 2.1) A mapping $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\psi_1,\psi_2)$ is said to be perfect if,

(i) f is continuous i.e. f is τ_1-ψ_1-continuous and τ_2-ψ_2-continuous.

(ii) f is compact i.e. the inverse image of every point of Y is τ_1-compact,τ_2-compact and pairwise compact.

(iii) f is closed i.e. the image of every τ_1-closed (resp. τ_2-closed) subset of X is ψ_1-closed (resp. ψ_2-closed) subset of Y.

Let $f:(X,\tau_1,\tau_2,\mathfrak{I})\longrightarrow(Y,\psi_1,\psi_2,\mathfrak{J})$ be a function, then we denote $f(\mathfrak{I})=\{f(I)|I\in\mathfrak{I}\}$ and $f^{-1}(\mathfrak{J})=\{f^{-1}(J)|J\in\mathfrak{J}\}$. Hence, $f(\mathfrak{I})$ and $f^{-1}(\mathfrak{J})$ are ideals of Y and X respectively.

Theorem 4.1 (i) Let $f:(X,\tau_1,\tau_2,\mathfrak{I})\longrightarrow(Y,\psi_1,\psi_2)$ be a continuous function and surjection. If $(X,\tau_1,\tau_2,\mathfrak{I})$ is p-$\mathfrak{I}$-generator, then (Y,ψ_1,ψ_2) is also p-$f(\mathfrak{I})$-generator.

(ii) Let $f:(X,\tau_1,\tau_2,\mathfrak{I})\longrightarrow(Y,\psi_1,\psi_2)$ be a continuous function and surjection. If $(X,\tau_1,\tau_2,\mathfrak{I})$ is p_1-$\mathfrak{I}$-generator, then (Y,ψ_1,ψ_2) is also p_1-$f(\mathfrak{I})$-generator.

Proof.(i) It is enough to show, if $(X,\tau_1,\tau_2,\mathfrak{I})$ is τ_i-$\mathfrak{I}$-generator, then (Y,ψ_1,ψ_2) is also ψ_i-$f(\mathfrak{I})$-generator.

Let $\mathcal{U}=\{U_\alpha|\alpha\in\Delta\}$ be any ψ_i-open cover of Y. Then by Definition 4.1, $\mathcal{V}=\{f^{-1}(U_\alpha)|\alpha\in\Delta\}$ is τ_i-open cover of X. So, we have a subcollection $\{f^{-1}(U_{\alpha_n})|n\in N\}$ such that $X\setminus\bigcup_{n\in N}f^{-1}(U_{\alpha_n})\in\mathfrak{I}$. Suppose $f^{-1}(Y\setminus\bigcup_{n\in N}U_{\alpha_n})=I$. So, $(Y\setminus\bigcup_{n\in N}U_{\alpha_n})=f(I)\in f(\mathfrak{I})$ as $I\in\mathfrak{I}$. Thus, we have the proof

(ii) It can be established following the technique used in establishing part(i).

We state the following result without proof.

Theorem 4.2. Let $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\psi_1,\psi_2,\mathfrak{J})$ be a perfect, open and surjective function. Then,

(i) if $(Y,\psi_1,\psi_2,\mathfrak{J})$ is p-$\mathfrak{J}$-generator, then (X,τ_1,τ_2) is p-$f^{-1}(\mathfrak{J})$-generator.

(ii) if $(Y,\psi_1,\psi_2,\mathfrak{J})$ is p_1-$\mathfrak{J}$-generator, then (X,τ_1,τ_2) is p_1-$f^{-1}(\mathfrak{J})$-generator.

Lemma 4.1. If $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\psi_1,\psi_2,\mathfrak{J})$ be an open function and surjective. If $\mathfrak{J}$ is ψ_i-codense, then $f^{-1}(\mathfrak{J})$ is τ_i-codense.

Proof. Let $f^{-1}(\mathfrak{J})$ is not τ_i-codense. Let $f^{-1}(J)\in f^{-1}(\mathfrak{J})\cap\tau_i\neq\{\emptyset\}$. Then,

$f^{-1}(J) \in \tau_i \setminus \{\emptyset\}$. Due to surjective and open; $f(f^{-1}(J)) = J \in \psi_i \setminus \{\emptyset\}$. This contradicts the fact that $\mathcal{J}$ is ψ_i-codense. Hence the proof.

Corollary 4.1. Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \psi_1, \psi_2)$ be a perfect,open and surjective function. Then,

(i) if (Y, ψ_1, ψ_2) is p-Lindelöf, then (X, τ_1, τ_2) is p-Lindelöf .

(ii) if (Y, ψ_1, ψ_2) is p_1-Lindelöf, then (X, τ_1, τ_2) is p_1-Lindelöf .

Proof. (i) (Y, ψ_1, ψ_2) is p-Lindelöf implies it is p-$\{\emptyset\}$-generator. Then, the proof follows from Theorem 4.2(i) and Remark 3.1.

(ii) Proof follows similar to the case (i)

Applying Theorem 3.3 and Lemma 4.1, one can get the following result.

Corollary 4.2. Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \psi_1, \psi_2)$ be a perfect, open and surjective function.

(i) If (Y, ψ_1, ψ_2) is pairwise weakly Lindelöf, then (X, τ_1, τ_2) is pairwise weakly Lindelöf .

(ii) If (Y, ψ_1, ψ_2) is pairwise non-weakly Lindelöf, then (X, τ_1, τ_2) is pairwise non-weakly Lindelöf .

5. On product bitopology

It is well known, every continuous mapping between p-compact spaces is p-compact in bitopological space. One may refer to Datta ([27], page no: 124)

Theorem 5.1. (i) If $(X, \tau_1, \tau_2, \mathcal{I})$ is p-$\mathcal{I}$-generator and (Y, ψ_1, ψ_2) is p-compact, then $(X \times Y, \tau_1 \times \psi_1, \tau_2 \times \psi_2)$ is p-$\pi^{-1}(\mathcal{I})$-generator; where $\pi : X \times Y \longrightarrow X$ is a projection map.

(ii) If $(X, \tau_1, \tau_2, \mathcal{I})$ is p_1-$\mathcal{I}$-generator and (Y, ψ_1, ψ_2) is p-compact, then $(X \times Y, \tau_1 \times \psi_1, \tau_2 \times \psi_2)$ is p_1-$\pi^{-1}(\mathcal{I})$-generator; where $\pi : X \times Y \longrightarrow X$ is a projection map.

Proof. The projection map is perfect. Hence, the rest follows from Theorem 4.2.

The following result is a consequence of Theorem 3.3, Lemma 4.1 and Theorem 5.1.

Corollary 5.1. (i) If (X, τ_1, τ_2) is pairwise weakly Lindelöf and (Y, ψ_1, ψ_2) is p-compact, then $(X \times Y, \tau_1 \times \psi_1, \tau_2 \times \psi_2)$ is pairwise weakly Lindelöf.

(ii) If (X, τ_1, τ_2) is pairwise non-weakly Lindelöf and (Y, ψ_1, ψ_2) is p-compact, then $(X \times Y, \tau_1 \times \psi_1, \tau_2 \times \psi_2)$ is pairwise non-weakly Lindelöf .

Corollary 5.2. (i) If (X, τ_1, τ_2) is p-Lindelöf and (Y, ψ_1, ψ_2) is p-compact, then $(X \times Y, \tau_1 \times \psi_1, \tau_2 \times \psi_2)$ is p-Lindelöf .

(ii) If (X, τ_1, τ_2) is p_1-Lindelöf and (Y, ψ_1, ψ_2) is p-compact, then $(X \times Y, \tau_1 \times \psi_1, \tau_2 \times \psi_2)$ is p_1-Lindelöf .

Proof. (i) By Remark 3.1, (X, τ_1, τ_2) is p-Lindelöf $\Leftrightarrow$ (X, τ_1, τ_1) is p-$\{\emptyset\}$-generator. By Theorem 5.1(i), $(X \times Y, \tau_1 \times \psi_1, \tau_2 \times \psi_2)$ is p-$\{\emptyset\}$-generator. Hence the proof.

6. Conclusion

In this paper, we have shown that p-Lindelöfness and p_1-Lindelöfness can be derived by defining new classes of sets in bitopological space. We also proved results related to perfect mapping of bitopological space and used them in the area of product bitopology. We used perfect mapping to prove various results. One can follow from literatures of bitopology that; various types of pairwise mappings play crucial roles to contradict results related to various pairwise concepts. Our idea may be extended on other types of Lindelöf spaces of a bitopological space. These methods give short and concrete ways to prove various results in product of Lindelöf spaces. We hope, this paper will attract attentions of topologists, economists and researchers of other branches. The connection between countability and p_1-Lindelöfness or p_1-$\{\emptyset\}$-generator may help economists to use bitopological space, Lindelöfness etc. in their respective research areas as one may refer to ([41],[43]), where authors studied utility functions and various results based on compactness, Lindelöfness, order and other properties of bitopology and general topology.

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S. Acharjee, Department of Mathematics, Pragjyotish College, Guwahati-781009, India.
E-mail address: `sacharjee326@gmail.com, santanuacharjee@rediffmail.com`

and

B. C. Tripathy, Department of Mathematics, Tripura University, Tripura-799022, India.
E-mail address: `tripathybc@rediffmail.com, tripathybc@yahoo.com`