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Unramified extensions of some cyclic quartic fields

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ABSTRACT: Let K be a cyclic quartic field such that its 2-class group $C_{K,2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. In this paper we give the generators of $C_{K,2}$ and we determine the fourteen unramified extensions of K.

Key Words: Unramified quadratic and biquadratic extensions, Hilbert 2-class field, cyclic quartic field, 2-class group, generators.

Contents

1. Introduction

Let K be an imaginary cyclic quartic extension of the rational field $\mathbb{Q}$, $K_2^{(1)}$ be the Hilbert 2-class field of K, $K_2^{(2)}$ be the Hilbert 2-class field of $K_2^{(1)}$, $K^{(*)}$ be the genus field of K, that is the maximal absolute abelian subfield of $K_2^{(1)}/K$ and let $C_{K,2}$ be the 2-class group of K. If $C_{K,2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, then A. Azizi and M. Talbi have studied this situation and answered concretely to the capitulation problem of $C_{K,2}$ in the three subfields of $K_2^{(1)}/K$ (see [1,2,3,4]...). Let $K=\mathbb{Q}(\sqrt{-2p\varepsilon\sqrt{\ell}})$ where $\ell\equiv 5 \pmod 8$ and $p\equiv 1 \pmod 4$ are different primes and ε is the fundamental unit of $k=\mathbb{Q}(\sqrt{\ell})$. If $\left(\frac{p}{l}\right)=-1$, then $C_{K,2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and the capitulation problem has been studied in [3]. But if $\left(\frac{p}{l}\right)=1$, then $K^{(*)}\subsetneq K_2^{(1)}$ and there exist two prime ideals $\mathcal{B}_1$, $\mathcal{B}_2$ of k such that $\mathcal{B}_1\mathcal{B}_2=(p)$. If h_0 denotes the class number of k, then $\mathcal{B}_1^{h_0}=(a+b\sqrt{\ell})$ and $\mathcal{B}_2^{h_0}=(a-b\sqrt{\ell})$; one can show that four prime ideals of k ramify in K which are: the prime ideal of k above 2, $(\sqrt{\ell})$, $\mathcal{B}_1$ and $\mathcal{B}_2$ (see [8]), so $C_{K,2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ if and only if $\left(\frac{2}{p}\right)=-\left(\frac{p}{\ell}\right)_4$ (see [6, Theorme 4, p. 68]). By class field theory, there are seven unramified quadratic fields over K and seven unramified biquadratic fields over K which are contained in Hilbert 2-class field $K_2^{(1)}$. The following diagram illustrates the situation.

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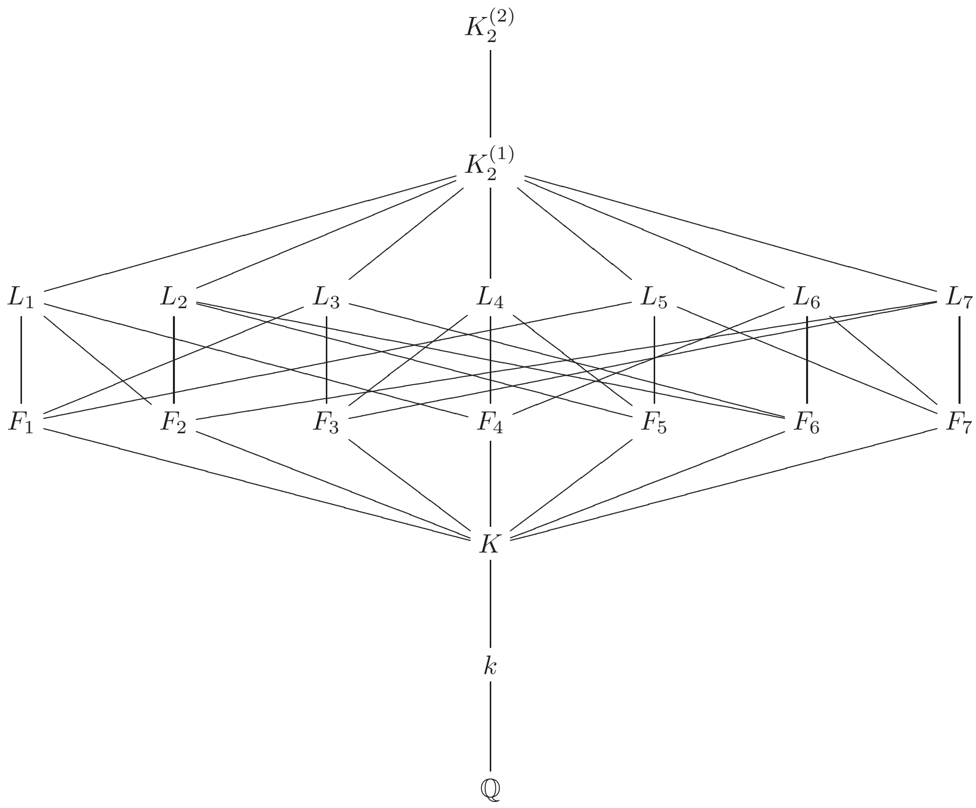

Diagram

In [5], the authors studied the capitulation problem of the 2-classes of the biquadratic fields $\mathbb{Q}(\sqrt{p_1p_2q}, \sqrt{-1})$ with 2-class group isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, in its 14 unramified abelian extension within the first Hilbert 2-class field. In the case where $K = \mathbb{Q}(\sqrt{-2p\varepsilon\sqrt{\ell}})$ with $C_{K,2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, we give, in this paper, the generators of $C_{K,2}$ and we will build the fourteen unramified abelian extensions within $K_2^{(1)}$ using [7] and [2,3] .

2. The generators of $C_{K,2}$

Lemma 2.1. *Let $p \equiv 1 \pmod 4$ and $\ell \equiv 5 \pmod 8$ be different primes. Put $k = \mathbb{Q}(\sqrt{\ell})$ and denote by ε its fundamental unit. Let $K = \mathbb{Q}(\sqrt{-2p\varepsilon\sqrt{\ell}})$, then the genus field of K is $K^{(*)} = K(\sqrt{p}, \sqrt{2})$.*

Proof: As ℓ and p are the unique primes of $\mathbb{Q}$ different from 2 which ramify in K, of ramification index $e_\ell = 4$ and $e_p = 2$ respectively, since the relative discriminant of K/k is $\Delta_{K/k} = (8p\sqrt{\ell})$; then, according to [9, Theorem 4, p. 48 – 49], we have $K^{(*)} = M_\ell M_p K$ where M_ℓ (respectively M_p) is the unique subfield of the ℓ-th (respectively p-th) cyclotomic number field $\mathbb{Q}(\xi_\ell)$ (respectively $\mathbb{Q}(\xi_p)$) of degree $e_\ell = 4$ (respectively $e_p = 2$). Moreover, it is known that $M_\ell = \mathbb{Q}(\sqrt{-\varepsilon\sqrt{\ell}})$ (see [10, Proposition 5.9, p. 160]) and $M_p = \mathbb{Q}(\sqrt{p})$. Thus $K^{(*)} = K(\sqrt{p}, \sqrt{2})$. □

Lemma 2.2. *Let $k = \mathbb{Q}(\sqrt{\ell})$ where ℓ is a prime number such that $\ell \equiv 5 \pmod 8$, h_0 be the class number of k, and p be a prime number such that $p \equiv 1 \pmod 4$. Assume that $\left(\frac{p}{\ell}\right) = 1$, then $p^{h_0} = \pi_1\pi_2$ with $\pi_1 = a + b\sqrt{\ell}$ and $\pi_2 = a - b\sqrt{\ell}$.*

1. *If $\left(\frac{\ell}{p}\right)_4 \neq \left(\frac{p}{\ell}\right)_4$, then the equation $-\pi_i \equiv x^2 \pmod 4$ admits solution in k;*
2. *If $\left(\frac{\ell}{p}\right)_4 = \left(\frac{p}{\ell}\right)_4$, then the equation $\pi_i \equiv x^2 \pmod 4$ admits solution in k.*

Proof: See [2, p. 277–280]. □

Theorem 2.3. *Let $K = \mathbb{Q}(\sqrt{-2p\varepsilon\sqrt{\ell}})$ where ε is the fundamental unit of $k = \mathbb{Q}(\sqrt{\ell})$, $\ell \equiv 5 \pmod 8$ and $p \equiv 1 \pmod 4$ are different primes such that $\left(\frac{p}{\ell}\right) = 1$ and $\left(\frac{2}{p}\right) = -\left(\frac{p}{\ell}\right)_4$. Then $C_{K,2} = \langle [\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are the prime ideals in K above p and $\mathcal{H}$ is the prime ideal in K above 2.*

Proof: Recall that $p\mathcal{O}_K = \mathcal{P}_1^2\mathcal{P}_2^2 = \pi_1\pi_2\mathcal{O}_K$ with $\mathcal{O}_K$ the ring of integers of K, $\pi_1 = a + b\sqrt{\ell}$ and $\pi_2 = a - b\sqrt{\ell}$.

The class $[\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}]$ is of order 2. In fact, if $\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0} = (\alpha)$ for any α in K, this is equivalent to $(p^{h_0}) = (\alpha^2)$ in K. Then there exists ε' a unit of K such that $p^{h_0}\varepsilon' = \alpha^2$, thus $p^{h_0}\varepsilon' = (c + d\sqrt{-2p\varepsilon\sqrt{\ell}})^2 = c^2 - 2p\varepsilon\sqrt{l}d^2 + 2cd\sqrt{-2p\varepsilon\sqrt{\ell}}$ with c and d in k, and as $\{\varepsilon\}$ is a fundamental system of units of K and $\sqrt{-1} \notin K$, then $p^{h_0}\varepsilon' \in k$ therefore $c = 0$ or $d = 0$. If $d = 0$, then $p^{h_0}\varepsilon' = c^2$, thus $\pm p^{h_0} = c^2$ or $p^{h_0}\varepsilon = c^2$ in k, which gives that $\sqrt{\pm p} \in k$ in the first case and $\sqrt{-1} \in \mathbb{Q}$ in the second case, which is impossible, and similarly, if $c = 0$ we find that $\pm\ell$ is a square in $\mathbb{Q}$, which is not the case.

The class $[\mathcal{P}_i^{h_0}]$ is of order 2. In fact, suppose that $\mathcal{P}_i^{h_0} = (\alpha)$ for any α in K, then $\mathcal{P}_i^{2h_0} = (\alpha)^2$, this is equivalent to $(\pi_i) = (\alpha^2)$ in K. Then there exists ε' a unit of K such that $\pi_i\varepsilon' = \alpha^2 = (c+d\sqrt{-2p\varepsilon\sqrt{\ell}})^2 = c^2 - 2p\varepsilon\sqrt{\ell}d^2 + 2cd\sqrt{-2p\varepsilon\sqrt{\ell}}$ with c and d in k, then $c = 0$ or $d = 0$. If $d = 0$, multiplying by π_j for $j \neq i$, we get $p^{h_0}\varepsilon' = \pi_j c^2$. So by applying the norm in $k/\mathbb{Q}$, we find that $\pm p^{h_0} = N_{k/\mathbb{Q}}(c)^2$, this means that $\pm p$ is a square in $\mathbb{Q}$, which is impossible. In a similar way if $c = 0$, we get $\pm\ell p$ is a square in $\mathbb{Q}$, which is impossible.

The class $[\mathcal{H}]$ is of order 2. In fact, suppose that $\mathcal{H} = (\alpha)$ for any α in K, then $\mathcal{H}^2 = (\alpha)^2$, this is equivalent to $(2) = (\alpha^2)$ in K. Then there exists ε' a unit of K such that $2\varepsilon' = \alpha^2 = (c+d\sqrt{-2p\varepsilon\sqrt{\ell}})^2 = c^2 - 2p\varepsilon\sqrt{\ell}d^2 + 2cd\sqrt{-2p\varepsilon\sqrt{\ell}}$ with c and d in k, then $c = 0$ or $d = 0$. If $d = 0$, then $2\varepsilon' = c^2$, thus $\pm 2 = c^2$ or $2\varepsilon = c^2$ in k, which gives that $\sqrt{\pm 2} \in k$ in the first case and $\sqrt{-1} \in \mathbb{Q}$ in the second case, which is impossible. If $c = 0$, then $2\varepsilon' = -2p\varepsilon\sqrt{\ell}d^2$, so by applying the norm in $k/\mathbb{Q}$, we find that $\pm l$ is a square in $\mathbb{Q}$, which is impossible.

The class $[\mathcal{H}\mathcal{P}_i^{h_0}]$ is of order 2. In fact, suppose that $\mathcal{H}\mathcal{P}_i^{h_0} = (\alpha)$ for any α in K, then $\mathcal{H}^2\mathcal{P}_i^{2h_0} = (\alpha)^2$, this is equivalent to $(2\pi_i) = (\alpha^2)$ in K. Then there exists ε' a unit of K such that $2\pi_i\varepsilon' = \alpha^2 = (c+d\sqrt{-2p\varepsilon\sqrt{\ell}})^2 = c^2 - 2p\varepsilon\sqrt{\ell}d^2 + 2cd\sqrt{-2p\varepsilon\sqrt{\ell}}$ with c and d in k, then $c = 0$ or $d = 0$. If $d = 0$, so by applying the norm in $k/\mathbb{Q}$, we find that $\pm p$ is a square in $\mathbb{Q}$, which is impossible. If $c = 0$, we get $\pm\ell p$ is a square in $\mathbb{Q}$, which is impossible.

The class $[\mathcal{H}\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}]$ is of order 2. In fact, if $\mathcal{H}\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0} = (\alpha)$ for any α in K, this is equivalent to $(2p^{h_0}) = (\alpha^2)$ in K. Then there exists ε' a unit of K such that $2p^{h_0}\varepsilon' = \alpha^2 = (c + d\sqrt{-2p\varepsilon\sqrt{\ell}})^2 = c^2 - 2p\varepsilon\sqrt{l}d^2 + 2cd\sqrt{-2p\varepsilon\sqrt{\ell}}$ with c and d in k, since $2p^{h_0}\varepsilon' \in k$ therefore $c = 0$ or $d = 0$. If $d = 0$, then $2p^{h_0}\varepsilon' = c^2$, thus $\pm 2p^{h_0} = c^2$ or $2p^{h_0}\varepsilon = c^2$ in k, which gives that $\sqrt{\pm 2p} \in k$ in the first case and $\sqrt{-1} \in \mathbb{Q}$ in the second case, which is impossible. If $c = 0$, we find that $\pm\ell$ is a square in $\mathbb{Q}$, which is absurd.

Thus $\langle[\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}]\rangle$ is of type $(2, 2, 2)$, then $C_{K,2} = \langle[\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}]\rangle$. □

3. The unramified extensions of K

Let $M = N(\sqrt{\alpha})$ be an extension of a number field N contains the 2-roots of unity, where α is a square free element of N coprime to 2, it is well known that M is unramified extension of N if and only if the principal ideal generated by α is the square of an ideal of N and the equation $\alpha \equiv x^2 \pmod 4$ admits solution in N (see [7]).

Lemma 2.1 and Lemma 2.2 allow us to deduce the following Theorem:

Theorem 3.1. *Let $K = \mathbb{Q}(\sqrt{-2p\varepsilon\sqrt{\ell}})$ where ε is the fundamental unit of $\mathbb{Q}(\sqrt{\ell})$, $\ell \equiv 5 \pmod 8$ and $p \equiv 1 \pmod 4$ are different primes such that $\left(\frac{p}{\ell}\right) = 1$ and $\left(\frac{2}{p}\right) = -\left(\frac{p}{\ell}\right)_4$. Then the fourteen unramified extensions of the imaginary cyclic quartic field K are given by:*

1. *If $\left(\frac{\ell}{p}\right)_4 \neq \left(\frac{p}{\ell}\right)_4$, then*
   - *The unramified quadratic extensions of K are:*
   $$F_1 = K(\sqrt{-\pi_1}) \quad \simeq \quad F_2 = K(\sqrt{-\pi_2}), \quad F_3 = K(\sqrt{2}), \quad F_4 = K(\sqrt{p}),$$
   $$F_5 = K(\sqrt{2p}) \quad \text{and} \quad F_6 = K(\sqrt{-2\pi_1}) \quad \simeq \quad F_7 = K(\sqrt{-2\pi_2}).$$
   - *The unramified biquadratic extensions of K are:*
   $$L_1 = F_1F_2 = F_1F_4 = F_2F_4, \qquad L_2 = F_2F_6 = F_2F_5 = F_6F_5,$$
   $$L_3 = F_1F_3 = F_1F_6 = F_3F_6, \quad L_4 = K^{(*)}, \quad L_5 = F_1F_7 = F_1F_5 = F_7F_5,$$
   $$L_6 = F_6F_7 = F_6F_4 = F_7F_4 \quad \text{and} \quad L_7 = F_2F_3 = F_2F_7 = F_3F_7.$$

2. *If* $\left(\frac{\ell}{p}\right)_4 = \left(\frac{p}{\ell}\right)_4$, *then*

   - *The unramified quadratic extensions of* K *are:*

$$F_1 = K(\sqrt{\pi_1}) \quad \simeq \quad F_2 = K(\sqrt{\pi_2}), \quad F_3 = K(\sqrt{2}), \quad F_4 = K(\sqrt{p}),$$
$$F_5 = K(\sqrt{2p}) \quad \text{and} \quad F_6 = K(\sqrt{2\pi_1}) \quad \simeq \quad F_7 = K(\sqrt{2\pi_2}).$$

   - *The unramified biquadratic extensions of* K *are:*

$$L_1 = F_1F_2 = F_1F_4 = F_2F_4, \qquad L_2 = F_2F_6 = F_2F_5 = F_6F_5,$$
$$L_3 = F_1F_3 = F_1F_6 = F_3F_6, \quad L_4 = K^{(*)}, \quad L_5 = F_1F_7 = F_1F_5 = F_7F_5,$$
$$L_6 = F_6F_7 = F_6F_4 = F_7F_4 \quad \text{and} \quad L_7 = F_2F_3 = F_2F_7 = F_3F_7.$$

Note that π_i *is defined in the Lemma 2.2.*

Proof: According to Lemma 2.1, $K^{(*)} = K(\sqrt{p}, \sqrt{2})$, then $F_3 = K(\sqrt{2})$, $F_4 = K(\sqrt{p})$ and $F_5 = K(\sqrt{2p})$.

1. If $\left(\frac{\ell}{p}\right)_4 \neq \left(\frac{p}{\ell}\right)_4$, then, from Lemma 2.2, the equation $-\pi_i \equiv x^2 \pmod 4$ admits solution in k i.e. admits solution in K. Since $\mathcal{B}_i^{h_0} = (\pi_i)$ and $\mathcal{B}_i$ ramifies in K, then $(-\pi_i) = (\mathcal{P}_i^{h_0})^2$ with $\mathcal{P}_i$ an ideal of K. Therefore $K(\sqrt{-\pi_i})$ is an unramified quadratic extension of K, this implies that $F_i = K(\sqrt{-\pi_i})$ for $i = 1, 2$. Since $K(\sqrt{-2\pi_i})$ is a subfield of $K(\sqrt{2}, \sqrt{-\pi_i})$ which is unramified over K, then $K(\sqrt{-2\pi_i})$ is unramified over K, thus $F_6 = K(\sqrt{-2\pi_1})$ and $F_7 = K(\sqrt{-2\pi_2})$. On the other hand, the extensions F_i are pairwise different for $i = 1, 2, 6, 7$, because for example if $F_1 = F_2$, then there exists $t \in K$ such that $\pi_1 = t^2\pi_2$, this yields that $p^{h_0} = t^2\pi_2^2$, which is not the case, since $\sqrt{p} \notin K$. Similarly, we show the other cases. Also $F_i \neq F_j$ for $(i, j) \in \{1, 2, 6, 7\} \times \{3, 4, 5\}$ (see the following Remark). It is easy to see that $F_1 \simeq F_2$ and $F_6 \simeq F_7$.

2. We proceed as in 1. We conclude easily that

$$L_1 = F_1F_2 = F_1F_4 = F_2F_4, \qquad L_2 = F_2F_6 = F_2F_5 = F_6F_5,$$
$$L_3 = F_1F_3 = F_1F_6 = F_3F_6, \quad L_4 = K^{(*)}, \quad L_5 = F_1F_7 = F_1F_5 = F_7F_5,$$
$$L_6 = F_6F_7 = F_6F_4 = F_7F_4 \quad \text{and} \quad L_7 = F_2F_3 = F_2F_7 = F_3F_7.$$

□

Remark 3.1. *The base field* K *admits tree unramified quadratic extensions absolutely abelian of type* $(2, 4)$, *which are intermediate fields between* K *and its genus field* $K^{(*)}$, *and four unramified quadratic extensions absolutely non-Galois which are* F_1, F_2, F_6 *and* F_7. *Moreover, the field* K *admits tree unramified biquadratic extensions absolutely Galois which are* L_1, L_4 *and* L_6, *and four unramified biquadratic extensions absolutely non-Galois which are* L_2, L_3, L_5 *and* L_7.

Example 3.1. *Let* $K = \mathbb{Q}(\sqrt{-2.17\varepsilon\sqrt{13}})$ *where* $\varepsilon = \frac{3+\sqrt{13}}{2}$. *As* $13 \equiv 5 \pmod 8$, $17 \equiv 1 \pmod 4$ *and* $\left(\frac{2}{17}\right) = -\left(\frac{17}{13}\right)_4 = \left(\frac{13}{17}\right)_4 = 1$, *then*

$$F_1 = K(\sqrt{-\pi_1}), \qquad F_2 = K(\sqrt{-\pi_2}), \qquad F_3 = K(\sqrt{2}), \qquad F_4 = K(\sqrt{17}),$$

$$F_5 = K(\sqrt{2.17}), \qquad F_6 = K(\sqrt{-2\pi_1}) \qquad \textit{and} \qquad F_7 = K(\sqrt{-2\pi_2}),$$

$$\textit{with} \qquad \pi_1 = 15 + 4\sqrt{13} \qquad \textit{and} \qquad \pi_2 = 15 - 4\sqrt{13}.$$

Moreover,

$$L_1 = K(\sqrt{-\pi_1}, \sqrt{-\pi_2}), \qquad L_2 = K(\sqrt{-2\pi_1}, \sqrt{-\pi_2}), \qquad L_3 = K(\sqrt{-\pi_1}, \sqrt{2}),$$

$$L_4 = K^{(*)} = K(\sqrt{2}, \sqrt{17}), \quad L_5 = K(\sqrt{-\pi_1}, \sqrt{-2\pi_2}), \quad L_6 = K(\sqrt{-2\pi_1}, \sqrt{-2\pi_2}),$$

$$L_7 = K(\sqrt{2}, \sqrt{-\pi_2}) \qquad \textit{and} \qquad K_2^{(1)} = K(\sqrt{2}, \sqrt{17}, \sqrt{-\pi_i}).$$

Example 3.2. *Let* $K = \mathbb{Q}(\sqrt{-2.89\varepsilon\sqrt{5}})$ *where* $\varepsilon = \frac{1+\sqrt{5}}{2}$. *As* $5 \equiv 5 \pmod 8$, $89 \equiv 1 \pmod 4$ *and* $\left(\frac{2}{89}\right) = -\left(\frac{89}{5}\right)_4 = -\left(\frac{5}{89}\right)_4 = 1$, *then*

$$F_1 = K(\sqrt{\pi_1}), \qquad F_2 = K(\sqrt{\pi_2}), \qquad F_3 = K(\sqrt{2}), \qquad F_4 = K(\sqrt{89}),$$

$$F_5 = K(\sqrt{2.89}), \qquad F_6 = K(\sqrt{2\pi_1}) \qquad \textit{and} \qquad F_7 = K(\sqrt{2\pi_2}),$$

$$\textit{with} \qquad \pi_1 = 13 + 4\sqrt{5} \qquad \textit{and} \qquad \pi_2 = 13 - 4\sqrt{5}.$$

Moreover,

$$L_1 = K(\sqrt{\pi_1}, \sqrt{\pi_2}), \qquad L_2 = K(\sqrt{2\pi_1}, \sqrt{\pi_2}), \qquad L_3 = K(\sqrt{\pi_1}, \sqrt{2}),$$

$$L_4 = K^{(*)} = K(\sqrt{2}, \sqrt{89}), \quad L_5 = K(\sqrt{\pi_1}, \sqrt{2\pi_2}), \quad L_6 = K(\sqrt{2\pi_1}, \sqrt{2\pi_2}),$$

$$L_7 = K(\sqrt{2}, \sqrt{\pi_2}) \qquad \textit{and} \qquad K_2^{(1)} = K(\sqrt{2}, \sqrt{89}, \sqrt{\pi_i}).$$

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