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On (Weakly) Precious Rings Associated To Central Polynomials

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ABSTRACT: Let R be an associative ring with identity and let $g(x)$ be a fixed polynomial over the center of R. We define R to be (weakly) $g(x)$-precious if for every element $a \in R$, there are a zero s of $g(x)$, a unit u and a nilpotent b such that $(a = \pm s + u + b)$ $a = s + u + b$. In this paper, we investigate many examples and properties of (weakly) $g(x)$-precious rings. If a and b are in the center of R with $b - a$ is a unit, we give a characterizations for (weakly) $(x - a)(x - b)$-precious rings in terms of (weakly) precious rings. In particular, we prove that if 2 is a unit, then a ring is precious if and only it is weakly precious. Finally, for $n \in \mathbb{N}$, we study (weakly) $(x^n - x)$-precious rings and clarify some of their properties.

Key Words: clean ring, $g(x)$-clean rings, $g(x)$-nil clean rings, precious rings, $g(x)$-precious rings .

Contents

1. Introduction

Throughout this paper all rings are considered associative with an identity. We use $U(R)$, $N(R)$ and $Id(R)$ to denote the the group of units, the set of nilpotent elements and the set of idempotents in R respectively. In 1977 Nicholson, [19], introduced the concept of clean rings as rings in which every element is a sum of a unit and an idempotent. He showed that clean rings must be exchange and they contain unit-regular and semiperfect rings. In [1], Ahn and Anderson defined a ring R to be weakly clean if each element $r \in R$ can be written as $r = u + e$ or $r = u - e$ for $u \in U(R)$ and $e \in Id(R)$. As a variant of (weakly) clean rings, A ring R is said to be (weakly) nil-clean if for every element $r \in R$, there are $b \in N(R)$ and $e \in Id(R)$ such that $(r = b + e$ or $r = b - e)$ $r = b + e$. Nil clean rings are in fact a stronger concept than clean rings and it was defined as early as 1988 by Hirano and others, [16]. Then (weakly) clean and nil clean rings were studied extensively by many authors, see for example [15,20,2,13,7,9,12].

For a ring R, we let $C(R)$ denotes the center of R and $g(x)$ be a polynomial in $C(R)[x]$. Camillo and Simon, [11], defined a ring R to be $g(x)$-clean if for every $r \in R$, $r = s + u$ where $u \in U(R)$ and $g(s) = 0$. Clearly, clean rings are precisely

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the $(x^2 - x)$-clean rings. Many properties of $g(x)$-clean rings have been studied by Fan and Yang, [14]. For $a, b \in C(R)$ with $b - a \in U(R)$, they characterized clean rings as $(x-a)(x-b)$-clean rings. Weakly $g(x)$-clean rings were defined and studied by Ashrafi and Ahmadi, [4] in an analogous way to weakly clean rings. As a special class of $g(x)$-clean rings, Khashan and Handam, [17], defined and studied $g(x)$-nil clean rings as rings in which every element is a sum of a nilpotent and a zero of $g(x)$. More recently, Ashrafi, Shebani and Chen, [5], have defined an element a in a ring R to be precious (weakly precious) if there exist an idempotent e, a unit u and a nilpotent b such that $a = e+u+b$ ($a = e+u+b$ or $a = -e+u+b$). A ring R is called (weakly) precious if every element in R is (weakly) precious. It has been proved that (weakly) precious rings are proper generalizations of both (weakly) clean and (weakly) nil clean rings. Moreover, it is clear that (weakly) clean rings and (weakly) precious rings coincide if the rings are reduced or commutative. Prompted by this definition, we define and study (weakly) $g(x)$-precious rings as a generalization of (weakly) precious, (weakly) $g(x)$-clean and $g(x)$-nil clean rings. R is called $g(x)$-precious if there exist a zero s of $g(x)$, $u \in U(R)$ and $b \in N(R)$ such that $a = s + u + b$ (equivalently, $a = -s + u + b$). As a general concept, we define a ring R to be weakly $g(x)$-precious if there are a zero s of $g(x)$, $u \in U(R)$ and $b \in N(R)$ such that $a = s + u + b$ or $a = -s + u + b$ (in short, we write $a = \pm s + u + b$).

In section 2, we first determine some conditions (weaker than commutativity) under which (weakly) $g(x)$-clean rings and (weakly) $g(x)$-precious rings are the same. Then we investigate the structure theorems of (weakly) $g(x)$-precious rings similar to those of (weakly) $g(x)$-clean and $g(x)$-nil clean rings. We deduce the (weakly) $g(x)$-preciousness of quotient rings, direct product of rings, triangular matrix rings and full matrix rings. Moreover, we characterize (weakly) $g(x)$-precious rings by modules.

In section 3, we study (weakly) $g(x)$-precious rings for some special kinds of polynomials $g(x) \in C(R)[x]$. We study $(x-a)(x-b)$-precious and weakly $x(x-b)$-precious rings where $a, b \in C(R)$ and characterize them in terms of precious and weakly precious rings. Among other results, we prove that if $2 \in U(R)$, then the statements R is precious and R is weakly $(x^2 - 1)$-precious are equivalent. Finally, for $n \in \mathbb{N}$, we study some particular properties of (weakly) $(x^n - x)$-precious rings.

2. Basic Properties of (Weakly) $g(x)$-Precious Rings

The purpose of this section is to give some examples and investigate some basic properties of (weakly) $g(x)$-precious rings.

Definition 2.1. *Let R be a ring and let $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $a \in R$ is called $g(x)$-precious (respectively, weakly $g(x)$-precious) if there exist a zero s of $g(x)$, $u \in U(R)$ and $b \in N(R)$ such that $a = s+u+b$ (respectively, $a = \pm s + u + b$). Moreover, R is called (weakly) $g(x)$-precious if every element in R is (weakly) $g(x)$-precious.*

It is clear that any precious ring is weakly precious. The converse is not true since for example simple computations show that both the rings $\mathbb{Z}_3$ and $T_2(\mathbb{Z}_3)$ are weakly (x^3-1)-precious which are not (x^3-1)-precious. While obviously, (weakly) precious rings are (weakly) (x^2-x)-precious, there are (weakly) $g(x)$-precious rings that are not (weakly) precious. For example, consider the group ring $\mathbb{Z}_{(7)}C_3$ where C_3 is the cyclic group of order 3 and $\mathbb{Z}_{(7)} = \left\{\frac{m}{k} : m,k \in \mathbb{Z} \text{ and } \gcd(7,k)=1\right\}$. We will see in the next section that $\mathbb{Z}_{(7)}C_3$ is (x^4-x)-precious which is not precious. It is also clear that every $g(x)$-clean element in a ring is $g(x)$-precious. However, the converse is not true in general. For example, it was shown in Theorem (3) in [3] that for the matrix $A = \begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix} \in M_2(\mathbb{Z})$, the element $(A,-A) \in M_2(\mathbb{Z}) \times M_2(\mathbb{Z})$ is (x^2-x)-precious which is not (x^2-x)-clean. For a non trivial example, one can easily verify that the ring $\mathbb{Z}_6$ is (weakly) (x^2-2x)-precious which is not (weakly) (x^2-2x)-clean. Since $g(x)$-nil clean rings are $g(x)$-clean by Proposition (2.3) in [17], then every $g(x)$-nil clean ring is $g(x)$-precious. The converse is also not true since for example we can verify that the ring $\mathbb{Z}_{10}$ is an (x^7-x)-precious which is not (x^7-x)-nil clean.

For any commutative ring, it is well known that the sum of a nilpotent and a unit is a unit. Therefore, the class of commutative (weakly) $g(x)$-precious rings is precisely the class of commutative (weakly) $g(x)$-clean rings.

An element a in a ring R is called strongly nilpotent if for every sequence $a = a_0, a_1, ..., a_i, ...$ such that $a_{i+1} \in a_i R a_i$, there exists an integer n with $a_n = 0$. It is well known that the prime radical $P(R)$ of R is exactly the set of all strongly nilpotent elements of R. A ring R is called 2-primal if every nilpotent element of R is strongly nilpotent, see [18]. A ring R is called a left (right) quasi-duo ring if every maximal left (right) ideal of R is an ideal. In the following proposition, we clarify other conditions under which $g(x)$-clean rings and $g(x)$-precious rings coincide.

Proposition 2.2. *Let R be a 2-primal ring or a left (right) quasi-duo ring and $g(x) \in C(R)[x]$. Then R is (weakly) $g(x)$-clean if and only if R is (weakly) $g(x)$-precious.*

Proof. $\Rightarrow)$: Clear

$\Leftarrow)$: Suppose R is a 2-primal $g(x)$-precious ring and let $a \in R$. Choose a zero s of $g(x)$, $u \in U(R)$ and $b \in N(R)$ such that $a = s+u+b$. Then $a = s+u(1+u^{-1}b)$. Since R is 2-primal, then $b \in P(R)$. As $P(R)$ is a nil ideal of R, then $1+u^{-1}b$ is a unit. Therefore, a is a $g(x)$-clean element and R is $g(x)$-clean. Similarly, if R is a left (right) quasi-duo ring, then by Lemma (2.3) in [26], $b \in J(R)$. So, again $u(1+u^{-1}b) \in U(R)$ and R is $g(x)$-clean. The result follows similarly if R is weakly $g(x)$-precious ring. □

A ring R is called nil-semicommutative if for any $a, b \in N(R)$, $ab = 0$ implies that $aRb = 0$. By Lemma (2.7) in [18], any nil-semicommutative ring is 2-primal. Therefore, the equivalence in Proposition 2.2 holds also if R is nil-semicommutative.

For any two rings R and S, we let $\varphi : C(R) \to C(S)$ be a ring homomorphism with $\varphi(1_R) = 1_S$. If $g(x) = \sum_{i=0}^{n} r_i x^i \in C(R)[x]$, we let $g_\varphi(x) := \sum_{i=0}^{n} \varphi(r_i) x^i \in$

$C(S)[x]$. In particular, if $g(x) \in \mathbb{Z}[x]$, then $g_\varphi(x) = g(x)$.

Proposition 2.3. *Let R and S be two rings and $\varphi : R \to S$ be an epimorphism. Then S is (weakly) $g_\varphi(x)$-precious whenever R is (weakly) $g(x)$-precious.*

Proof. Suppose that R is $g(x)$-precious. Assume that $g(x) = \sum_{i=0}^{n} r_i x^i \in C(R)[x]$ and let $t \in S$. Choose an element $a \in R$ such that $\varphi(a) = t$. Since R is $g(x)$-precious, then $a = s + u + b$ for some zero s of $g(x)$, $u \in U(R)$ and $b \in N(R)$. Thus, $t = \varphi(a) = \varphi(s) + \varphi(u) + \varphi(b)$ where $\varphi(u) \in U(R)$ and $\varphi(b) \in N(R)$. Moreover, we have $g_\varphi(\varphi(s)) = \sum_{i=0}^{n} \varphi(r_i)(\varphi(s))^i = \sum_{i=0}^{n} \varphi(r_i)\varphi(s^i) = \varphi\left(\sum_{i=0}^{n} r_i s^i\right) = \varphi(g(s)) = \theta(0_R) = 0_S$. It follows that S is $g_\varphi(x)$-precious. The case when R is weakly $g(x)$-precious is similar. □

Corollary 2.4. *Let $\{R_\alpha\}_{\alpha\in\Lambda}$ be a family of rings and let $R = \prod_{\alpha\in\Lambda} R_\alpha$. If R is $g(x)$-precious, then for any $\alpha \in \Lambda$, R_α is $g_\varphi(x)$-precious where φ is the α-projection homomorphism.*

The converse of Corollary 2.4 is true in the case of finite direct product.

Proposition 2.5. *Let $R_1, R_2, ..., R_n$ be rings and consider $R = \prod_{i=1}^{k} R_i$. For each $i \in \{1, 2, ..., n\}$, let $g_i(x) = a_0^{(i)} + a_1^{(i)} x + ... + a_k^{(i)} x^k \in C(R_i)[x]$. If $g(x) = \left\{a_0^{(i)}\right\}_{i=1}^{n} + \left\{a_1^{(i)}\right\}_{i=1}^{n} x + ... + \left\{a_k^{(i)}\right\}_{i=1}^{n} x^k \in C(R)[x]$, then R is $g(x)$-precious if and only if R_i is $g_i(x)$-precious for all i. In particular if $g(x) \in \mathbb{Z}[x]$, then R is $g(x)$-precious if and only if R_i is $g(x)$-precious for all i.*

Proof. $\Rightarrow)$: Corollary 2.4.

$\Leftarrow)$: Let $\{x_i\}_{i=1}^{n} \in \prod_{i=1}^{n} R_i$. For each i, we can find a zero s_i of $g_i(x)$, $u_i \in U(R_i)$ and $b_i \in N(R_i)$ such that $x_i = s_i + u_i + b_i$. If we let $s = \{s_i\}_{i=1}^{n}$, $u = \{u_i\}_{i=1}^{n}$ and $b = \{b_i\}_{i=1}^{n}$, then clearly we have $g(s) = 0$, $u \in U(R)$ and $b \in N(R)$. Therefore, R is $g(x)$-precious. □

On the other side, we next prove the following proposition in the case of weakly $g(x)$-precious rings.

Proposition 2.6. *Let $R = \prod_{\alpha\in\Lambda} R_\alpha$ be a finite direct product of rings and let $g(x) \in \mathbb{Z}[x]$. Then R is weakly $g(x)$-precious if and only if each R_α is weakly $g(x)$-precious and at most one R_α is not $g(x)$-precious.*

Proof. $\Rightarrow)$: Suppose R is weakly $g(x)$-precious. Then for $\alpha \in \Lambda$, R_α is weakly $g(x)$-precious as it is a homomorphic image of R. Suppose for some α_1 and α_2, $\alpha_1 \neq \alpha_2$, R_{α_1} and R_{α_2} are not $g(x)$-precious. Choose $r_1 \in R_{\alpha_1}$ not of the form $s_1 + u_1 + b_1$

and $r_2 \in R_{\alpha_2}$ not of the form $-s_2 + u_2 + b_2$ for $u_i \in U(R_{\alpha_i})$, $b_i \in N(R_{\alpha_i})$ and $g(s_i) = 0$, $i = 1, 2$. Then clearly, (r_1, r_2) is not weakly $g(x)$-precious in $R_{\alpha_1} \times R_{\alpha_2}$, a contradiction. Therefore, at most one R_α is not $g(x)$-precious.

$\Leftarrow$) : Conversely, we assume that R_β is a weakly $g(x)$-precious that is not $g(x)$-precious for a fixed index $\beta \in \Lambda$ and R_α is $g(x)$-precious for all $\alpha \neq \beta$. Let $r = (r_\alpha) \in R$. Then there exist in R_β a unit u_β, a nilpotent b_β and a zero s_β of $g(x)$ such that $r_\beta = \pm s_\beta + u_\beta + b_\beta$. If $r_\beta = s_\beta + u_\beta + b_\beta$, for each $\alpha \in \Lambda \setminus \{\beta\}$, write $r_\alpha = s_\alpha + u_\alpha + b_\alpha$ where $u_\alpha \in U(R_\alpha)$, $b_\alpha \in N(R_\alpha)$ and $g(s_\alpha) = 0$. Therefore, $r = (s_\alpha) + (u_\alpha) + (b_\alpha)$ is a (weakly) $g(x)$-precious decomposition of r. Similarly, if $r_\beta = -s_\beta + u_\beta + b_\beta$, then for each $\alpha \in \Lambda \setminus \{\beta\}$, write $r_\alpha = -s_\alpha + u_\alpha + b_\alpha$ where $u_\alpha \in U(R_\alpha)$, $b_\alpha \in N(R_\alpha)$ and $g(s_\alpha) = 0$. Consequently, $r = -(s_\alpha) + (u_\alpha) + (b_\alpha)$ is again a weakly $g(x)$-precious decomposition of r. Therefore, r is weakly $g(x)$-precious in R as required. □

Proposition 2.7. *Let R be a ring, I be an ideal of R and consider the canonical homomorphism $\varphi : R \to R/I$.*

(1) If R is a (weakly) $g(x)$-precious ring, then R/I is (weakly) $g_\varphi(x)$-precious.

(2) If R/I is (weakly) $g_\varphi(x)$-precious where I is nil and roots of $g_\varphi(x)$ left modulo I, then R is (weakly) $g(x)$-precious.

Proof. (1) It is clear by Proposition 2.3.

(2) suppose $\overline{R} = R/I$ is $g_\varphi(x)$-precious and let $a \in R$. Then $\overline{a} = a + I = \overline{s} + \overline{u} + \overline{b}$ where $g_\varphi(\overline{s}) = \overline{0}$, $\overline{u} \in U(\overline{R})$ and $\overline{b} \in N(\overline{R})$. Since I is a nil ideal of R, we can easily check that $u \in U(R)$ and $b \in N(R)$. As the root $\overline{s}$ of $g_\varphi(x)$ lifts modulo I, we may assume also that $s \in R$ with $g(s) = 0$. Hence, $a = s + u + b + i$ for some $i \in I$. If we choose an integer $k \geq 1$ such $b^k = 0$, then we get $(b + i)^k \in I$. Therefore, $b + i \in N(R)$ and so R is $g(x)$-precious. If R/I is weakly $g_\varphi(x)$-precious, then R is weakly $g(x)$-precious by a similar approach. □

Proposition 2.8. *Let R be a ring and $g(x) \in C(R)[x]$. Then the formal power series $R[[t]]$ is (weakly) $g(x)$-precious if and only if R is (weakly) $g(x)$-precious.*

Proof. $\Rightarrow$) : Clear as R is a homomorphic image of $R[[t]]$.

$\Leftarrow$) : Suppose R is $g(x)$-precious and let $f = a_0 + a_1x + a_2x^2 + ... \in R[[t]]$. Write $a_0 = s + u + b$ where $u \in U(R)$, $b \in N(R)$ and $g(s) = 0$. Then $f = s + (u + a_1x + a_2x^2 + ...) + b$ where clearly $u + a_1x + a_2x^2 + ... \in U(R[[t]])$ and $b \in N(R[[t]])$. Therefore, $R[[t]]$ is $g(x)$-precious. The case of weakly $g(x)$-preciousness is similar □

Since for a commutative clean ring R, the ring of polynomials $R[t]$ is not clean, [15], then $R[t]$ is not precious (and so is not an $(x^2 - x)$-precious).

Let R be a ring and $n \in \mathbb{N}$. Then the matrix ring $M_n(R)$ is a $C(R)$-algebra. Indeed, we can define $\pi : C(R) \to M_n(R)$ by $\pi(a) = aI_n$ where I_n is the identity matrix. In the next two propositions, we determine when the upper triangular matrix ring $T_n(R)$ and the full matrix ring $M_n(R)$ are (weakly) $g(x)$-precious for $g(x) \in C(R)[x]$.

Proposition 2.9. *Let R be a ring and let $g(x) \in C(R)[x]$. The following are equivalent.*

(1) R is $g(x)$-precious.

(2) $T_n(R)$ is $g(x)$-precious for all $n \in \mathbb{N}$.

(3) $T_n(R)$ is weakly $g(x)$-precious for some $n \geq 2$.

Proof. (1)⇒(2): Suppose R is $g(x)$-precious, $n \in \mathbb{N}$ and $A = [a_{ij}] \in T_n(R)$. For each i, choose a zero s_{ii} of $g(x)$, $u_{ii} \in U(R)$ and $b_{ii} \in N(R)$ such that $a_{ii} = s_{ii} + u_{ii} + b_{ii}$. Then one can easily see that $A = S + U + B$ where $U = \begin{bmatrix} u_{11} & & & * \\ & u_{22} & & \\ & & \ddots & \\ 0 & & & u_{nn} \end{bmatrix} \in U(T_n(R))$, $B = \begin{bmatrix} b_{11} & & & 0 \\ & b_{22} & & \\ & & \ddots & \\ 0 & & & b_{nn} \end{bmatrix} \in N(T_n(R))$ and $S = \begin{bmatrix} s_{11} & & 0 \\ & s_{22} & \\ 0 & & s_{nn} \end{bmatrix}$ is a zero of $g(x)$. Hence, $T_n(R)$ is $g(x)$-precious.

(2)⇒(3): Trivial.

(3)⇒(1): Suppose $T_n(R)$ is weakly $g(x)$-precious for some $n \geq 2$. Let $a \in R$ and choose $A = \begin{bmatrix} a & & & & 0 \\ & -a & & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix}$. By assumption, we may write $A = \pm S + U + B$ where $U = \begin{bmatrix} u_1 & & & * \\ & u_2 & & \\ & & \ddots & \\ 0 & & & u_n \end{bmatrix} \in U(T_n(R))$, $B = \begin{bmatrix} b_1 & & & * \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_n \end{bmatrix} \in N(T_n(R))$ and $S = \begin{bmatrix} s_1 & & & * \\ & s_2 & & \\ & & \ddots & \\ 0 & & & s_n \end{bmatrix}$ is a zero of $g(x)$. If $A = S+U+B$, then $a = s_1 + u_1 + b_1$. Similarly, if $A = -S + U + B$, then $a = s_2 - u_2 - b_2$ where clearly $u_1, u_2 \in U(R)$, $b_1, b_2 \in N(R)$ and s_1, s_2 are zeros of $g(x)$. Therefore, R is $g(x)$-precious. □

Proposition 2.10. *Let R be a ring, $n \in \mathbb{N}$ and $g(x) \in C(R)[x]$. If R is $g(x)$-precious, then so is the matrix ring $M_n(R)$.*

Proof. We use the induction on n. For $n = 1$, the result is clear. Let $n \gneqq 1$ and suppose the statement is true for $M_{n-1}(R)$. Let $A \in M_n(R)$ and write $A = \begin{bmatrix} A_* & X \\ Y & a \end{bmatrix}$ in block form where $a \in R$ and $A_* \in M_{n-1}(R)$. Since $M_{n-1}(R)$

is $g(x)$-precious, then we can find $U_* \in U(M_{n-1}(R))$, $B_* \in N(M_{n-1}(R))$ and a zero S_* of $g(x)$ such that $A_* = S_* + U_* + B_*$. Now, $a - Y\ U_*^{-1}X \in R$. Hence, $a - Y\ U_*^{-1}X = s + u + b$ where $g(s) = 0$, $u \in U(R)$ and $b \in N(R)$. Now, $A = \begin{bmatrix} S_* + U_* + B_* & X \\ Y & s + u + b + YU_*^{-1}X \end{bmatrix} = \begin{bmatrix} S_* & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} U_* & X \\ Y & u + YU_*^{-1}X \end{bmatrix} +$ $+ \begin{bmatrix} B_* & 0 \\ 0 & b \end{bmatrix} = S + U + B$. Since

$$\begin{bmatrix} I_{n-1} & 0 \\ -YU_*^{-1} & 1 \end{bmatrix} \begin{bmatrix} U_* & X \\ Y & u + YU_*^{-1}X \end{bmatrix} \begin{bmatrix} I_{n-1} & -U_*^{-1}X \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} U_* & 0 \\ 0 & u \end{bmatrix},$$

then U is a unit in $M_n(R)$. Since also clearly $B \in N(M_n(R))$ and $g(S) = 0$, then $M_n(R)$ is $g(x)$-precious. □

Let R be a ring and let M be an R-R-bimodule. The trivial extension of R by M is the ring $R(M) = R \oplus M$ with the usual addition and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This ring is isomorphic to the ring of all matrices of the form $\begin{bmatrix} r & m \\ 0 & r \end{bmatrix}$ where $r \in R$ and $m \in M$ with usual matrix addition and multiplication. It is clear that R naturally embeds into $R(M)$ via $r \to \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$. Thus any polynomial $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$ can be written as $g(x) = \sum_{i=0}^{n} \begin{bmatrix} a_i & 0 \\ 0 & a_i \end{bmatrix} x^i \in C(R(M))[x]$ and conversely.

Proposition 2.11. *Let R be a ring, M be an R-R-bimodule and $g(x) \in C(R)[x]$. Then R is (weakly) $g(x)$-precious if and only if $R(M)$ is (weakly) $g(x)$-precious.*

Proof. $\Leftarrow)$: If $R(M)$ is weakly $g(x)$-precious, then R is so since $\varphi : R(M) \to R$ defined by $\varphi(\begin{bmatrix} r & m \\ 0 & r \end{bmatrix}) = r$ is a ring epimorphism. Conversely, suppose R is weakly $g(x)$-precious and let $\begin{bmatrix} r & m \\ 0 & r \end{bmatrix} \in R(M)$. Write $r = \pm s + u + b$ where $g(s) = 0$, $u \in U(R)$ and $b \in N(R)$. Then $\begin{bmatrix} r & m \\ 0 & r \end{bmatrix} = \begin{bmatrix} \pm s + u + b & m \\ 0 & \pm s + u + b \end{bmatrix} =$ $\pm \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} + \begin{bmatrix} b & m \\ 0 & b \end{bmatrix}$. Clearly $\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$ is a zero of $g(x)$ and $\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \in U(R(M))$. Moreover, simple computations show that $\begin{bmatrix} b & m \\ 0 & b \end{bmatrix} \in N(R(M))$. Therefore, $R(M)$ is weakly $g(x)$-precious. The $g(x)$-preciousness equivalent is similar. □

3. Special Types of (Weakly) $g(x)$-Precious Rings

In this section we study some special types of (weakly) $g(x)$-precious rings. First, we consider an $(x-a)(x-b)$-precious (weakly $x(x-b)$-precious) ring R where $a, b \in C(R)$ and give a characterization of such a ring in terms of precious (weakly precious) rings. Other types of polynomials such as $x^n - x$, $x^n + x$ and $x^n - 1$ are also considered.

We start by the following main theorem.

Theorem 3.1. *Let R be a ring and $a, b \in C(R)$.*

(1) If $b-a \in U(R)$, then R is precious if and only if R is $(x-a)(x-b)$-precious.

(2) If $b \in U(R)$, then R is weakly precious if and only if R is weakly $x(x-b)$-precious.

Proof. (1) $\Rightarrow$) : Suppose R is precious and let $r \in R$. As $b-a \in U(R)$, then we may write $\frac{r-a}{b-a} = e + u + n$ where $e \in Id(R)$, $u \in U(R)$ and $n \in N(R)$. Thus, $r = e(b-a)+a+u(b-a)+n(b-a)$ where clearly $u(b-a) \in U(R)$ and $n(b-a) \in N(R)$. Moreover, $[e(b-a)+a-a)][e(b-a)+a-b] = e^2(b-a)^2 - e(b-a)^2 = 0$ and so $e(b-a)+a$ is a zero of $(x-a)(x-b)$. Therefore, R is $(x-a)(x-b)$-precious.

$\Leftarrow$) : Suppose R is $(x-a)(x-b)$-precious and let $r \in R$. Write $r(b-a)+a = s+u+n$ where $u \in U(R)$, $n \in N(R)$ and $(s-a)(s-b) = 0$. Then, $r = (\frac{s-a}{b-a}) + \frac{u}{b-a} + \frac{n}{b-a}$ where $\frac{u}{b-a} \in U(R)$ and $\frac{b}{b-a} \in N(R)$. Moreover,

$$\left(\frac{s-a}{b-a}\right)^2 = \frac{(s-a)(s-b+b-a))}{(b-a)^2} = \frac{(s-a)(s-b)+(s-a)(b-a))}{(b-a)^2} = \frac{s-a}{b-a}.$$

Therefore, R is precious.

(2) Suppose R is weakly precious with $b \in U(R)$. For $r \in R$, we write $\frac{r}{b} = \pm e + u + n$ where e is an idempotent, u is a unit and $n \in N(R)$. Then, $r = \pm eb + ub + nb$ where clearly $ub \in U(R)$, $nb \in N(R)$ and eb is a root of $x(x-b)$. Hence, R is weakly $x(x-b)$-precious. The other implication is almost similar and left to the reader. □

If we take $b = -1$ in the previous theorem, we conclude that R is (weakly) precious if and only if R is (weakly) $x(x+1)$-precious.

We note that the above equivalence of weakly precious rings and weakly $(x-a)(x-b)$-precious rings is a ring property. This means that this equivalence holds for a ring R but it may fail for a single element. For example, 3 is weakly $(x-1)(x-2)$-precious which is not weakly precious in $\mathbb{Z}$.

We recall that a unit in a ring R is called unipotent if it is of the form $1+b$ where $b \in N(R)$. For any ring R and $n \in \mathbb{N}$, the set of all elements of R that can be written as a sum of no more than n units is denoted by $U_n(R)$. If $R = U_n(R)$, then R is called an n-good ring. Such rings were studied extensively by many authors, see for example [21] and [22]. Next, we generalize this concept as follows.

Definition 3.2. *Let R be a ring. Then R is said to be involution 3-good if for every $r \in R$, we have $r = u + w + v$ where $u \in U(R)$, $w \in 1 + N(R)$ and $v^2 = 1$.*

Lemma 3.3. *Let R be a ring and $k \in \mathbb{N}$. Then R is involution 3-good if and only if for every $r \in R$, we have $r = u + w + v$ where $u \in U(R)$, $w \in 1 + N(R)$ and $(v - k)^2 = 1$.*

Proof. Suppose R is involution 3-good and let $k \in \mathbb{N}$. For $r \in R$, write $r - k = u + w + v$ where $u \in U(R)$, $w \in 1 + N(R)$ and $v^2 = 1$. Then, $r = u + w + (v + k)$ where $(v + k - k)^2 = v^2 = 1$. Conversely, let $r \in R$ and write $r + k = u + w + v$ where $u \in U(R)$, $w \in 1 + N(R)$ and $(v - k)^2 = 1$. Then $r = u + w + (v - k)$ where $(v - k)^2 = 1$. □

In the following Theorem, we characterize involution 3-good rings in term of precious and $g(x)$-precious rings.

Theorem 3.4. *Let R be a ring for which $2 \in U(R)$. The following are equivalent:*
(1) R is precious.
(2) R is $(x^2 - 2x)$-precious.
(3) R is $(x^2 + 2x)$-precious.
(4) R is $(x^2 - 1)$-precious.
(5) R is weakly $(x^2 - 1)$-precious.
(6) R is involution 3-good.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) : Theorem 3.1.

(2) $\Leftrightarrow$ (4) : Suppose R is (x^2-2x)-precious and let $a \in R$. Then $1+a = s+u+b$ where $u \in U(R)$, $b \in N(R)$ and $s^2 - 2s = 0$. Hence, $a = (s - 1) + u + b$ where $(s - 1)^2 - 1 = s^2 - 2s = 0$. Therefore, R is $(x^2 - 1)$-precious. Conversely, suppose R is $(x^2 - 1)$-precious and let $a \in R$. Then $1 - a = s + u + b$ where $u \in U(R)$, $b \in N(R)$ and $s^2 - 1 = 0$. So, $a = 1 - s + (-u) + (-b)$ where $(1 - s)^2 - 2(1 - s) = 0$ and R is $(x^2 - 2x)$-precious.

(4) $\Rightarrow$ (5) : Clear.

(5) $\Rightarrow$ (6) : Suppose R is weakly $(x^2 - 1)$-precious and let $a \in R$. Write $a - 2 = \pm s + u + b$ where $u \in U(R)$, $b \in N(R)$ and $s^2 - 1 = 0$. If $a - 2 = s + u + b$, then $a = (s + 1) + u + (b + 1)$ where $(s + 1 - 1)^2 = 1$. If $a - 2 = -s + u + b$, then $a = (1 - s) + u + (b + 1)$ where $(1 - s - 1)^2 = 1$. Therefore, R is involution 3-good by Lemma 3.3.

(6) $\Rightarrow$ (4) : Suppose R is involution 3-good and let $a \in R$. By Lemma 3.3, we may write $a + 2 = u + w + v$ where u is a unit, w is a unipotent and $(v - 1)^2 = 1$. Then $a = (v - 1) + u + (w - 1)$ is an $(x^2 - 1)$-precious decomposition of a. □

The equivalence of statements (1),(2) and (3) in Theorem 3.4 can be generalized if $n \in U(R)$ for any $n \in \mathbb{N}$. The proof is similar.

Theorem 3.5. *Let R be a ring and $n \in \mathbb{N}$ for which $n \in U(R)$. The following are equivalent:*
(1) R is (weakly) precious.
(2) R is (weakly) $(x^2 - nx)$-precious.
(3) R is (weakly) $(x^2 + nx)$-precious.

Example 3.6. *For any Continuous or discrete R-module M where $n \in U(R)$, the endomorphism $End_R(M)$ is clean, [10], (and so precious). Hence, $End_R(M)$ is $(x^2 - nx)$-precious.*

Example 3.7. *For any strongly zero dimensional topological space X, the ring of all continuous real valued functions $C(X)$ is clean with $n \in U(C(X))$, [6]. So, $C(X)$ and $M_k(C(X))$ are $(x^2 - nx)$-precious for any $k \in \mathbb{N}$ by Theorem 3.5 and proposition 2.10.*

Example 3.8. *let $n, k \in \mathbb{N}$, F be a field with character char $F = c \nmid n$ and V be an infinite dimensional vector space over F. Let R be a subring of $End_F(V)$ generated by the identity and the finite rank transformations. Then R is clean with $n \in U(R)$, [15], and so R and $M_k(R)$ are $(x^2 - nx)$-precious.*

In the remaining results of this section, we will concern about $(x^n - x)$-precious rings where $n \in \mathbb{N}$. An element p in a ring R is called potent if $p^n = p$ for some $n \in \mathbb{N}$. It is clear that any idempotent element in R is potent and so any precious ring is $(x^n - x)$-precious. However, the converse is not true in general. For example consider the group ring $\mathbb{Z}_{(7)}C_3$ where C_3 is the cyclic group of order 3 and $\mathbb{Z}_{(7)} = \left\{ \frac{m}{k} : m,k \in \mathbb{Z} \text{ and } \gcd(7, k) = 1 \right\}$. Then by Theorem (3.1) in [25], $\mathbb{Z}_{(7)}C_3$ is $(x^4 - x)$-clean (and so $(x^4 - x)$-precious). On the other hand $\mathbb{Z}_{(7)}C_3$ is not clean by Example (1) in [15]. Since also $\mathbb{Z}_{(7)}C_3$ is reduced, then $\mathbb{Z}_{(7)}C_3$ is not precious. A ring R is said to be semipotent if each left ideal of R that is not contained in the Jacobson radical contains a non zero idempotent. In [23], Wood proved also that $\mathbb{Z}_{(7)}C_3$ is not semipotent and so an $(x^4 - x)$-precious need not be semipotent.

Recall that for a ring R and $n \in \mathbb{N}$, R is called n-clean ring if for any $a \in R$, there is $e \in Id(R)$ and $u_1, u_2, ..., u_n \in U(R)$ such that $a = e + u_1 + u_2 + ... + u_n$.

Proposition 3.9. *Let R be a ring and let $n \in \mathbb{N}$. If R is $(x^n - x)$-precious, then R is 3-clean. If moreover, $2 \in U(R)$, then R is 4-good.*

Proof. Suppose R is $(x^n - x)$-precious and let $a \in R$. Then $a - 1 = s + u + b$ where $u \in U(R)$, $b \in N(R)$ and $s = s^n$ is a potent element of R. Since it is well known that every potent element is clean, then $s = e + v$ where $e \in Id(R)$ and $v \in U(R)$. Therefore, $a = e + v + u + (b + 1)$ is a 3-clean element in R and R is 3-clean. Now, suppose $2 \in U(R)$ and write $\frac{a+1}{2} - 1 = s + u + b$ where $u \in U(R)$, $b \in N(R)$ and $s = s^n$, then similarly we can write $a = (2e - 1) + 2v + 2u + 2(b + 1)$ where $u, v \in U(R)$, $b \in N(R)$ and $e \in Id(R)$. It is clear that the four terms in this expansion of a are units and so R is 4-good. □

Proposition 3.10. *Let R be a ring, $n \in \mathbb{N}$ and $g(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 \in C(R)[x]$ where $a_0 \in U(R)$. If R is (weakly) $g(x)$-precious, then R is a 3-good ring. In particular, if R is (weakly) $(x^{n-2} + x^{n-3} + ... + x + 1)$-precious, then R is (weakly) $(x^n - x)$-precious.*

Proof. Let $a \in R$ and write $a - 1 = \pm s + u + b$ where $u \in U(R)$, $b \in N(R)$ and $s^n + a_{n-1}s^{n-1} + ... + a_1s + a_0 = 0$. Then $a = \pm s + u + (b + 1)$ where

$b+1 \in U(R)$. Since $s(s^{n-1}+a_{n-1}s^{n-2}+...+a_1) = -a_0 \in U(R)$, then $s \in U(R)$. Therefore, $a \in U_3(R)$ and R is 3-good. Now, suppose in particular that R is (weakly) $(x^{n-2}+x^{n-3}+...+x+1)$-precious. For any $a \in R$, write $a = \pm s+u+b$ where $u \in U(R)$, $b \in N(R)$and $s^{n-2}+s^{n-3}+...+s+1 = 0$. Then $s^n - s = s(s-1)(s^{n-2}+s^{n-3}+...+s+1) = 0$ and so R is (weakly) $(x^n - x)$-precious. □

In particular, if R is a (weakly) (x^2+x+1)-precious ring, then R is a 3-good and (weakly) $(x^4 - x)$-precious.

In the following proposition, we can see that (weakly) $(x^n - x)$-precious rings are the same as (weakly) $(x^n + x)$-precious rings for any even positive integer n. Thus, we conclude that the ring $\mathbb{Z}_{(7)}C_3$ is also $(x^4 + x)$-precious.

Proposition 3.11. *Let R be a ring, $k \in \mathbb{N}$ and $a, b \in R$. Then R is (weakly) $(ax^{2k} + bx)$-precious if and only if R is (weakly) $(ax^{2k} - bx)$-precious.*

Proof. $\Rightarrow)$: Suppose R is $(ax^{2k} + bx)$-precious and let $r \in R$. Then $-r = s+u+n$ where $u \in U(R)$, $n \in N(R)$ and $as^{2k} + bs = 0$. Hence, $r = (-s)+(-u)+(-n)$ where $-u \in U(R)$, $-n \in N(R)$ and $a(-s)^{2k} - b(-s) = as^{2k} + bs = 0$. Therefore, R is $(ax^{2k} - bx)$-precious.

$\Leftarrow)$: Conversely, Suppose R is $(ax^{2k} - bx)$-precious and let $r \in R$. Write $-r = s+u+n$ where $u \in U(R)$, $n \in N(R)$ and $as^{2k} - bs = 0$. Then $r = (-s)+(-u)+(-n)$ where $-u \in U(R)$, $-n \in N(R)$ and $a(-s)^{2k} + b(-s) = as^{2k} - bs = 0$. Therefore, R is $(ax^{2n} + bx)$-precious. The weakly case is almost the similar. □

The equivalence in proposition 3.11 need not be true for odd powers. For example it is easy to check that the field $\mathbb{Z}_3$ is an $(x^3 - x)$-precious which is not $(x^3 + x)$-precious.

Theorem 3.12. *Let R be a (weakly) $(x^n - x)$-precious ring, $a \in R$ and $n \geq 2$. Then either a is a (weakly) $(x^{n-1} - 1)$-precious element or there is $b \in N(R)$ such that $(a-b)R$ and $R(a-b)$ contain non zero idempotents.*

Proof. Write $a = \pm s+u+b$ where $u \in U(R)$, $b \in N(R)$ and $s^n = s$. Now, $a-b = \pm s+u$ and then $(a-b)s^{n-1} = \pm s+us^{n-1}$. So $(a-b)(1-s^{n-1}) = u(1-s^{n-1})$ and then $u(1-s^{n-1})u^{-1} = (a-b)(1-s^{n-1})u^{-1} \in (a-b)R$. Since clearly $1-s^{n-1}$ is an idempotent in R, $u(1-s^{n-1})u^{-1}$ is also idempotent. If a is not (weakly) $(x^{n-1} - 1)$-precious, then $1 - s^{n-1} \neq 0$ and so $u(1-s^{n-1})u^{-1}$ is a non trivial idempotent in $(a-b)R$. Similarly, $R(a-b)$ contains a non trivial idempotent. □

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