



Numerical solution of Fractional differential equation by Wavelets and Hybrid functions

A.H. Refahi Sheikhan and M. Mashoof

ABSTRACT: In this paper, we introduce methods based on operational matrix of fractional order integration for solving a typical n-term non-homogeneous fractional differential equation (FDE). We use Block pulse wavelets matrix of fractional order integration where a fractional derivative is defined in the Caputo sense. Also we consider Hybrid of Block-pulse functions and shifted Legendre polynomials to approximate functions. By the use of these methods we translate an FDE to an algebraic linear equations which can be solved. Methods have been tested by some numerical examples.

Key Words: fractional order differential equation, wavelet, Block pulse, Hybrid function, operational matrices.

Contents

1 Introduction	231
2 Preliminaries	232
3 Operational Matrix of Fractional Order Integration	234
4 Estimation of the error	235
5 Numerical Examples	236
6 Conclusion	242

1. Introduction

In recent decades, the fractional calculus and fractional differential equations have been attracted much attention and increasing interest. Fractional differential equations are generalized from integer order ones, which are achieved by replacing integer order derivatives by fractional ones. In recent years, studies on application of the FDE in science has attracted increasing attention [1,2,4,7,17]. For instance, Bagley and Torvik formulated the motion of a rigid plate immersing in a Newtonian fluid [9]. It shows that the use of fractional derivatives for the mathematical modeling of viscoelastic materials is quite natural [9]. It should be mentioned that the main reasons for the theoretical development are mainly the wide use of polymers in various fields of engineering [9]. Also in 1991, S. Westerlund suggested

using fractional derivatives for the description of propagation of plane electromagnetic waves in an isotropic and homogeneous, lossy dielectric and in the paper on electrochemically polarizable media, published in 1993, [9]. Caputo suggested the fractional-order version of the relationship between electric field and electric flux density [9]. An FDE in time domain can be described as the following for

$$a_n({}_a D_t^{\alpha_n} y(t)) + \cdots + a_1({}_a D_t^{\alpha_1} y(t)) + a_0({}_a D_t^{\alpha_0} y(t)) = u(t), \quad (1.1)$$

subject to the initial conditions

$$y^{(i)}(a) = d_i, i = 0, \dots, n, \quad (1.2)$$

where $a_i \in R, 0 < \alpha_1 < \alpha_2 < \dots < \alpha_n$, and ${}_a D_t^{\alpha_i} y(t)$ denotes the caputo fractional derivative of order α_i .

We can see the conditions of existence and uniqueness of solutions to the FDE in [9]. Moreover several numerical methods have been used to approximate the solution of fractional differential equations, such as finite difference method [11], collocation [12] method and other methods [13,5,18].

Any *time function* can be synthesized completely to a tolerable degree of accuracy by using set of orthogonal functions. For such accurate representation of a time function, the orthogonal set should be “complete” [10]. In this paper we will apply Block-pulse and Hybrid functions based on Block-pulse wavelet and Shifted Legendre polynomials to approximate the solution of (1.1) under conditions (1.2). We begin by introducing some necessary definitions and theorems of the fractional calculus theory and wavelets. In Section 3 the operational matrices of fractional order integration for Block-pulse wavelets and Hybrid functions are obtained. Also in Section 4 the proposed methods are applied to several examples. The conclusion is given at the end.

2. Preliminaries

In this Section, we present some basic definitions and properties of fractional calculus and wavelets [9,10,6,3].

Definition 2.1. The shifted Legendre polynomials are defined on the interval $[0, 1]$ and can be determined with the aid of the following recurrence formulae [13]:

$$P_{i+1}(x) = \frac{(2i+1)(2x-1)}{i+1} P_i(x) - \frac{i}{i+1} P_{i-1}(x), \quad i = 1, 2, \dots,$$

where $P_0(x) = 1$ and $P_1(x) = 2x - 1$.

Definition 2.2. The m-set of block-pulse functions on $[0, \eta)$ is defined as:

$$b_i(t) = \begin{cases} 1 & ; \quad \frac{\eta i}{m} \leq t < \frac{\eta(i+1)}{m}, \\ 0 & ; \quad \text{otherwise,} \end{cases}$$

where $i = 0, 1, 2, \dots, m-1$.

The functions b_i are disjoint and orthogonal [10].

Theorem 2.1. A function $f(x) \in L^2([0, T])$ may be expanded by the Block-pulse functions as:

$$f(x) \simeq \sum_{i=1}^{m_1} f_i b_i(t) = F^T B_m(x), \quad (2.1)$$

where

$$F = (f_1 \ \cdots \ f_m) \text{ and } B_m(x) = (b_1(x) \ \cdots \ b_m(x)).$$

The Block-pulse coefficients f_i are obtained as

$$f_i = \frac{T}{h} \int_{(i-1)h}^{ih} f(x) dx. \quad (2.2)$$

Proof: In [10]. □

Now we define the Hybrid functions of Block-Pulse and shifted Legendre polynomials.

Definition 2.3. Hybrid function $hy_{i,j}(x)$, $i = 0, \dots, m-1$ and $j = 0, \dots, n-1$ are defined on the interval $[0, T]$ as

$$hy_{i,j}(x) = \begin{cases} P_j(\frac{m}{T}x - i) & ; \quad \frac{iT}{m} \leq x < \frac{(i+1)T}{m}, \\ 0 & ; \quad \text{otherwise,} \end{cases} \quad (2.3)$$

where $P_j(t)$ is the j^{th} shifted Legendre polynomials on $[0, 1]$.

Now for approximate the function $f(x)$ we can set [15,16]

$$f(x) \simeq C^T Hy_{n,m}(x) \quad (2.4)$$

where

$$C^T = (c_{0,0} \ \cdots \ c_{0,n-1} \ c_{(m-1),(n-1)})$$

and

$$Hy_{n,m}(x) = (hy_{0,0}(x) \ \cdots \ hy_{0,n-1}(x) \ \cdots \ hy_{(m-1),(n-1)}(x))$$

and $c_{i,j} = \frac{\langle f(x), hy_{i,j} \rangle}{\langle hy_{i,j}, hy_{i,j} \rangle}$ where $\langle u(x), v(x) \rangle = \int_0^T u(x)v(x)dx$.

Definition 2.4. A real function $f(x)$, $x \geq 0$ is said to be in space C_μ , $\mu \in R$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in [0, \infty)$, and it is said to be in the space C_μ^m iff $f^m \in C_\mu$, $m \in \mathbb{N}$.

Definition 2.5. The Riemann-Liouville fractional derivative of order α with respect to the variable x and with the starting point at $x = a$ is

$${}_a D_t^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(1+m-\alpha)} \frac{d^{m+1}}{dx^{m+1}} \int_a^x (x-\tau)^{m-\alpha} f(\tau) d\tau & ; \quad 0 \leq m \leq \alpha < m+1, \\ \frac{d^{m+1}}{dx^{m+1}} & ; \quad \alpha = m+1 \in \mathbb{N}. \end{cases}$$

Definition 2.6. The Riemann-Liouville fractional integral of order α is

$$I^\alpha(f(x)) = {}_a D_t^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0.$$

Definition 2.7. The fractional derivative of $f(x)$ by means of Caputo sense is defined as

$${}_a D_x^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \int_a^x (x - \tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau,$$

where $n - 1 < \alpha \leq n, n \in \mathbb{N}, x > 0, f \in C_{-1}^n$.

For the Caputo's derivative we have $D_t^\alpha C = 0$, C is a constant and

$$D_x^\alpha x^n = \begin{cases} 0 & ; \quad n \in \mathbb{N}, n < \lceil \alpha \rceil \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^n & ; \quad n \in \mathbb{N}, n < \lfloor \alpha \rfloor \end{cases} \quad (2.5)$$

The relation between the Riemann-Liouville operator and Caputo operator has given by the following expressions [14]:

$${}_a D_x^\alpha I^\alpha f(x) = f(x), \quad (2.6)$$

$$I^\alpha {}_a D_x^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(a^+) \frac{(x-a)^k}{k!}, \quad x > 0. \quad (2.7)$$

3. Operational Matrix of Fractional Order Integration

In this Section we will introduce operational matrix methods based on Block-pulse and Hybrid functions of Block-pulse and shifted Legendre polynomials to numerical solution of fractional order differential equations.

Fractional integration of the block-pulse function vector has given as

$$(I^\alpha B_m)(t) = F^{(\alpha)} B_m(t) \quad (3.1)$$

where $F^{(\alpha)}$ is the Block-pulse operational matrix of the fractional order integration and [14]

$$F^{(\alpha)} = \left(\frac{T}{m}\right)^\alpha \frac{1}{\Gamma(\alpha+2)} \begin{pmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \quad (3.2)$$

where $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$.

Now let $Hy_{n,m} \simeq \Phi B_{mn}(x)$ and $I^\alpha Hy_{n,m}(x) = Q^{(\alpha)} Hy_{n,m}(x)$ then we can construct operational matrix of fractional order integration for Hybrid functions as:

$$Q^{(\alpha)} = \Phi F^{(\alpha)} \Phi^{-1}. \quad (3.3)$$

4. Estimation of the error

In this section, we analyze the errors of a function is expanded in terms of Bluck-pulse or Hybrid functions of Block-Pulse and shifted Legendre polynomials.

Theorem 4.1. *Let $f(t)$ be an arbitrary real bounded function, which is square integrable in the interval $[0, 1)$, and $e(t) = f(t) - F^T B_m(x)$. Then*

$$\|e(t)\| \leq ch. \quad (4.1)$$

Proof: In [19]. □

Let $L^2[0, T]$ be the space of square integrable functions on $[0, T]$ and $X = \text{Span}\{hy_{i,j}(x) : i = 0, \dots, m-1 \text{ and } j = 0, \dots, n-1\}$. It is clear that $hy_{i,j}(x)$, is at most a function of degree $n-1$. Now let $f \in L^2[0, T]$. Since X is a finite dimensional vector space, f has the unique best approximation out of X such as $p \in X$, that is

$$\exists p \in X \forall q \in X : \|f - p\|_2 \leq \|f - q\|_2,$$

where $\|f\|_2 = \sqrt{\langle f, f \rangle}$. Therefore there exist the unique coefficients such that

$$f(x) \simeq p = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} c_{i,j} hy_{i,j}(x),$$

where $c_{i,j}$ are defined in Definition 2.3, for more details refer to [20].

Theorem 4.2. *Let $f \in L^2[0, T]$ is n times continuously differentiable and $f^{(n)}(x) < M$ on $[0, T]$. If $\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} c_{i,j} hy_{i,j}(x) = C^T Hy_{n,m}(x)$ is the best approximation of f out of X , then we have*

$$\|f - C^T Hy_{n,m}(x)\|_2 \leq \frac{M\sqrt{T^3}}{\sqrt{3}n!m}.$$

Proof: Let f_i be the Taylor polynomial of order $n-1$ for f on $[\frac{iT}{m}, \frac{(i+1)T}{m})$, therefore

$$f_i(x) = \sum_{k=0}^{n-1} f\left(\frac{iT}{m}\right) \frac{(x - \frac{iT}{m})^k}{k!},$$

also for each i , there exist $\zeta_i \in (\frac{iT}{m}, \frac{(i+1)T}{m})$ such that

$$|f(x) - f_i(x)| \leq |f^{(n)}(\zeta_i)| \frac{(x - \frac{iT}{m})^n}{n!}. \quad (4.2)$$

Since $C^T Hy_{n,m}(x)$, is the best approximation f out of X , $f_i \in X$, from above equation we have

$$\|f - C^T Hy_{n,m}\|_2^2 = \int_0^T |f(x) - C^T Hy_{n,m}(x)|^2 dx$$

$$= \sum_{i=0}^{m-1} \int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} |f(x) - C^T H y_{n,m}(x)|^2 dx$$

$$\leq \sum_{i=0}^{m-1} \int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} |f(x) - f_i(x)|^2 dx$$

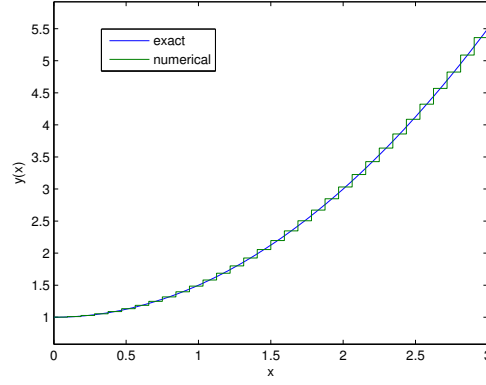
$$\leq \sum_{i=0}^{m-1} \int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} (|f^{(n)}(\zeta_i)| \frac{(x - \frac{iT}{m})^n}{n!})^2 dx \leq \frac{M^2 T^3}{3(n!)^2 m^2}.$$

□

Theorems 4.2 shows that the error of Hybrid functions of Block-Pulse and shifted Legendre polynomials reduces to zero very fast as n and m increase.

5. Numerical Examples

In order to show the efficiency of operational matrix of fractional order integration by Block-pulse and Hybrid functions of Block-pulse and shifted Legendre polynomials for solving initial value problems as (1.1), we apply it to solve different types of FDE which exact solutions are known.



a. Numerical solution

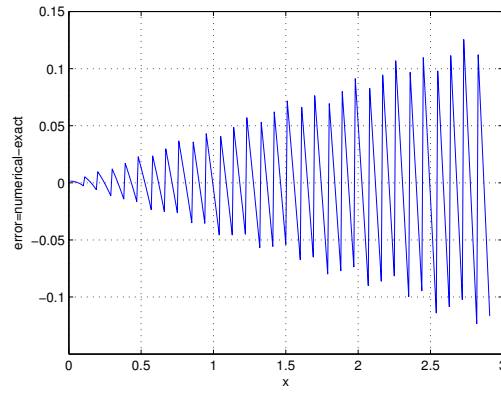
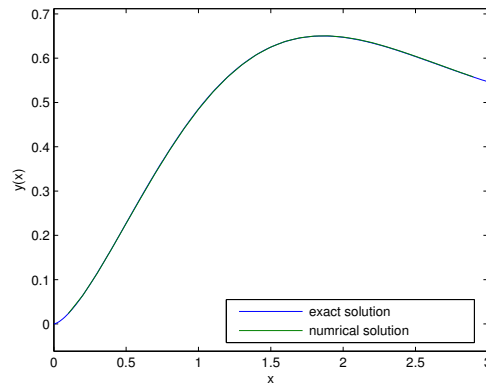
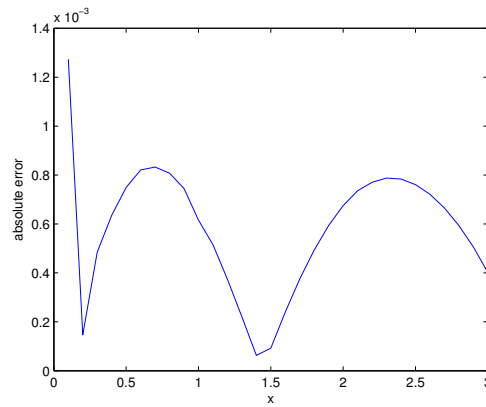
b. Generated error of $c_y B_m(t)$ for $y(t)$

Figure 1: Fig 1.a. shows the comparison of the exact solution with numerical solution generated by $B_{32}(x)$ and Fig 1.b. present the error generated by $B_{32}(x)$, for example 4.1



*a.*By $Hy_{8,3}(x)$



*b.*Absolute error

Figure 2: Fig 2.a. shows exact and numerical solutions for $a = 2$, $\alpha = 1.5$, by $Hy_{8,3}(x)$ and fig 2.b. represent the absolute error in example 4.2, case 1.

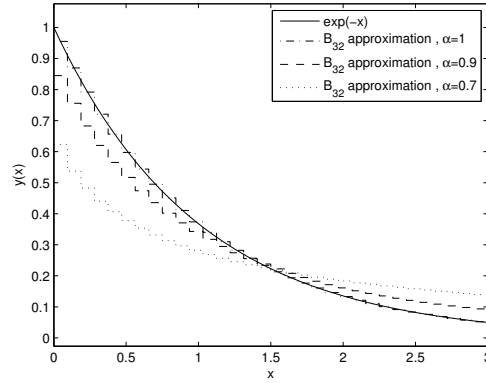
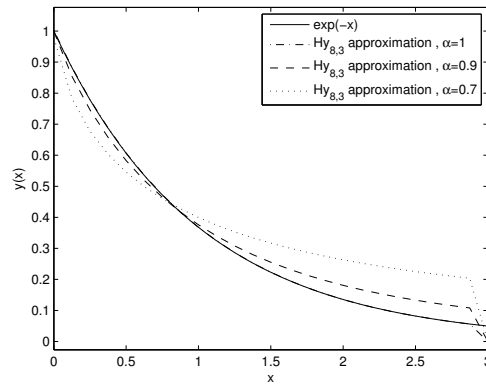

 By $B_{32}(x)$

 By $Hy_{8,3}(x)$

 Figure 3: Numerical solution of example 4.2, case 2, by $B_{32}(x)$ and $Hy_{8,3}(x)$ with $\alpha = 1, 0.9, 0.7$.

Example 5.1. Consider the equation

$${}_0D_x^2 y(x) + 3{}_0D_x y(x) + 2{}_0D_x^{q_2} y(x) + {}_0D_x^{q_1} y(x) + 5y(x) = f(x) ; y(0) = 1, y'(0) = 0, \quad (5.1)$$

with

$$f(x) = 1 + 3x + \frac{2}{\Gamma(3 - q_2)} x^{2-q_2} + \frac{1}{\Gamma(3 - q_1)} x^{2-q_1} + 5\left(1 + \frac{1}{2}t^2\right)$$

when $q_1 = 0.0159$ and $q_2 = 0.1379$, The exact solution of this problem is $1 + \frac{1}{2}t^2$. By integrating from both sides of order q_2 , we have

$$\begin{aligned} y(x) + 3I(y(x)) + 2I^{2-q_2}(y(x)) + I^{2-q_1}(y(x)) + 5I^2(y(x)) \\ = I^2(f(x)) + 1 + 3I(1) + 2I^{2-q_2}(1) + I^{2-q_1}(1). \end{aligned} \quad (5.2)$$

In this example we use the operational matrix of fractional order integration with respect to Block-pulse wavelet and Hybrid functions of Block-Pulse and shifted Legendre polynomials. From Section 3, we can approximate solution $y(x)$, $f(x)$ and 1 as follows

$$\begin{aligned} y(x) &= C_y^T W(x) \\ f(x) &= C_f^T W(x) \\ 1 &= C_1^T W(x). \end{aligned} \quad (5.3)$$

If we use Block-pulse wavelet then $W(x) = B_m(x)$, if we apply Hybrid functions of Block-Pulse and shifted Legendre polynomials then we have $W(x) = Hy_{n,m}(x)$. By substituting above equations in (16) we have

$$C_y(I + 3G + 2G^{(2-q_2)} + G^{(2-q_1)} + 5G^{(2)}) = C_f G^{(2)} + C_1(I + 3G + 2G^{(2-q_2)} + G^{(2-q_1)}) \quad (5.4)$$

where $G^{(\alpha)} = F^{(\alpha)}$ or $Q^{(\alpha)}$, if we use $B_m(x)$ or $Hy_{n,m}(x)$, respectively. By solving above algebraic equations by existing methods [8,21], we can find C_y . Results are shown in fig .1 for $W(x) = B_{32}(x)$. From Fig. 1.a we can see that the numerical solution (block-pulse simulation) is coincide with the exact solution in much points. Also from Fig. 1.b we found that the error ranges between -0.15 and 0.15 .

Example 5.2. (Bagley-Torvik equation) Consider the following initial value problems Bagley-Torvik equation

$${}_0D_x^2 y(x) + {}_0D_x^{\frac{3}{2}} y(x) + y(x) = 1 + x; \quad y(0) = 1, \quad y'(0) = 1. \quad (5.5)$$

The exact solution is $y(x) = 1 + x$, [13]. By taking I^2 from both side of (19) we have

$$I^2({}_0D_x^2 y(x) + {}_0D_x^{\frac{3}{2}} y(x) + y(x)) = I^2(1 + x), \quad (5.6)$$

now from (8), (9) and initial conditions we can see that above equation becomes to

$$y(x) - x - 1 + I^{\frac{1}{2}}(y(x)) + I^{\frac{1}{2}}(-x - 1) + I^2(y(x)) = I^2(1 + x). \quad (5.7)$$

which is the integral representation of (5.5).

In this example we use the operational matrix of fractional order integration with respect to Block-pulse wavelet. By applying theorem 2.1, we can approximate solution $y(x)$ and $1 + x$ as follows

$$\begin{aligned} y(x) &= C_b^T B_m(x) \\ 1 + x &= C_1^T B_m(x) \end{aligned} \quad (5.8)$$

by substituting (22) in (21) and using operational matrices we have

$$C_b^T(I_m + F^{(\frac{1}{2})} + F^{(2)}) = C_1^T(I_m + F^{(\frac{1}{2})} + F^{(2)}). \quad (5.9)$$

From (3.2) we can see that the entries of principal diagonal of upper triangular matrix $I_m + F^{(\frac{1}{2})} + F^{(2)}$ are positive and thus the matrix $I_m + F^{(\frac{1}{2})} + F^{(2)}$ is

nonsingular. This shows that the algebraic equations (23) have a unique solution as $C_b = C_1$. But

$$y(x) = C_b^T B_m(x) = C_1^T B_m(x) \simeq 1 + x,$$

and also theorem 4.1 shows that $C_1^T B_m(x) \rightarrow 1 + x$ as $m \rightarrow \infty$. Therefore the numerical solution can be regarded as $1 + x$, which is the exact solution.

Table 1: Absolute error for $\alpha = 1.5$, $a = 12$ for example 4.2, case 1.

x	$ C_b B_{32}(x) - y(x) $	$ C_{hy} Hy_{8,3}(x) - y(x) $
0.2	9.1×10^{-3}	1.4×10^{-3}
0.5	8.1×10^{-4}	7.6×10^{-4}
0.8	8.6×10^{-4}	1.2×10^{-3}
1.1	6.0×10^{-4}	1.4×10^{-4}
1.4	5.2×10^{-4}	5.0×10^{-4}
1.7	1.5×10^{-4}	6.6×10^{-5}
2	5.3×10^{-5}	1.3×10^{-4}
2.3	3.2×10^{-5}	4.8×10^{-5}
2.6	3.6×10^{-6}	2.9×10^{-5}
2.9	1.4×10^{-5}	1.9×10^{-5}

Example 5.3. (Relaxation-oscillation Equation)

In this example we consider an FDE appearing in applied problems [9].

$$\begin{aligned} {}_0D_x^\alpha y(x) + ay(x) &= f(x) \quad , \quad t > 0, \\ y^k(0) &= a_k \quad , \quad k = 0, 1, \dots, n-1, \end{aligned} \quad (5.10)$$

where $n-1 < \alpha \leq n$. For $0 < \alpha \leq 2$ and $a_k = 0$ this equation is called the Relaxation-oscillation equation.

Case 1. Consider $a_k = 0$ and $f(x) \equiv H(x)$, where $H(x)$ is the Heaviside function. In this case The analytical solution of [9] is

$$y(x) = \int_0^x G(x-\tau)f(\tau), \quad G(x) = x^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha). \quad (5.11)$$

The integral representation of (5.10) for $1 < \alpha < 2$ is

$$y(x) + aI^\alpha(y(x)) = I^\alpha(f(x)), \quad (5.12)$$

we solve the problem, by applying the Hybrid functions of Block-Pulse and shifted Legendre polynomials described in the previous Sections on $[0, 3]$ for $\alpha = 1.5$. The algebraic equations corresponding to (5.12) are of the form $C(I + aG^{(\alpha)}) = C_f G^{(\alpha)}$ where $G^{(\alpha)}$ is defined in example 5.1. Fig. 2 shows the numerical results and absolute error generated by Hybrid functions ($Hy_{8,3}(x)$) for $a = 2$ with $\alpha = 1.5$. Also the absolute error for $a = 12$ are shown in Table 1 generated by Block pulse and

Hybrid functions. From Table 1, we can see that the operational matrix methods achieve a good approximation with the exact solution.

Case 2. In this case, we consider $f(x) = 0$, $0 \leq \alpha \leq 1$ and $a_0 = 1$, $a_1 = 0$, the analytical solution is [13]

$$y(x) = \sum_{k=0}^{\infty} \frac{(-x^\alpha)^k}{\Gamma(\alpha k + 1)}, \quad (5.13)$$

for $\alpha = 1$ we have from (5.13), $y(x) = \exp(-x)$. The exact solution $\exp(-x)$, with $\alpha = 1$ and numerical solution by Block-pulse wavelets and Hybrid function for $\alpha = 1, 0.9$ and 0.7 are shown in Fig. 3. From Fig 3 we can see that the numerical solution converges to $\exp(-x)$ as $\alpha \rightarrow 1$. Also The absolute error for $\alpha = 1$ are shown in Table 2. Table 2 shows that the Hybrid operational matrix method gives an efficient numerical solution for $\alpha = 1$.

Table 2: Absolute error for $\alpha = 1$, by $B_{32}(x)$ and $Hy_{8,3}(x)$, for example 4.3, case 2.

x	$ C_b B_{32}(x) - e^{-x} $	$ C_{hy} Hy_{8,3}(x) - e^{-x} $
0.2	2.6×10^{-2}	8.5×10^{-4}
0.5	9.0×10^{-3}	3.9×10^{-4}
0.8	1.6×10^{-3}	1.1×10^{-4}
1.1	7.5×10^{-3}	4.3×10^{-5}
1.4	1.0×10^{-2}	1.2×10^{-4}
1.7	6.2×10^{-3}	1.6×10^{-4}
2	2.1×10^{-3}	1.7×10^{-4}
2.3	2.5×10^{-4}	1.6×10^{-4}
2.6	1.6×10^{-3}	1.5×10^{-4}
2.9	2.2×10^{-3}	1.3×10^{-4}

6. Conclusion

The fractional differential equations play an important role in physics, chemical mixing, chaos theory, and biological system as well. The fundamental goal of this work has been to apply an efficient method for the solution of FDE with initial values. In this paper we presented the operational matrix of fractional order integration method to solve FDE with initial values. This method transforms FDE into algebraic equations. Examples show that the method has been successfully applied to find the approximate solutions of the FDE with initial values. Figures and Tables show that this method is extremely effective and practical for this sort of approximate solutions.

References

1. Aminikhah, H., Refahi Sheikhan, A. and Rezazadeh, H., *Stability analysis of distributed order fractional chen system*, Scientific World Journal. 2013, 1-13, (2013).

2. Aminikhah, H., Refahi Sheikhan, A. and Rezazadeh, H., *Stability analysis of linear distributed order system with multiple time delays*, U.P.B. Sci. Bull. Series A. 77, 207-218, (2015).
3. Ansari, A., Refahi Sheikhan, A. and Kordrostami, S., *On the generating function $e^{xt+y\phi(t)}$ and its fractional calculus*, Cent. Eur. J. Phys. 11, 1457-1462, (2013).
4. Aminikhah, H., Refahi Sheikhan, A. H. and Rezazadeh, H., *Exact solutions of some nonlinear systems of partial differential equations by using the functional variable method*, MATHEMATICA, 56, 103-116, (2014).
5. Ansari, A., Refahi Sheikhan, A., *New identities for the wright and the mittag-leffler functions using the laplace transform*, Asian-European Journal of Mathematics., 7, 1-8, (2014).
6. Aminikhah, H., Refahi Sheikhan, A. H. and Rezazadeh, H., *Approximate analytical solutions of distributed order fractional Riccati differential equation*, Ain Shams Engineering Journal, Article in press.
7. Aminikhah, H., Refahi Sheikhan, A. H. and Rezazadeh, H., *Sub-equation method for the fractional regularized long-wave equations with conformable fractional derivatives*, Scientia Iranica, 23, 1048-1054, (2016).
8. Saberi Najafi, H., Edalatpanah, S., A., Refahi sheikhan, A., H., *Convergence Analysis of Modified Iterative Methods to Solve Linear Systems*, Mediterranean Journal of Mathematics, 11, 1019-1032, (2014).
9. Podlubny, I., *Fractional differential equations. Mathematics in Science and Engineering*, Academic Press, New York, NY, USA., (1999).
10. Deb, A., Ghosh, S., *Power electronic Systems: Walsh analysis With Matlabr*, CRC Press, Taylor & Francis Group, LLC., (2014).
11. Meerschaert, M. M., Tadjeran, C., *Finite difference approximations for two-sided space-fractional partial differential equations*, Applied Numerical Mathematics, 56, 80-90, (2006).
12. Öztürk, Y., AnapalJ, A., Gülsu, M. and Sezer, M., *A Collocation Method for Solving Fractional Riccati Differential Equation*, Journal of Applied Mathematics, 2013, 1-8, (2013).
13. Saadatmandi, A., Dehghan, M., *A new operational matrix for solving fractiona-order differential equations*, Comput. Math. Appl., 59, 1326-1336, (2010).
14. Yi, M., Huang, J. and Wei, J., *Block pulse operational matrix method for solving fractional partial differential equation*, Applied Mathematics and Computation., 221, 121-131, (2013).
15. Yang, C., *Numerical Solution of Nonlinear Fredholm Integro differential Equations of Fractional Order by Using Hybrid of Block-Pulse Functions and Chebyshev Polynomials*, Mathematical Problems in Engineering, 2011, 1-11, (2011).
16. Patra, A. and Rao, G. P. *General Hybrid Orthogonal Functions and Their Applications in Systems and Control*, Springer, LNCIS 213, London, (1996).
17. Rezazadeh, H., Aminikhah, H., Refahi Sheikhan, A., *Stability analysis of Hilfer fractional differential systems*, Math. Commun., 21, 45-64, (2016).
18. Aminikhah, H., Refahi Sheikhan, A. and Rezazadeh, H., *Travelling wave solutions of nonlinear systems of PDEs by us ing the functional variable method*, Bol.Soc.Paran.Mat., 34, 213-229, (2016).
19. Maleknejad, K., Khodabin, M., Rostami, M., *A numerical method for solving m-dimensional stochastic ItˆoVolterra integral equations by stochastic operational matrix*, Comput. Math. Appl, 63, 133-143, (2012).
20. Canuto, C., Quarteroni, A., Hussaini, M. Y., A. Zang, T. A., *Spectral Methods, Fundamentals in Single Domains*, Springer-Verlag, Berlin, Heidelberg, (2006).
21. Saberi Najafi, H., Refahi, A., *A new restarting method in the Lanczos algorithm for generalized eigenvalue problem*, Applied Mathematics and Computation, 184, 421-428, (2007).

A.H. Refahi Sheikhani
Department of Applied Mathematics
Faculty of Mathematical Sciences
Lahijan Branch
Islamic Azad University
Lahijan, Iran
E-mail address: ah_refahi@liau.ac.ir

and

M. Mashoof
Department of Applied Mathematics
Faculty of Mathematical Sciences
Lahijan Branch
Islamic Azad University
Lahijan, Iran
E-mail address: mashoof.mohammad@yahoo.com