Bol. Soc. Paran. Mat. (3s.) **v. 36** 1 (2018): 125–150.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v36i1.29440

Existence of entropy solutions for nonlinear elliptic equations in Musielak framework with L^1 data

M.S.B. Elemine Vall, A. Ahmed, A. Touzani and A. Benkirane

Abstract: We prove existence of solutions for strongly nonlinear elliptic equations of the form

$$\begin{cases} A(u) + g(x,u,\nabla u) = f + \operatorname{div}(\phi(u)) & \text{in } \Omega, \\ u \equiv 0 & \partial\Omega. \end{cases}$$

Where $A(u) = -\operatorname{div}(a(x,u,\nabla u))$ be a Leray-Lions operator defined in $D(A) \subset W_0^1 L_\varphi(\Omega) \to W^{-1}L_\psi(\Omega)$, the right hand side belongs in $L^1(\Omega)$, and $\phi \in C^0(\mathbb{R},\mathbb{R}^N)$, without assuming the Δ_2-condition on the Musielak function.

Key Words: Musielak Orlicz spaces, elliptic problem, Musielak Orlicz function.

Contents

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, we consider the following nonlinear boundary problem

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) + g(x,u,\nabla u) = f - \operatorname{div}(\phi(u)) & \text{in } \Omega, \\ u \equiv 0, \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $A(u) = -div(a(x,u,\nabla u))$ is an operator of Leray-Lions type, g is a nonlinearity with the sign condition but any restriction on its growth, $f \in L^1(\Omega)$ and $\phi \in C^0(\mathbb{R},\mathbb{R}^N)$. The notion of entropy solution, used in [14], allows us to give a meaning to a possible solution of (1.1).

In the classical Sobolv spaces, Boccardo in [14] has proved the existence and regularity of an entropy solution u of problem (1.1) for $2 - \frac{1}{N} < p < N$, in the particular case where $g \equiv 0$, see also [13,17] for related topics.

Submitted October 11, 2015. Published January 22, 2016

In the sitting of Orlicz spaces, A.Benkirane and J.Bennouna in [6] have studied the existence of entropy solution of (1.1) where $g \equiv 0$, Aharouch and Azroul [1] studied the problem (1.1), where $g \equiv 0$, for more results see [11,12].

In the Sobolev variable exponent, E.Azroul, H.Hjiaj, and A.Touzani [4] have proved the existence and some regularity result for the problem (1.1), Bendahmane and Wittbold in [5] proved the existence and uniqueness of renormalized solution to the problem (1.1) in the particular case $a(x,s,\xi) = |\xi|^{p(x)-2}\xi$, $g \equiv 0$, $\phi = 0$.

In Musielak Orlicz framework, M. Ait Khellou, A. Benkirane, S.M. Douiri (see [3]) have proved the existence of entropic solution of (1.1) in the variational case where $\phi = 0$, M. L. Ahmed Oubeid, A. Benkirane, M. Sidi El Vally in [2] proved the existence of entropy solution of (1.1) where $g \equiv 0$, $\phi = 0$ and the right hand side is a measure data, recently A.Benkirane, F.Blali and M.Sidi El Vally in [7] have solved (1.1) in the case where the Musielak-orlicz function complementary to φ satisfies the Δ_2-condition. For some existing results for strongly nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces [10, 22].

Our purpose is to generalize the result [3] and we prove the existence of entropy solution of (1.1). We first give a proof of a Poincaré-type inequality allowing us to prove our result (Lemma 4.4).

This article is organized as follows. In the second section we are going to recall some important definitions and results of Musielak-Orlicz-Sobolev spaces. We introduce in the third section some assumptions on $a(x,s,\xi)$ and $g(x,s,\xi)$ for which our problem has a solution. The fourth section contains some important lemmas useful to prove our main results. The section 5 will be devoted to show the existence of entropy solutions for the problem (1.1).

2. Preliminary

Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$, and satisfiying the following conditions :

a) $\varphi(x,.)$ is an N-function $\Big($convex, increasing, continous, $\varphi(x,0)=0, \varphi(x,t)>0$, $\forall t>0$, $\frac{\varphi(x,t)}{t} \longrightarrow 0$ as $t \longrightarrow 0$, $\frac{\varphi(x,t)}{t} \longrightarrow \infty$ as $t \longrightarrow \infty\Big)$.

b) $\varphi(.,t)$ is a measurable function.

A function φ, which satisfies the conditions $a)$ and $b)$ is called Musielak-Orlicz function.

For a Musielak-Orlicz function φ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its nonnegative reciprocal function φ_x^{-1}, with respect to t that is

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x, \varphi_x^{-1}(t)) = t.$$

The Musielak-Orlicz function φ is said to satisfy the Δ_2-condition if for some $k > 0$; and a non negative function h; integrable in Ω we have

$$\varphi(x,2t) \leq k\varphi(x,t) + h(x) \text{ for all } x \in \Omega \text{ and } t \geq 0. \tag{2.1}$$

When (2.1) holds only for $t \geq t_0 > 0$; then φ said satisfies Δ_2 near infinity.
Let φ and γ be two Musielak-orlicz functions, we say that φ dominate γ, and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x \in \Omega$

$$\gamma(x,t) \leq \varphi(x,ct) \text{ for all } t \geq t_0, \quad (\text{ resp. for all } t \geq 0 \text{ i.e. } t_0 = 0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity), and we write $\gamma \prec\prec \varphi$, If for every positive constant c we have

$$\lim_{t \longrightarrow 0} \left(\sup_{x\in\Omega} \frac{\gamma(x,ct)}{\varphi(x,t)} \right) = 0, \quad (\text{resp. } \lim_{t \longrightarrow \infty} \left(\sup_{x\in\Omega} \frac{\gamma(x,ct)}{\varphi(x,t)} \right) = 0).$$

Remark 2.1. *[8] If $\gamma \prec\prec \varphi$ near infinity, then $\forall \varepsilon > 0$ there exist $k(\varepsilon) > 0$ such that for almost all $x \in \Omega$ we have*

$$\gamma(x,t) \leq k(\varepsilon)\varphi(x,\varepsilon t), \qquad \textit{for all } t \geq 0. \tag{2.2}$$

We define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx.$$

where $u : \Omega \longrightarrow \mathbb{R}$ a Lebesgue measurable function. In the following the measurability of a function $u : \Omega \longrightarrow \mathbb{R}$ means the Lebesgue measurability.
The set

$$K_\varphi(\Omega) = \Big\{ u : \Omega \longrightarrow \mathbb{R} \textit{ measurable} \Big/ \rho_{\varphi,\Omega}(u) < +\infty \Big\}.$$

is called the generalized Orlicz class.
The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$.
Equivalently

$$L_\varphi(\Omega) = \Big\{ u : \Omega \longrightarrow \mathbb{R} \textit{ measurable} \Big/ \rho_{\varphi,\Omega}\Big(\frac{|u(x)|}{\lambda}\Big) < +\infty, \text{ for some } \lambda > 0 \Big\}.$$

Let

$$\psi(x,s) = \sup_{t \geq 0} \{st - \varphi(x,t)\}.$$

that is, ψ is the Musielak-Orlicz function complementary to φ in the sens of Young with respect to the variable s.
In the space $L_\varphi(\Omega)$ we define the following two norms :

$$\|u\|_{\varphi,\Omega} = \inf \Big\{ \lambda > 0 / \int_\Omega \varphi\Big(x, \frac{|u(x)|}{\lambda}\Big) dx \leq 1 \Big\}.$$

which is called the Luxemburg norm and the so called Orlicz norm by :

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_{\psi,\Omega} \leq 1} \int_\Omega |u(x)v(x)|dx.$$

where ψ is the Musielak Orlicz function complementary to φ. There two norms are equivalent [21].
The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space [21].
We say that sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\Big(\frac{u_n - u}{k}\Big) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \Big\{u \in L_\varphi(\Omega) : \forall|\alpha| \le m,\ D^\alpha u \in L_\varphi(\Omega)\Big\}.$$

and

$$W^m E_\varphi(\Omega) = \Big\{u \in E_\varphi(\Omega) : \forall|\alpha| \le m,\ D^\alpha u \in E_\varphi(\Omega)\Big\}.$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + ... + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak Orlicz Sobolev space.
Let

$$\overline{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\le m} \rho_{\varphi,\Omega}\Big(D^\alpha u\Big) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\Big\{\lambda > 0 : \overline{\rho}_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) \le 1\Big\}.$$

For $u \in W^m L\varphi(\Omega)$ there functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\Big(W^m L\varphi(\Omega), \|\|^m_{\varphi,\Omega}\Big)$ is a Banach space if φ satisfies the following condition [21] :

$$\text{there exist a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x, 1) \ge c. \tag{2.3}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\le m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.
We denote by $D(\Omega)$ the space of infinitely smooth functions with compact support in Ω and by $D(\overline{\Omega}))$ the restriction of $D(\mathbb{R}^N)$ on Ω.
Let $W_0^m L\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$.
Let $W^m E\varphi(\Omega)$ the space of functions u such that u and its distribution derivatives up to order m lie in $E_\varphi(\Omega)$, and $W_0^m E\varphi(\Omega)$ is the (norm) closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$.
The following spaces of distributions will also be used :

$$W^{-m} L_\psi(\Omega) = \Big\{f \in D'(\Omega);\ f = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega)\Big\}.$$

and

$$W^{-m} E_\psi(\Omega) = \Big\{f \in D'(\Omega);\ f = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega)\Big\}.$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\rho}_{\varphi,\Omega}\Big(\frac{u_n - u}{k}\Big) = 0.$$

For two Musielak Orlicz functions φ and ψ the following inequality is called the Young inequality [21]:

$$ts \leq \varphi(x,t) + \psi(x,s), \quad \forall t, s \geq 0, x \in \Omega. \tag{2.4}$$

This inequality implies the inequality

$$\||u|\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1. \tag{2.5}$$

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} > 1. \tag{2.6}$$

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} \leq 1. \tag{2.7}$$

For two complementary Musielak Orlicz functions φ and ψ let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$ we have the Hölder inequality [21]

$$\left|\int_\Omega u(x)v(x)dx\right| \leq \|u\|_{\varphi,\Omega}\||v|\|_{\psi,\Omega}. \tag{2.8}$$

3. Essential assumptions

Let Ω be a bounded open subset of $\mathbb{R}^N$ satisfying the segment property. Throughout this paper, we assume that φ and ψ are two Musielak complementary functions, such that

$$\varphi(x,t) \text{ decreases with respect to one of coordinate of } x. \tag{3.1}$$

Let $A : D(A) \subset W_0^1 L_\varphi(\Omega) \longrightarrow W^{-1}L_\psi(\Omega)$ be a mapping given by $A(u) = -div(a(x,u,\nabla u))$, where a is a function satisfying the following conditions

$$a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N \text{ is a Carathéodory function.} \tag{3.2}$$

There are two Musielak Orlicz functions φ and γ such that $\gamma \prec\prec \varphi$, a positive function $h_1(\cdot) \in E_\psi(\Omega)$ and positive constants ν, β such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$

$$|a(x,s,\xi)| \leq \beta\Big(h_1(x) + \psi_x^{-1}\gamma(x,\nu|s|) + \psi_x^{-1}\varphi(x,\nu|\xi|)\Big) \tag{3.3}$$

$$\Big(a(x,s,\xi) - a(x,s,\xi')\Big)(\xi - \xi') > 0 \tag{3.4}$$

$$a(x,s,\xi).\xi \geq \alpha\varphi(x,|\xi|) \tag{3.5}$$

Let furthermore $g(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}$ be a Carathéodory function such that for a.e. $x\in\Omega$, and for all $s\in\mathbb{R}$, $\xi\in\mathbb{R}^N$, the following growth condition

$$|g(x,s,\xi)| \leq b(|s|)\Big(h_2(x)+\varphi(x,|\xi|)\Big), \tag{3.6}$$

is satisfied, where $b:\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous positive function which belongs to $L^1(\Omega)$ and $h_2(\cdot)\in L^1(\Omega)$.
And g also satisfies the following sign condition

$$g(x,s,\xi)s \geq 0, \quad \forall s\in\mathbb{R}, \forall\xi\in\mathbb{R}^N, \text{ a.e. } x\in\Omega. \tag{3.7}$$

Let

$$f\in L^1(\Omega). \tag{3.8}$$

Et

$$\phi:\mathbb{R}\longrightarrow\mathbb{R}^N \text{ continuous.} \tag{3.9}$$

4. Some technical Lemmas

Lemma 4.1. *[9]. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

i) There exist a constant $c>0$ such that $\inf_{x\in\Omega}\varphi(x,1)\geq c$.

ii) There exist a constant $A>0$ such that for all $x,y\in\Omega$ with $|x-y|\leq\frac{1}{2}$ we have

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\overline{\frac{A}{log\left(\frac{1}{|x-y|}\right)}}}, \quad \forall t\geq 1. \tag{4.1}$$

iii)

$$\textit{If } D\subset\Omega \textit{ is a bounded measurable set, then } \int_D \varphi(x,1)dx<\infty. \tag{4.2}$$

iv) There exist a constant $C>0$ such that $\psi(x,1)\leq C$ a.e in Ω.

Under these assumptions, $D(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $D(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for the modular convergence and $D(\overline{\Omega})$ is dense in $W^1L_\varphi(\Omega)$ the modular convergence.

Consequently, the action of a distribution S in $W^{-1}L_\psi(\Omega)$ on an element u of $W_0^1L_\varphi(\Omega)$ is well defined. It will be denoted by $<S,u>$.
Truncation operator. For $k>0$ we define the truncation at height k: $T_k:\mathbb{R}\longrightarrow\mathbb{R}$ by:

$$T_k(s)=\begin{cases} s & \text{if } |s|\leq k.\\ k\frac{s}{|s|} & \text{if } |s|>k.\end{cases} \tag{4.3}$$

Lemma 4.2. *[8]. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak- Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$. Then $F(u) \in W_0^1 L_\varphi(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & \text{in } \{x \in \Omega : u(x) \notin D\}. \\ 0 & \text{in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 4.3. *[3]. Let $(f_n), f \in L^1(\Omega)$ such that*

i) *$f_n \geq 0$ a.e in Ω.*

ii) *$f_n \longrightarrow f$ a.e in Ω.*

iii) *$\int_\Omega f_n(x)(dx \longrightarrow \int_\Omega f(x)dx$.*

Then $f_n \longrightarrow f$ strongly in $L^1(\Omega)$.

Lemma 4.4. *Under the assumptions of lemma 4.1, and by assuming that $\varphi(x, t)$ decreases with respect to one of coordinate of x, there exists a constant $c > 0$ which depends only on Ω such that*

$$\int_\Omega \varphi(x, |u(x)|)dx \leq \int_\Omega \varphi(x, c|\nabla u(x)|)dx \quad \forall u \in W_0^1 L_\varphi(\Omega). \tag{4.4}$$

Proof: Since $\varphi(x, t)$ decreases with respect to one of coordinate of x , there exists $i_0 \in \{1, \ldots, N\}$ such that the function $\sigma \longrightarrow \varphi(x_1, ..., x_{i_0-1}, \sigma, x_{i_0+1}, ..., x_N, t)$ is decreasing for every $x_1, ..., x_{i_0-1}, x_{i_0+1}, ..., x_N \in \mathbb{R}$ and $\forall t > 0$.
To prove our result, it suffices to show that

$$\int_\Omega \varphi(x, |u(x)|)dx \leq \int_\Omega \varphi\Big(x, 2d\Big|\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)dx, \quad \forall u \in W_0^1 L_\varphi(\Omega). \tag{4.5}$$

where $d = \max\Big(diam(\Omega), 1\Big)$ and $diam(\Omega)$ is the diameter of Ω.
First suppose that $u \in D(\Omega)$, then

$$\begin{aligned}
&\varphi(x, |u(x_1, ..., x_N)|) \\
&\leq \varphi\Big(x, \int_{-\infty}^{x_{i_0}} \Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x_1, ..., x_{i_0-1}, \sigma, x_{i_0+1}, ..., x_N)d\sigma\Big), \\
&\leq \frac{1}{d}\int_{-\infty}^{+\infty} \varphi\Big(x, d\Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x_1, ..., x_{i_0-1}, \sigma, x_{i_0+1}, ..., x_N)\Big)d\sigma \\
&\leq \frac{1}{d}\int_{-\infty}^{+\infty} \varphi\Big(x_1, ..., x_{i_0-1}, \sigma, x_{i_0+1}, ..., x_N, d\Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x_1, ..., x_{i_0-1}, \sigma, x_{i_0+1}, ..., x_N)\Big)d\sigma.
\end{aligned}$$

By integrating with respect to x, we get

$$\begin{aligned}
&\int_\Omega \varphi(x, |u(x_1, ..., x_N)|)dx \\
&\leq \int_\Omega \frac{1}{d}\int_{-\infty}^{+\infty} \varphi\Big(x_1, ..., x_{i_0-1}, \sigma, x_{i_0+1}, ..., x_N, d\Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x_1, ..., x_{i_0-1}, \sigma, x_{i_0+1}, ..., x_N)\Big)d\sigma dx,
\end{aligned}$$

since $\varphi\Big(x_1,...,x_{i_0-1},\sigma,x_{i_0+1},...,x_N,d\Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x_1,...,x_{i_0-1},\sigma,x_{i_0+1},...,x_N)\Big)$ independent of x_{i_0}, we can get it out of the integral to respect of x_{i_0} and by the fact that σ is arbitrary, then by Fubini's Theorem we get

$$\int_\Omega \varphi(x,|u(x)|)dx \leq \int_\Omega \varphi\Big(x,d\Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x)\Big)dx, \quad \forall u\in D(\Omega). \tag{4.6}$$

For $u\in W_0^1L_\varphi(\Omega)$ according to Lemma 4.1, we have the existence of $u_n\in D(\Omega)$ and $\lambda>0$ such that

$$\overline{\varrho}_{\varphi,\Omega}\Big(\frac{u_n-u}{\lambda}\Big)=0, \quad \text{as } n\longrightarrow+\infty,$$

hence

$$\begin{cases} \int_\Omega \varphi\Big(x,\frac{|u_n-u|}{\lambda}\Big)dx\longrightarrow 0, \quad \text{as } n\longrightarrow+\infty,\\ \int_\Omega \varphi\Big(x,\frac{|\nabla u_n-\nabla u|}{\lambda}\Big)dx\longrightarrow 0, \quad \text{as } n\longrightarrow+\infty,\\ u_n\longrightarrow u \quad \text{a.e in } \Omega, \quad (\text{ for a subsequence still denote } u_n). \end{cases}$$

Then, we have

$$\begin{aligned} \int_\Omega \varphi\Big(x,\frac{|u(x)|}{2d\lambda}\Big)dx &\leq \liminf_{n\longrightarrow+\infty}\int_\Omega \varphi\Big(x,\frac{|u_n(x)|}{2d\lambda}\Big)dx\\ &\leq \liminf_{n\longrightarrow+\infty}\int_\Omega \varphi\Big(x,\frac{1}{2\lambda}\Big|\frac{\partial u_n}{\partial x_{i_0}}(x)\Big|\Big)dx\\ &= \liminf_{n\longrightarrow+\infty}\int_\Omega \varphi\Big(x,\frac{1}{2\lambda}\Big|\frac{\partial u_n}{\partial x_{i_0}}(x)-\frac{\partial u}{\partial x_{i_0}}(x)+\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)dx\\ &\leq \frac{1}{2}\liminf_{n\longrightarrow+\infty}\int_\Omega \varphi\Big(x,\frac{1}{\lambda}\Big|\frac{\partial u_n}{\partial x_{i_0}}(x)-\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)dx\\ &+ \frac{1}{2}\int_\Omega \varphi\Big(x,\frac{1}{\lambda}\Big|\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)dx\\ &\leq \int_\Omega \varphi\Big(x,\frac{1}{\lambda}\Big|\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)dx. \end{aligned}$$

Hence

$$\int_\Omega \varphi\Big(x,|u(x)|\Big)dx \leq \int_\Omega \varphi\Big(x,2d\Big|\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)dx, \quad \forall u\in W_0^1L_\varphi(\Omega).$$

□

Lemma 4.5 (The Nemytskii Operator). *[3]. Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak Orlicz functions. Let $f:\Omega\times\mathbb{R}^p\longrightarrow\mathbb{R}^q$ be a Carathéodory function such that for a.e. $x\in\Omega$ and all $s\in\mathbb{R}^p$:*

$$|f(x,s)|\leq c(x)+k_1\psi_x^{-1}\varphi(x,k_2|s|), \tag{4.7}$$

where k_1 and k_2 are real positives constants and $c(.) \in E_\psi(\Omega)$.
Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is continuous from

$$\left(\mathcal{P}(E_\varphi(\Omega), \frac{1}{k_2}\right)^p = \prod \left\{u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{k_2}\right\}.$$

into $(L_\psi(\Omega))^q$ for the modular convergence.
Furthermore, if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$, then N_f is strongly continuous from $\left(\mathcal{P}(E_\varphi(\Omega), \frac{1}{k_2}\right)^p$ to $(E_\gamma(\Omega))^q$.

Lemma 4.6 (Technical Lemma). *Assume that (3.2)...(3.5) are satisfied, and let $(z_n)_n$ be a sequence in $W_0^1 L_\varphi(\Omega)$ such that*

i) $z_n \rightharpoonup z$ in $W_0^1 L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$.

ii) $(a(\cdot, z_n, \nabla z_n))_n$ is bounded in $(L_\psi(\Omega))^N$.

iii) $\int_\Omega \Big(a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z \chi_s)\Big)(\nabla z_n - \nabla z \chi_s)dx \longrightarrow 0$ as $n, s \longrightarrow \infty$.
where χ_s is the characteristic function of $\Omega_s = \{x \in \Omega : |\nabla z| \leq s\}$.

Then, we have

$$z_n \longrightarrow z \text{ for the modular convergence in } W_0^1 L_\varphi(\Omega).$$

Proof: Let $s > 0$ and $\Omega_s = \{x \in \Omega : |\nabla z| \leq s\}$ and denote by χ_s the Characteristic function of Ω_s.
Fix $r > 0$ and let $s > r$, we have

$$\begin{aligned} 0 &\leq \int_{\Omega_r} \Big(a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z)\Big)(\nabla z_n - \nabla z)dx \\ &\leq \int_{\Omega_s} \Big(a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z)\Big)(\nabla z_n - \nabla z)dx \\ &= \int_{\Omega_s} \Big(a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z \chi_s)\Big)(\nabla z_n - \nabla z \chi_s)dx \\ &\leq \int_{\Omega} \Big(a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z \chi_s)\Big)(\nabla z_n - \nabla z \chi_s)dx. \end{aligned}$$

By *iii*), we obtain

$$\lim_{n \longrightarrow \infty} \int_{\Omega_r} \Big(a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z)\Big)(\nabla z_n - \nabla z)dx = 0.$$

So as in [18], we have

$$\nabla z_n \longrightarrow \nabla z \quad \text{a.e. in } \Omega. \tag{4.8}$$

On the one hand, we have

$$\begin{aligned}\int_\Omega a(x,z_n,\nabla z_n)\nabla z_n dx &= \int_\Omega \Big(a(x,z_n,\nabla z_n)-a(x,z_n,\nabla z\chi_s)\Big)(\nabla z_n-\nabla z\chi_s)dx \\ &+ \int_\Omega a(x,z_n,\nabla z\chi_s)(\nabla z_n-\nabla z\chi_s)dx \\ &+ \int_\Omega a(x,z_n,\nabla z_n)\nabla z\chi_s dx. \end{aligned} \tag{4.9}$$

Since $(a(\cdot,z_n,\nabla z_n))_n$ is bounded in $(L_\psi(\Omega))^N$ and using the almost every where convergence of the gradients we obtain

$$a(x,z_n,\nabla z_n) \rightharpoonup a(x,z,\nabla z) \text{ weakly in } (L_\psi(\Omega))^N \text{ for } \sigma(\Pi L_\psi,\Pi E_\varphi).$$

Which implies that

$$\int_\Omega a(x,z_n,\nabla z_n)\nabla z\chi_s dx \longrightarrow \int_\Omega a(x,z,\nabla z)\nabla z\chi_s dx. \tag{4.10}$$

Letting $s\longrightarrow\infty$, we obtain

$$\int_\Omega a(x,z,\nabla z)\nabla z\chi_s dx \longrightarrow \int_\Omega a(x,z,\nabla z)\nabla z dx. \tag{4.11}$$

On the other hand, it is easy to see that second term of the right hand side of (4.9) tends to 0, as $n\longrightarrow\infty$. Consequently, from iii), (4.10) and (4.11), we have

$$\int_\Omega a(x,z_n,\nabla z_n)\nabla z_n dx \longrightarrow \int_\Omega a(x,z,\nabla z)\nabla z dx. \tag{4.12}$$

Using (3.5) and the convexity of φ, we have

$$\alpha\varphi\Big(x,\frac{|\nabla z_n-\nabla z|}{2}\Big) \leq \frac{1}{2}a(x,z_n,\nabla z_n)\cdot\nabla z_n+\frac{1}{2}a(x,z,\nabla z)\cdot\nabla z.$$

Then by (4.12), we get

$$\lim_{\text{meas}(E)\longrightarrow 0}\sup_{n\in\mathbb{N}}\int_E \varphi\Big(x,\frac{|\nabla z_n-\nabla z|}{2}\Big)dx=0.$$

Then by using Vitali's theorem, one has

$$z_n\longrightarrow z \text{ for the modular convergence in } W_0^1L_\varphi(\Omega).$$

$\square$

5. Main results

In the sequel we assume that Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, and let φ and ψ be a two complementary Musielak Orlicz functions. We consider the following boundary value problem

$$(P)\begin{cases} A(u) + g(.,u,\nabla u) = f + div(\phi(u)) \quad \text{in } \Omega, \\ u \equiv 0 \quad \text{on } \partial\Omega. \end{cases}$$

We will prove the following existence theorem

Theorem 5.1. *Let φ et ψ be two complementary Musielak Orlicz functions satisfying the assumptions of Lemma 4.1, we assume that (3.1)-(3.9) hold true. Then the problem*

$$\begin{cases} T_k(u) \in W_0^1L_\varphi(\Omega), \quad \forall k > 0 \\ \int_\Omega a(x,u,\nabla u)\cdot\nabla T_k(u-v)dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)dx \\ \leq \int_\Omega fT_k(u-v)dx + \int_\Omega \phi(u)\nabla T_k(u-v)dx \\ \forall v \in W_0^1L_\varphi(\Omega)\cap L^\infty(\Omega), \end{cases} \tag{5.1}$$

has at least one solution.

Proof:

Step 1 : Approximate problems.

We consider the following approximate problems

$$(P_n)\begin{cases} -div\Big(a(x,u_n,\nabla u_n) + \phi_n(u_n)\Big) + g_n(.,u_n,\nabla u_n) = f_n \quad \text{in } \Omega, \\ u_n \in W_0^1L_\varphi(\Omega), \end{cases}$$

where $(f_n)_n$ is a sequence in $W^{-1}E_\psi(\Omega)\cap L^1(\Omega)$ such that $f_n \longrightarrow f$ in $L^1(\Omega)$ with $\|f_n\|_1 \leq \|f\|_1$, $\phi_n(s) = \phi(T_n(s))$ and $g_n(x,s,\xi) = T_n(g(x,s,\xi))$.
From A. Benkirane, M. Sidi El Vally in [8], then the problem (P_n) has at least one weak solution $u_n \in W_0^1L_\varphi(\Omega)$.

Step 2 : A priori estimates

Taking $T_k(u_n)$ as a test function in (P_n), we obtain

$$\begin{aligned} \int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)dx \quad &+ \quad \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)dx \\ &= \quad \int_\Omega f_nT_k(u_n)dx + \int_\Omega \phi_n(u_n)\cdot\nabla T_k(u_n)dx \end{aligned}$$

Thanks to (3.7), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)dx \leq \int_\Omega f_nT_k(u_n)dx + \int_\Omega \phi_n(u_n)\cdot\nabla T_k(u_n)dx$$

$$\int_\Omega f_n T_k(u_n)dx \leq \|f\|_{L^1(\Omega)} \cdot k.$$

Taking $\tilde{\phi}_n(s) = \int_0^s \phi_n(\tau)d\tau$, then $\tilde{\phi}_n(0) = 0_{\mathbb{R}^N}$ and $\tilde{\phi}_n \in C^1(\mathbb{R}^N)$. By the Divergence Theorem (see also [15]), we obtain

$$\begin{aligned}\int_\Omega \phi_n(u_n)\cdot\nabla T_k(u_n)dx &= \int_\Omega div(\tilde{\phi}_n(T_k(u_n)))dx\\ &= \int_{\partial\Omega} \tilde{\phi}_n(T_k(u_n)).\overrightarrow{n}\,dx\\ &= \sum_{i=1}^N \Big(\int_{\partial\Omega} \tilde{\phi}^i_n(T_k(u_n)).n_i dx\Big) = 0,\end{aligned}$$

since $u = 0$ on $\partial\Omega$, with $\tilde{\phi}_n = (\tilde{\phi}^1_n, ..., \tilde{\phi}^N_n)$ and $\overrightarrow{n} = (n_1, ..., n_N)$ the normal vector on $\partial\Omega$.
Then, we have

$$\int_\Omega a(x, u_n, \nabla u_n)\cdot\nabla T_k(u_n)dx \leq \|f\|_{L^1(\Omega)} \cdot k \tag{5.2}$$

Hence, by using (3.5), we have

$$\int_\Omega \varphi(x, |\nabla T_k(u_n)|)dx \leq \frac{\|f\|_{L^1(\Omega)}}{\alpha}\cdot k \tag{5.3}$$

Assuming the existence of a positive function M such that $\lim_{t\longrightarrow+\infty} \frac{M(t)}{t} = +\infty$ and $M(t) \leq \operatorname{ess\,inf}_{x\in\Omega}\varphi(x,t), \quad \forall t \geq 0$.
Then, we have

$$\begin{aligned} M\Big(\frac{k}{c}\Big)\text{meas}\Big\{|u_n| > k\Big\} &= \int_{\{|u_n|>k\}} M\Big(\frac{k}{c}\Big)dx\\ &\leq \int_\Omega M\Big(\frac{|T_k(u_n)|}{c}\Big)dx\\ &\leq \int_\Omega \varphi\Big(x, \frac{|T_k(u_n)|}{c}\Big)dx\\ &\leq \int_\Omega \varphi\Big(x, |\nabla T_k(u_n)|\Big)dx, \quad (\text{ using Lemma 4.4})\\ &\leq \frac{\|f\|_{L(\Omega)}}{\alpha}\cdot k, \quad (\text{using (5.3)}),\end{aligned}$$

where this c is the constant of Lemma 4.4.
Then

$$\text{meas}\Big\{|u_n| > k\Big\} \leq \frac{\|f\|_{L(\Omega)}\cdot k}{\alpha M\Big(\frac{k}{c}\Big)} \longrightarrow 0, \quad \text{as } k \longrightarrow +\infty \tag{5.4}$$

Since $\forall\delta > 0$

$$\begin{aligned}\text{meas}\Big\{|u_n - u_m| > \delta\Big\} &\leq \text{meas}\Big\{|u_n| > k\Big\} + \text{meas}\Big\{|u_m| > k\Big\}\\ &+ \text{meas}\Big\{|T_k(u_n) - T_k(u_m)| > \delta\Big\}\end{aligned}$$

Using (5.4), we get $\forall\varepsilon > 0$, there exists $k_0 > 0$ such that

$$\text{meas}\Big\{|u_n| > k\Big\} \leq \frac{\varepsilon}{3}, \quad \text{meas}\Big\{|u_m| > k\Big\} \leq \frac{\varepsilon}{3}, \quad \forall k \geq k_0(\varepsilon), \tag{5.5}$$

By using (5.3) the sequence $(T_k(u_n))_n$ is bounded in $W_0^1L_\varphi(\Omega)$, then there exists $v_k \in W_0^1L_\varphi(\Omega)$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup v_k \text{ in } W_0^1L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \\ T_k(u_n) \longrightarrow v_k \text{ strongly in } E_\varphi(\Omega). \end{cases} \tag{5.6}$$

Therefore, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω, then for all $k > 0$ and $\delta, \varepsilon > 0$ there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\text{meas}\Big\{|T_k(u_n) - T_k(u_m)| > \delta\Big\} \leq \frac{\varepsilon}{3}, \quad \forall m, n \geq n_0 \tag{5.7}$$

Combining (5.5) and (5.7), we obtain that for all $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$ such that

$$\text{meas}\Big\{|u_m - u_m| > \delta\Big\} \leq \varepsilon, \quad \forall n, m \geq n_0.$$

It follows that $(u_n)_n$ is a Cauchy sequence in measure, the there exists a function u such that

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) \text{ in } W_0^1L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \\ T_k(u_n) \longrightarrow T_k(u) \text{ strongly in } E_\varphi(\Omega). \end{cases} \tag{5.8}$$

Step 3 : Boundness of $(a(x, T_k(u_n), \nabla T_k(u_n)))_n$ in $(L_\psi(\Omega))^N$

Let $w \in (E_\varphi(\Omega)^N$ be arbitrary such that $\|w\|_{\varphi,\Omega} \leq 1$, by (3.4) we have

$$\Big(a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \frac{w}{\nu})\Big)(\nabla T_k(u_n) - \frac{w}{\nu}) > 0.$$

hence

$$\begin{aligned} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\frac{w}{\nu}dx \quad &\leq \quad \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)dx \\ &- \quad \int_\Omega a(x, T_k(u_n), \frac{w}{\nu})(\nabla T_k(u_n) - \frac{w}{\nu})dx. \end{aligned} \tag{5.9}$$

On the one hand, thinks to (5.2), we have

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)dx \leq ck.$$

On the other hand, for λ large enough ($\lambda > \beta$), we have by using (3.3).
$\int_\Omega \psi_x\Big(\frac{a(x,T_k(u_n),\frac{w}{\nu})}{3\lambda}\Big)dx$

$$
\begin{aligned}
&\leq \int_\Omega \psi_x\Big(\frac{\beta\Big(h_1(x)+\psi_x^{-1}(\gamma(x,\nu|T_k(u_n)|))+\psi_x^{-1}(\varphi(x,|w|))\Big)}{3\lambda}\Big)dx\\
&\leq \frac{\beta}{\lambda}\int_\Omega \psi_x\Big(\frac{h_1(x)+\psi_x^{-1}(\gamma(x,\nu|T_k(u_n)|))+\psi_x^{-1}(\varphi(x,|w|))}{3}\Big)dx\\
&\leq \frac{\beta}{3\lambda}\Big(\int_\Omega \psi_x(h_1(x))dx+\int_\Omega \gamma(x,\nu|T_k(u_n)|)dx+\int_\Omega \varphi(x,|w|)dx\Big)\\
&\leq \frac{\beta}{3\lambda}\Big(\int_\Omega \psi_x(h_1(x))dx+\int_\Omega \gamma(x,\nu k)dx+\int_\Omega \varphi(x,|w|)dx\Big).
\end{aligned}
$$

Now, since γ grows essentially less rapidly than φ near infinity ad by using the Remark 2.1, there exists $r(k)>0$ such that $\gamma(x,\nu k)\leq r(k)\varphi(x,1)$ and so we have
$\int_\Omega \psi_x\Big(\frac{a(x,T_k(u_n),\frac{w}{\nu})}{3\lambda}\Big)dx$

$$
\leq \frac{\beta}{3\lambda}\Big(\int_\Omega \psi_x(h_1(x))dx+r(k)\int_\Omega \varphi(x,1)dx+\int_\Omega \varphi(x,|w|)dx\Big).
$$

hence $a(x,T_k(u_n),\frac{w}{\nu})$ is bounded in $(L_\psi(\Omega))^N$.
Which implies that second term of the right hand side of (5.9) is bounded, consequently, we obtain

$$
\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))w\,dx \leq c_4(k), \qquad \text{for all } w\in (L_\varphi(\Omega))^N \text{ with } \|w\|_{\varphi,\Omega}\leq 1.
$$

Hence by the theorem of Banach Steinhous, the sequence $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ remains bounded in $(L_\psi(\Omega))^N$.
Which implies that, for all $k>0$ there exists a function $l_k\in (L_\psi(\Omega))^N$ such that

$$
a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup l_k \text{ weakly in } (L_\psi(\Omega))^N \text{ for } \sigma(\Pi L_\psi,\Pi E\varphi). \tag{5.10}
$$

Step 4 : Mean convergence of truncations

In the sequel, we denote by $\varepsilon_i(n)$, $i=1,\cdots$, various real functions which converge to 0 as n tends to infinity.
Let $\eta_k(s)=s\exp(\sigma s^2)$ where $\sigma=\Big(\frac{b_k}{2\alpha}\Big)^2$ and $b_k=\sup\{b(s):|s|\leq k\}$ it is obvious that

$$
\eta_k'(s)-\frac{b_k}{\alpha}|\eta_k(s)|\geq \frac{1}{2}, \quad \forall s\in\mathbb{R} \tag{5.11}
$$

Let $\Omega_s=\{x\in\Omega:|\nabla T_k(u(x))|\leq s\}$ and denote by χ_s the characteristic function of Ω_s, it is easy to show that $\Omega_s\subset\Omega_{s+1}$ and $meas(\Omega\setminus\Omega_s)\longrightarrow 0$ as s tend to infinity, and χ_s denotes the characteristic function of the subset Ω_s.

Denote by $\varepsilon_i(n,s)(i=0,1,2,...)$ various sequences of real numbers which tend to 0 when n and $s \longrightarrow \infty$, i.e

$$\lim_{s\longrightarrow\infty}\lim_{n\longrightarrow\infty}\varepsilon_i(n,s)=0.$$

We consider $h>k>0$ and $M=4k+h$, we set

$$w_n = T_{2k}\Big(u_n - T_h(u_n) + T_k(u_n) - T_k(u)\Big).$$

Taking $\eta_k(w_n)$ as a test function in (P_n), we obtain

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla w_n\eta_k'(w_n)dx + \int_\Omega g_n(x,u_n,\nabla u_n)\eta_k(w_n)dx\\ &= \int_\Omega f_n\eta_k(w_n)dx + \int_\Omega \phi_n(u_n)\cdot\nabla w_n\eta_k'(w_n)dx.\end{aligned}$$

It is easy to see that $\nabla w_n = 0$ on $\{|u_n|>M\}$ and since $g_n(x,u_n,\nabla u_n)\cdot\eta_k(w_n)\geq 0$ on $\{|u_n|>k\}$, we have

$$\begin{aligned}&\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla w_n\eta_k'(w_n)dx + \int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\eta_k(w_n)dx\\ &\leq \int_\Omega f_n\eta_k(w_n)dx + \int_\Omega \phi_n(u_n)\cdot\nabla w_n\eta_k'(w_n)dx.\end{aligned} \tag{5.12}$$

We have
$\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla w_n\eta_k'(w_n)dx$

$$\begin{aligned}&= \int_{\{|u_n|\leq k\}} a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_{2k}(u_n - T_k(u))\eta_k'(w_n)dx\\ &+ \int_{\{|u_n|>k\}} a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla T_{2k}\Big(u_n - T_h(u_n) + T_k(u_n) - T_k(u)\Big)\eta_k'(w_n)dx.\end{aligned} \tag{5.13}$$

On the one hand, since $|u_n - T_k(u)|\leq 2k$ on $\{|u_n|\leq k\}$, we have

$$\begin{aligned}&\int_{\{|u_n|\leq k\}} a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_{2k}(u_n - T_k(u))\eta_k'(w_n)dx\\ &= \int_{\{|u_n|\leq k\}} a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla (T_k(u_n) - T_k(u))\eta_k'(w_n)dx\\ &= \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla (T_k(u_n) - T_k(u))\eta_k'(w_n)dx\\ &- \int_{\{|u_n|>k\}} a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla (T_k(u_n) - T_k(u))\eta_k'(w_n)dx.\end{aligned} \tag{5.14}$$

Since $1 \leq \eta_k'(w_n) \leq \eta_k'(2k)$, it follows that

$$\begin{aligned}
- & \int_{\{|u_n|>k\}} a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla (T_k(u_n) - T_k(u))\eta_k'(w_n)dx \\
& = \int_{\{|u_n|>k\}} a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u)\eta_k'(w_n)dx \\
& \leq \eta_k'(2k) \int_{\{|u_n|>k\}} |a(x, T_k(u_n), \nabla T_k(u_n))||\nabla T_k(u)|dx.
\end{aligned}$$

Then by (5.10), we have

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup l_k, \quad \text{ in } (L_\psi(\Omega))^N \text{ for } \sigma(\Pi L_\psi, \Pi E_\varphi).$$

Then

$$\int_{\{|u_n|>k\}} |a(x, T_k(u_n), \nabla T_k(u_n))||\nabla T_k(u)|dx \longrightarrow \int_{\{|u>k\}} l_k|\nabla T_k(u)|dx = 0,$$

and we obtain

$$\int_{\{|u_n|>k\}} a(x, T_k(u_n), \nabla T_k(u_n))\nabla\Big(T_k(u_n) - T_k(u)\Big)\eta_k'(w_n)dx \leq \varepsilon_0(n). \tag{5.15}$$

On the other hand, for second term on the right hand side of (5.13), and taking $y_n = u_n - T_h(u_n) + T_k(u_n) - T_k(u)$, we obtain

$$\begin{aligned}
& \int_{\{|u_n|>k\}} a(x, T_M(u_n), \nabla T_M(u_n))\eta_k'(w_n)\nabla T_{2k}(y_n)dx \\
& = \int_{\{|u_n|>k\}\cap\{|y_n|\leq 2k\}} a(x, T_M(u_n), \nabla T_M(u_n))\eta_k'(w_n)\nabla T_{2k}(y_n)dx \\
& = \int_{\{|u_n|>k\}\cap\{|y_n|\leq 2k\}} a(x, T_M(u_n), \nabla T_M(u_n))\eta_k'(w_n)\nabla\Big(u_n - T_h(u_n) + T_k(u_n) - T_k(u)\Big)dx \\
& = \int_{\{|u_n|>k\}\cap\{|y_n|\leq 2k\}} a(x, T_M(u_n), \nabla T_M(u_n))\eta_k'(w_n)\nabla\Big(u_n - T_k(u)\Big)\chi_{\{|u_n|>h\}}dx \\
& - \int_{\{|u_n|>k\}\cap\{|y_n|\leq 2k\}} a(x, T_M(u_n), \nabla T_M(u_n))\eta_k'(w_n)\nabla T_k(u)\chi_{\{|u_n|\leq h\}} \\
& = \int_{\{|u_n|>k\}\cap\{|y_n|\leq 2k\}} a(x, T_M(u_n), \nabla T_M(u_n))\nabla T_M(u_n)\eta_k'(w_n)\chi_{\{|u_n|>h\}}dx \\
& - \int_{\{|u_n|>k\}\cap\{|y_n|\leq 2k\}} a(x, T_M(u_n), \nabla T_M(u_n))\eta_k'(w_n)\nabla T_k(u)\chi_{\{|u_n|>h\}}dx \\
& - \int_{\{|u_n|>k\}\cap\{|y_n|\leq 2k\}} a(x, T_M(u_n), \nabla T_M(u_n))\eta_k'(w_n)\nabla T_k(u)\chi_{\{|u_n|\leq h\}} \\
& \leq - \int_{\{|u_n|>k\}} |a(x, T_M(u_n), \nabla T_M(u_n))||\nabla T_k(u)|\eta_k'(w_n)dx
\end{aligned} \tag{5.16}$$

By combining (5.13)-(5.16), we obtain

$$\begin{aligned}
&\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\eta'_k(w_n)\nabla w_n dx\\
&\geq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\eta'_k(w_n)\Big(\nabla T_k(u_n)-\nabla T_k(u)\Big)dx\\
&-\int_{\{|u_n|>k\}} |a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_M(u)|\eta'_k(w_n)dx-\varepsilon_0(n),
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big]\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big)\eta'_k(w_n)dx\\
&\leq \int_{\{|u_n|>k\}} |a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_M(u)|\eta'_k(w_n)dx\\
&+\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\nabla w_n\eta'_k(w_n)dx\\
&-\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big(\nabla T_k(u_n)-\nabla T_k(u)\Big)\eta'_k(w_n)dx\\
&-\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\Big(\nabla T_k(u)\chi_s-\nabla T_k(u)\Big)\eta'_k(w_n)dx\\
&+\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big(\nabla T_k(u)\chi_s-\nabla T_k(u)\Big)\eta'_k(w_n)dx+\varepsilon_0(n).
\end{aligned}$$

We obtain

$$\begin{aligned}
&\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big]\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big)\eta'_k(w_n)dx\\
&\leq \eta'_k(2k)\int_{\{|u_n|>k\}} |a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|dx\\
&+\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\nabla w_n\eta'_k(w_n)dx\\
&+\eta'_k(2k)\int_\Omega |a(x,T_k(u_n),\nabla T_k(u)\chi_s)||\nabla T_k(u_n)-\nabla T_k(u)|dx\\
&-\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\Big(\nabla T_k(u)\chi_s-\nabla T_k(u)\Big)dx\\
&+\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big(\nabla T_k(u)\chi_s-\nabla T_k(u)\Big)dx+\varepsilon_0(n).
\end{aligned}\tag{5.17}$$

Now, we study each term on the right hand side of the above inequality. For the first term, we have

$$|a(x,T_M(u_n),\nabla T_M(u_n))|\rightharpoonup |l_M| \quad \text{in } L_\psi(\Omega),$$

and since $\varphi(x,|\nabla T_k(u)|\chi_{\{|u_n|>k\}})\leq\varphi(x,|\nabla T_k(u)|)$ and $\varphi(x,|\nabla T_k(u)|\chi_{\{|u_n|>k\}})\longrightarrow 0$ a.e in Ω, by the Lebesgue dominated convergence theorem, we deduce that

$$\nabla T_k(u_n)\chi_{\{|u_n|>k\}}\longrightarrow 0 \quad \text{in } L_\psi(\Omega), \quad \text{as } n\longrightarrow\infty,$$

which implies the first term in the right hand side of (5.17) tends to 0 as n tends to infinity, and we can write

$$\eta_k'(2k)\int_{\{|u_n|>k\}}|a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|dx=\varepsilon_1(n). \tag{5.18}$$

For the third term of the right hand side of (5.17), we have by using (5.10) and the fact that $\nabla T_k(u_n)$ tends weakly to $\nabla T_k(u)$ in $(L_\varphi(\Omega))^N$ and by Lemma 4.5 $a(x,T_k(u_n),\nabla T_k(u)\chi_s)$ tends to $a(x,T_k(u),\nabla T_k(u)\chi_s)$ strongly in $(E_\psi(\Omega))^N$. Then we obtain

$$\eta_k'(2k)\int_\Omega|a(x,T_k(u_n),\nabla T_k(u)\chi_s)||\nabla T_k(u_n)-\nabla T_k(u)|dx\longrightarrow 0 \quad \text{as } n\longrightarrow\infty,$$

then

$$\eta_k'(2k)\int_\Omega|a(x,T_k(u_n),\nabla T_k(u)\chi_s)||\nabla T_k(u_n)-\nabla T_k(u)|dx=\varepsilon_2(n). \tag{5.19}$$

For the fourth term on the right hand side of (5.17) and by using (5.10)we have

$$\begin{aligned}&\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\Big(\nabla T_k(u)\chi_s-\nabla T_k(u)\Big)dx\\&\qquad\rightharpoonup\int_\Omega l_k\cdot\Big(\nabla T_k(u)\chi_s-\nabla T_k(u)\Big)dx.\end{aligned}$$

Then, by letting s to infinity, we obtain

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\Big(\nabla T_k(u)\chi_s-\nabla T_k(u)\Big)dx=\varepsilon_3(n,s). \tag{5.20}$$

For the fifth term on the right hand side of (5.17) and by using Lemma 4.5, we have

$$\begin{aligned}&\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big(\nabla T_k(u)\chi_s-\nabla T_k(u)\Big)dx\\&\qquad\rightharpoonup\int_\Omega a(x,T_k(u),\nabla T_k(u)\chi_s)\Big(\nabla T_k(u)\chi_s-\nabla T_k(u)\Big)dx\end{aligned}$$

Then by letting s to infinity, we obtain

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big(\nabla T_k(u)\chi_s-\nabla T_k(u)\Big)dx=\varepsilon_4(n,s). \tag{5.21}$$

By (5.17)-(5.21), we deduce that

$$\begin{aligned}&\int_\Omega\Big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big]\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big)\eta_k'(w_n)dx\\&\leq\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\nabla w_n\eta_k'(w_n)dx+\varepsilon_5(n,s).\end{aligned} \tag{5.22}$$

Now, we turn to the second term on the left hand side of (5.12). By (3.6) we have

$$
\begin{aligned}
&\Big|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\eta_k(w_n)dx\Big|\\
&\le \int_{\{|u_n|\le k\}} b(|u_n|)\Big[h_2(x)+\varphi(x,|\nabla T_k(u_n)|)\Big]|\eta_k(w_n)|dx\\
&\le b_k\int_{\{|u_n|\le k\}} \Big[h_2(x)+\varphi(x,|\nabla T_k(u_n)|)\Big]|\eta_k(w_n)|dx\\
&\le b_k\int_{\{|u_n|\le k\}} h_2(x)|\eta_k(w_n)|dx+\frac{b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|\eta_k(w_n)|dx\\
&\le b_k\int_{\{|u_n|\le k\}} h_2(x)|\eta_k(w_n)|dx\\
&+\frac{b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big)|\eta_k(w_n)|dx\\
&+\frac{b_k}{\alpha}\int_\Omega\Big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big]\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big)|\eta_k(w_n)|dx\\
&+\frac{b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi_s|\eta_k(w_n)|dx.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{b_k}{\alpha}\int_\Omega\Big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big]\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big)\eta_k(w_n)dx\\
&\ge\Big|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\eta_k(w_n)dx\Big|-b_k\int_{\{|u_n|\le k\}} h_2(x)|\eta_k(w_n)|dx\\
&-\frac{b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big)|\eta_k(w_n)|dx\\
&-\frac{b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi_s|\eta_k(w_n)|dx.
\end{aligned}
\tag{5.23}
$$

As in (5.20) and (5.21), we have

$$
\frac{b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big)|\eta_k(w_n)|dx=\varepsilon_6(n,s),
\tag{5.24}
$$

and

$$
\frac{b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi_s|\eta_k(w_n)|dx=\varepsilon_7(n,s).
\tag{5.25}
$$

By letting n to infinity, we obtain

$$
\int_{\{|u_n|\le k\}} h_2(x)|\eta_k(w_n)|dx\longrightarrow\int_{\{|u|\le k\}} h_2(x)|\eta_k(T_{2k}(u-T_h(u)))|dx=0.
\tag{5.26}
$$

Concerning the third term on the right hand side of (5.23), we have

$$
\begin{aligned}
&\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big)|\eta_k(w_n)|dx\\
&\le\eta_k(2k)\int_\Omega|a(x,T_k(u_n),\nabla T_k(u)\chi_s)||\nabla T_k(u_n)-\nabla T_k(u)\chi_s|dx,
\end{aligned}
$$

from (5.19), we have as n goes to infinity

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u)\chi_s)\Big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\Big)|\eta_k(w_n)|dx \longrightarrow 0. \tag{5.27}$$

For the last term of the right hand side of (5.23), by using (5.10) and the fact that

$$\nabla T_k(u)\chi_s|\eta_k(w_n)| \longrightarrow \nabla T_k(u)\chi_s|\eta_k(T_{2k}(u - T_h(u)))| = 0 \quad \text{in } (E_\varphi(\Omega))^N,$$

we have

$$\begin{aligned}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u)\chi_s|\eta_k(w_n)|dx \\ \longrightarrow \int_\Omega l_k \nabla T_k(u)\chi_s|\eta_k(T_{2k}(u - T_h(u)))|dx = 0.\end{aligned} \tag{5.28}$$

Combining (5.23)-(5.28), we obtain

$$\begin{aligned}&\frac{b_k}{\alpha}\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big]\Big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\Big)|\eta_k(w_n)|dx \\ &\geq \left|\int_{\{|u_n|\leq k\}} g_n(x, u_n, \nabla u_n)\eta_k(w_n)dx\right| + \varepsilon_8(n, s).\end{aligned} \tag{5.29}$$

Thanks to (5.12), (5.22) and (5.29), we obtain

$$\begin{aligned}&\int_\Omega \Big[a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)\Big]\Big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\Big) \\ &\quad\times\Big(\eta_k'(w_n) - \frac{b_k}{\alpha}|\eta_k(w_n)|\Big)dx \\ &\leq \int_\Omega a(x, T_M(u_n), \nabla T_M(u_n))\eta_k'(w_n)\nabla w_n dx - \left|\int_{\{|u_n|\leq k\}} g_n(x, u_n, \nabla u_n)\eta_k(w_n)dx\right| + \varepsilon_9(n, s) \\ &\leq \int_\Omega f_n\eta_k(w_n)dx + \int_{\{|u_n|\leq M\}} \phi_n(T_M(u_n))\eta_k'(w_n)\nabla w_n dx + \varepsilon_9(n, s).\end{aligned} \tag{5.30}$$

Using the fact that $w_n \rightharpoonup T_{2k}(u - T_h(u))$ weakly in $L^\infty(\Omega)$, then

$$\int_\Omega f_n\eta_k(w_n)dx \longrightarrow \int_\Omega f\eta_k(T_{2k}(u - T_h(u)))dx \quad \text{as } n \longrightarrow \infty, \tag{5.31}$$

and for n large enough (for exemple $n \geq M$), we deduce

$$\int_\Omega \phi_n(T_M(u_n))\eta_k'(w_n)\nabla w_n dx = \int_{\{|u_n|\leq M\}} \phi(T_M(u_n))\eta_k'(w_n)\nabla w_n dx.$$

It follows that

$$\begin{aligned}&\int_\Omega \phi_n(T_M(u_n))\eta_k'(w_n)\nabla w_n dx \\ &\quad\longrightarrow \int_\Omega \phi(T_M(u))\eta_k'(T_{2k}(u - T_h(u)))\nabla T_{2k}(u - T_h(u))dx \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{5.32}$$

Combining (5.11), (5.30) and (5.32), we obtain

$$\frac{1}{2}\int_{\Omega}\Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big]\Big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\Big)dx$$
$$\leq \int_{\Omega} f\eta_k(T_{2k}(u - T_h(u)))dx$$
$$+\int_{\Omega}\phi(T_M(u))\eta_k'(T_{2k}(u-T_h(u)))\nabla T_{2k}(u-T_h(u))dx + \varepsilon_9(n,s). \tag{5.33}$$

Taking $\overline{\phi}(s) = \int_0^s \phi(\tau)\eta'(\tau - T_h(\tau))d\tau$, then $\overline{\phi}(0) = 0_{\mathbb{R}^n}$ and $\overline{\phi} \in C^1(\mathbb{R}^n)$. By the Divergence Theorem (see also [15]), we obtain

$$\begin{aligned}
&\int_{\Omega}\phi(T_M(u)\eta_k'(T_{2k}(u-T_h(u)))\nabla T_{2k}(u-T_h(u))dx\\
&= \int_{h<u\leq 2k+h}\phi(u)\eta_k'(u-T_h(u))\nabla u dx\\
&= \int_{|u|\leq 2k+h}\phi(T_{2k+h}(u)\eta_k'(T_{2k+h}(u)-T_h(u))\nabla u dx\\
&- \int_{|u|\leq h}\phi(T_h(u)\eta_k'(T_h(u)-T_h(u))\nabla T_h(u)dx\\
&= \int_{\Omega} div\overline{\phi}(T_{2k+h}(u))dx - \int_{\Omega} div\overline{\phi}(T_h(u))dx\\
&= \int_{\partial\Omega}\overline{\phi}(T_{2k+h}(u)).\overrightarrow{n}dx - \int_{\partial\Omega}\overline{\phi}(T_h(u)).\overrightarrow{n}dx\\
&= \sum_{i=1}^{N}\Big(\int_{\partial\Omega}\overline{\phi}_i(T_{2k+h}(u)).n_i dx - \int_{\partial\Omega}\overline{\phi}_i(T_h(u)).n_i dx\Big) = 0,
\end{aligned}$$

since $u = 0$ on $\partial\Omega$, with $\overline{\phi} = (\overline{\phi}_1, ..., \overline{\phi}_N)$ and $\overrightarrow{n} = (n_1, ..., n_N)$ the normal vector on $\partial\Omega$.
Then, by letting h to infinity in (5.33), we obtain

$$\int_{\Omega}\Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big]$$
$$\times\Big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\Big)dx \longrightarrow 0, \tag{5.34}$$

as $n \longrightarrow \infty$.
Then by using Lemma 4.6, we obtain

$$\varphi\Big(\cdot, |\nabla T_k(u_n)|\Big) \longrightarrow \varphi\Big(\cdot, |\nabla T_k(u)|\Big) \text{ strongly in } L^1(\Omega). \tag{5.35}$$

Then by Lemma 4.4, we have

$$T_k(u_n) \longrightarrow T_k(u) \text{ in } W_0^1L_{\varphi}(\Omega) \text{ for the modular convergence.} \tag{5.36}$$

Step 5 : The equi-integrability of $g_n(x,u_n,\nabla u_n)$.

From (5.36), we have

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } \Omega. \tag{5.37}$$

To prove that

$$g_n(x,u_n,\nabla u_n) \longrightarrow g(x,u,\nabla u) \qquad \text{strongly in} \quad L^1(\Omega),$$

using the Vitali's theorem, it is sufficient to prove that $g_n(x,u_n,\nabla u_n)$ is uniformly equi-integrable. Indeed, taking $T_1(u_n - T_h(u_n))$ as a test function in (P_n), we obtain

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_1(u_n - T_h(u_n))\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_1(u_n - T_h(u_n))\,dx \\ = \int_\Omega f_n T_1(u_n - T_h(u_n))\,dx + \int_\Omega \phi_n(u_n)\nabla T_1(u_n - T_h(u_n))\,dx,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\int_{\{h<|u_n|\le h+1\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx + \int_{\{h\le|u_n|\}} g_n(x,u_n,\nabla u_n)T_1(u_n - T_h(u_n))\,dx \\ = \int_{\{h\le|u_n|\}} f_n T_1(u_n - T_h(u_n))\,dx + \int_{\{h<|u_n|\le h+1\}} \phi_n(u_n)\nabla u_n\,dx.\end{aligned}$$

Taking $\overline{\phi}_n(t) = \displaystyle\int_0^t \phi_n(\tau)\,d\tau$, then $\overline{\phi}_n(0) = 0_{I\!R^N}$ and $\overline{\phi}_n \in C^1(I\!R, I\!R^N)$, in view of the Divergence theorem, we obtain

$$\begin{aligned}\int_{\{h<|u_n|\le h+1\}} \phi_n(u_n)\nabla u_n\,dx &= \int_{\{|u_n|\le h+1\}} \phi_n(u_n)\nabla u_n\,dx \\ &\quad - \int_{\{|u_n|\le h\}} \phi_n(u_n)\nabla u_n\,dx \\ &= \int_\Omega \phi_n(T_{h+1}(u_n))\nabla T_{h+1}(u_n)\,dx \\ &\quad - \int_\Omega \phi_n(T_h(u_n))\nabla T_h(u_n)\,dx \\ &= \int_\Omega \operatorname{div}\overline{\phi}_n(T_{h+1}(u_n))\,dx - \int_\Omega \operatorname{div}\overline{\phi}_n(T_h(u_n))\,dx \\ &= \int_{\partial\Omega} \overline{\phi}_n(T_{h+1}(u_n)).\overrightarrow{n}\,d\sigma \\ &\quad - \int_{\partial\Omega} \overline{\phi}_n(T_h(u_n)).\overrightarrow{n}\,d\sigma = 0,\end{aligned}$$

since $u_n = 0$ on $\partial\Omega$, with $\overline{\phi}_n = (\overline{\phi}_{n,1},\ldots,\overline{\phi}_{n,N})$, and since

$$\int_{\{h<|u_n|\le h+1\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx \ge 0.$$

Then

$$\begin{aligned}\int_{\{h+1\leq|u_n|\}}|g_n(x,u_n,\nabla u_n)|\,dx &= \int_{\{h+1\leq|u_n|\}}|g_n(x,u_n,\nabla u_n)|T_1(u_n-T_h(u_n))\,dx\\ &\leq \int_{\{h\leq|u_n|\}}|g_n(x,u_n,\nabla u_n)|T_1(u_n-T_h(u_n))\,dx\\ &\leq \int_{\{h\leq|u_n|\}}|f_n||T_1(u_n-T_h(u_n))|\,dx\\ &\leq \int_{\{h\leq|u_n|\}}|f_n|\,dx.\end{aligned}$$

Thus, for all $\delta>0$, there exist $h(\delta)>0$ such that

$$\int_{\{h(\delta)\leq|u_n|\}}|g_n(x,u_n,\nabla u_n)|\,dx\leq\frac{\delta}{2}. \tag{5.38}$$

For any measurable subset $E\subset\Omega$, we have

$$\begin{aligned}\int_E|g_n(x,u_n,\nabla u_n)|\,dx \quad\leq\quad &\int_{E\cap\{|u_n|<h(\delta)\}}b(k)(h_2(x)\\ &+\varphi(x,|\nabla u_n|)\,dx+\int_{\{|u_n|\geq h(\delta)\}}|g_n(x,u_n,\nabla u_n)|\,dx.\end{aligned} \tag{5.39}$$

Thanks to (5.37), there exists $\beta(\delta)>0$ such that

$$\int_{E\cap\{|u_n|<h(\delta)\}}b(k).(h_2(x)+\varphi(x,|\nabla u_n|)\,dx\leq\frac{\delta}{2}\qquad\text{for}\quad meas(E)\leq\beta(\delta). \tag{5.40}$$

Finally, by combining (5.38) – (5.40), we obtain

$$\int_E|g_n(x,u_n,\nabla u_n)|\,dx\leq\delta,\qquad\qquad\text{with}\quad meas(E)\leq\beta(\delta). \tag{5.41}$$

Then $(g_n(x,u_n,\nabla u_n))_n$ is equi-integrable, and by the Vitali's Theorem we deduce that

$$g_n(x,u_n,\nabla u_n)\longrightarrow g(x,u,\nabla u)\qquad\text{in}\quad L^1(\Omega). \tag{5.42}$$

Step 6 : Passage to the limit.

Let $v\in W_0^1L_\varphi(\Omega)\cap L^\infty(\Omega)$ and $\lambda=k+\|v\|_\infty$ with $k>0$, we will show that

$$\liminf_{n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)\,dx\geq\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx.$$

If $|u_n|>\lambda$ then $|u_n-v|\geq|u_n|-\|v\|_\infty>k$, therefore $\{|u_n-v|\leq k\}\subseteq\{|u_n|\leq\lambda\}$, which implies that

$$\begin{aligned}a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)&=a(x,u_n,\nabla u_n)\nabla(u_n-v)\chi_{\{|u_n-v|\leq k\}}\\ &=a(x,T_\lambda(u_n),\nabla T_\lambda(u_n))(\nabla T_\lambda(u_n)-\nabla v)\chi_{\{|u_n-v|\leq k\}}.\end{aligned} \tag{5.43}$$

Then

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)\,dx &= \int_\Omega a(x,T_\lambda(u_n)\nabla T_\lambda(u_n))(\nabla T_\lambda(u_n)-\nabla v)\chi_{\{|u_n-v|\le k\}}\,dx\\ &= \int_\Omega (a(x,T_\lambda(u_n),\nabla T_\lambda(u_n))-a(x,T_\lambda(u_n),\nabla v))(\nabla T_\lambda(u_n)-\nabla v)\chi_{\{|u_n-v|\le k\}}\,dx\\ &\quad+\int_\Omega a(x,T_\lambda(u_n),\nabla v)(\nabla T_\lambda(u_n)-\nabla v)\chi_{\{|u_n-v|\le k\}}\,dx.\end{aligned} \tag{5.44}$$

We obtain

$$\begin{aligned}\liminf_{n\to+\infty}&\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)\,dx\\ &\ge \int_\Omega (a(x,T_\lambda(u),\nabla T_\lambda(u))-a(x,T_\lambda(u),\nabla v))(\nabla T_\lambda(u)-\nabla v)\chi_{\{|u-v|\le k\}}\,dx\\ &\quad+\lim_{n\to+\infty}\int_\Omega a(x,T_\lambda(u_n),\nabla v)(\nabla T_\lambda(u_n)-\nabla v)\chi_{\{|u_n-v|\le k\}}\,dx.\end{aligned} \tag{5.45}$$

The second term in the right hand side of (5.45) is equal to

$$\int_\Omega a(x,T_\lambda(u),\nabla v)(\nabla T_\lambda(u)-\nabla v)\chi_{\{|u-v|\le k\}}dx.$$

Finally, we get

$$\begin{aligned}\liminf_{n\to+\infty}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)\,dx &\ge \int_\Omega a(x,T_\lambda(u),\nabla T_\lambda(u))(\nabla T_\lambda(u)-\nabla v)\chi_{\{|u-v|\le k\}}\,dx,\\ &= \int_\Omega a(x,u,\nabla u)(\nabla u-\nabla v)\chi_{\{|u-v|\le k\}}\,dx\\ &= \int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx.\end{aligned}$$

Now, taking $T_k(u_n-v)$ as a test function in (P_n) and passing to the limit, we conclude the desired statement. □

References

1. L. Aharouch, E. Azroul; Existance and regurarity of entropy solutions for some nonlinear elliptic equations, (Capri, 2004), Electronic Journal of Differential Equations, Conference 11, 2004, pp 1-10.

2. M. L. Ahmed Oubeid, A. Benkirane, M. Sidi El Vally, Nonlinear Elliptic Equations Involving Measure Data in Musielak-Orlicz-Sobolev Spaces, Journal ofAbstract Diffrential Equations and Applications Volume 4, Number 1, pp. 43Ű57 (2013).

3. M. Ait Khellou, A. Benkirane, S.M. Douiri Existance of solutions for elliptic equations having naturel growth terms in Musielak Orlicz spaces, J. Math. Comput. Sci. 4 (2014), No. 4, 665-688.

4. E. Azroul, H. Hjiaj, A. Touzani; Existence and regularity of entropy solutions for strongly nonlinear $p(x)$-elliptic equations, Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 68, pp. 1Ű27.

5. M.Bendahmane, P. Wittbold, renormalized solutions for nonlinear elliptic equations with variable exponents and L^1 data, Nonlinear Anal. 70 (2009) 567-583.

6. A. Benkirane, J. Bennouna, Existence of Entropy Solutions for some Nonlinear problems in Orlicz spaces, Abstract and Applied Analysis 7:2 (2002) 85Ű102.

7. A. Benkirane, F. Blali, M. Sidi El Vally (Ould Mohamedhen Val), An existence theorem for a strongly Nonlinear elliptique problem in Musielak-Orlicz spaces, APPLICATIONS MATHEMATICAE 41, 2-3 (2014), pp. 175-184.

8. A. Benkirane, M. Sidi El Vally (Ould Mohamedhen val), Variational inequalities in Musielak-Orlicz-Sobolev spaces, Bull. Belg. Math. Soc. Simon Stevin 21 (2014), 787-811.

9. A. Benkirane, M. Sidi El Vally (Ould Mohamedhen Val), Some approximation properties in Musielak-Orlicz- Sobolev spaces, Thai.J. Math. 10 (2012), 371-381.

10. A. Benkirane, M. Sidi El Vally (Ould Mohamedhen Val), An existence result for nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces, Bull. Belg. Math. Soc. Simon Stevin Volume 20, Number 1 (2013), 57-75.

11. A. Benkirane, A. Elmahi, An existence theorem for a strongly non-linear elliptic problem in Orlicz spaces. Nonlinear Anal. Theory Meth. Appl. 1999;36:11Ű24.

12. A. Benkirane, A. Elmahi. Almost everywhere convergence of gradients of solutions to elliptic equations in Orlicz spaces and application. Nonlinear Anal. Theory Meth. Appl. 1997;28:1769-1784.

13. P.Bénilan, L.Boccardo, T.Galouëtt, R.Gariepy, M.Pierre, and J.L. Vázquez, An L^1 theory of existanece and uniqueness of solutions of nonlinear elliptic equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 22 (1995), no. 2, 241-273.

14. L. Boccardo; Some nonlinear Dirichlet problem in L^1 involving lower order terms in divergence form, Progress in elliptic and parabolic partial differential equations (Capri, 1994), Pitman Res. Notes Math. Ser., 350, p. 43-57, Longman, Harlow, 1996.

15. L. Boccardo, D. Giachetti, J. I. Di as, F. Murat; Existence and Regularity of Renormalized Solutions for Some Elliptic Problems Involving Derivations of Nonlinear terms, journal of differential equations., 106, (1993), 215-237.

16. L. Boccardo, D. Giachetti, J. I. Diaz, and F. Murat, Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms, J. Differential Equations 106 (1993), no. 2, 215-237.

17. R. J. DiPerna and P.L. Lions, On the Cauchy problem for Boltzmann equations: Global existence and weak stability, Ann. of Math. (2) 130 (1989), no. 2, 321-366.

18. J.-P. Gossez, Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients, Trans. Amer. Math. Soc. 190 (1974), 163-205.

19. P. Gwiazda, A. Swierczewska-Gwiazda, On non-newtonian fluids with the property of rapid thickening under different stimulus, Math. Models Methods Appl. Sci. 18 (2008), 1073-1092.

20. Lars Diening, Petteri Harjulehto, Peter Hästö, Michael Rùđièka, Lebesgue and Sobolev spaces with variable exponents, (2010).

21. J. Musielak ; Modular spaces and Orlicz spaces, Lecture Notes in Math. 1034 (1983).

22. M. Sidi El Vally; Strongly Nonlinear Elliptic Problems in MusielakŰOrliczŰSobolev Spaces, Advances in Dynamical Systems and Applications ISSN 0973-5321, Volume 8, Number 1, pp. 115Ű124 (2013).

M.S.B. Elemine Vall
University of Sidi Mohamed Ibn Abdellah, Faculty of Sciences Dhar El Mahraz, Laboratory LAMA, Department of Mathematics, B.P. 1796 Atlas Fez, Morocco
E-mail address: saad2012bouh@gmail.com

and

A. Ahmed
University of Sidi Mohamed Ibn Abdellah, Faculty of Sciences Dhar El Mahraz, Laboratory LAMA, Department of Mathematics, B.P. 1796 Atlas Fez, Morocco
E-mail address: ahmedmath2001@gmail.com

and

A. Touzani
University of Sidi Mohamed Ibn Abdellah, Faculty of Sciences Dhar El Mahraz, Laboratory LAMA, Department of Mathematics, B.P. 1796 Atlas Fez, Morocco
E-mail address: atouzani07@gmail.com

and

A. Benkirane
University of Sidi Mohamed Ibn Abdellah, Faculty of Sciences Dhar El Mahraz, Laboratory LAMA, Department of Mathematics, B.P. 1796 Atlas Fez, Morocco
E-mail address: abd.benkirane@gmail.com