Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) v. 36 1 (2018): 177–193.
ISSN-00378712 in press
doi:10.5269/bspm.v36i1.28123

Finite Integral Formulas Involving Aleph Function

Dinesh Kumar, R.K. Saxena and Jeta Ram

abstract: In the present work we derive various integral formulas involving ℵ-function multiplied with algebraic functions and special functions.

Key Words: Aleph function, I-function, H-function, Jacobi polynomials, Legendre function, Bessel Maitland function, Hypergeometric function.

Contents

0mathclass33C05, 33C20, 33C45, 33C60.

1. Introduction and preliminaries

Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}_0^-$ and $\mathbb{N}$ be sets of complex numbers, real and positive numbers, nonpositive and positive integers, respectively, $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. The Aleph (ℵ)-function, which is a very general higher transcendental function and was introduced by Südland *et al.* [22,23], is defined by means of Mellin-Barnes type integral in the following manner (see, *e.g.*, [14,15] and [18]):

$$\aleph[z] = \aleph^{m,n}_{p_k,q_k,\tau_k;r}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},[\tau_j(a_{jk},A_{jk})]_{n+1,p_k;r}\\(b_j,B_j)_{1,m},[\tau_j(b_{jk},B_{jk})]_{m+1,q_k;r}\end{matrix}\right.\right] := \frac{1}{2\pi i}\int_{\mathcal{L}}\Omega^{m,n}_{p_k,q_k,\tau_k;r}(s)\, z^{-s}\, ds, \tag{1.1}$$

Submitted June 10, 2015. Published January 21, 2016

where $z \in \mathbb{C} \setminus \{0\}$, $i = \sqrt{-1}$, and

$$\Omega\,{}^{m,n}_{p_k,q_k,\tau_k;r}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\,.\,\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{k=1}^{r}\tau_k\prod_{j=m+1}^{q_k}\Gamma(1-b_{jk}-B_{jk}s)\,.\,\prod_{j=n+1}^{p_k}\Gamma(a_{jk}+A_{jk}s)}. \tag{1.2}$$

Here Γ denotes the familiar Gamma function; The integration path $\mathcal{L} = \mathcal{L}_{i\,\gamma\,\infty}$ $(\gamma \in \mathbb{R})$ extends from $\gamma - i\infty$ to $\gamma + i\infty$; The poles of the Gamma functions $\Gamma(1-a_j-A_js)$ $(j,\,n \in \mathbb{N};\ 1 \le j \le n)$ do not coincide with those of $\Gamma(b_j+B_js)$ $(j,\,m \in \mathbb{N};\ 1 \le j \le m)$; The parameters $p_k,\,q_k \in \mathbb{N}_0$ satisfy the conditions $0 \le n \le p_k,\ \ 1 \le m \le q_k,\ \ \tau_k > 0\ (1 \le k \le r)$; The parameters $A_j,\,B_j,\,A_{jk},\,B_{jk} > 0$ and $a_j,\,b_j,\,a_{jk},\,b_{jk} \in \mathbb{C}$; The empty product in (1.2) is (as usual) understood to be unity. The existence conditions for the defining integral (1.1) are given below:

$$\varphi_l > 0 \quad \text{and} \quad |\arg(z)| < \frac{\pi}{2}\varphi_l \quad \left(l \in \overline{1,r}\right) \tag{1.3}$$

and

$$\varphi_l \ge 0, \quad |\arg(z)| < \frac{\pi}{2}\varphi_l \quad \text{and} \quad \Re\{\zeta_l\} + 1 < 0, \tag{1.4}$$

where

$$\varphi_l := \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_l\left(\sum_{j=n+1}^{p_l} A_{jl} + \sum_{j=m+1}^{q_l} B_{jl}\right) \tag{1.5}$$

and

$$\zeta_l := \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_l\left(\sum_{j=m+1}^{q_l} b_{jl} - \sum_{j=n+1}^{p_l} a_{jl}\right) + \frac{1}{2}(p_l - q_l) \quad \left(l \in \overline{1,r}\right). \tag{1.6}$$

Remark 1.1. *The expression in* (1.1) *of the Aleph-function does not follow completely the notational convention of the Fox's H-function* (*see* Remark 1.2). *Namely, in the $\aleph$-functions, the kernel $\Omega\,{}^{m,n}_{p_k,q_k,\tau_k;r}(s)$, parameter couples $(a_j,A_j)_{1,n}$, $(b_j,B_j)_{1,m}$ build the Gamma function terms exclusively in the numerator, and $[\tau_j(a_{jk},A_{jk})]_{n+1,p_k}$, $[\tau_j(b_{jk},B_{jk})]_{n+1,q_k}$ build the linear combination exclusively in the denominator, while, for the $H^{m,n}_{p,q}[z]$, both upper $(a_j,A_j)_{1,p}$ and lower couples of parameters $(b_j,B_j)_{1,q}$ play roles in forming both numerator and denominator terms according to m and n.*

Remark 1.2. *Setting $\tau_j = 1$ $(j \in \overline{1,r} := \{1,2,\ldots,r\})$ in* (1.1) *yields the I-function* (*see* [19]) *whose further special case when $r = 1$ reduces to the familiar H-function* (*see* [8,9]).

Definition 1. Gamma Function:
The simplest interpretation of the gamma function is simply the generalization of

the factorial for all real numbers. The definition of gamma function is given by

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt, \qquad (\forall\, z \in \Re)\,. \tag{1.7}$$

Definition 2. Beta Function:
Also known as the Euler Integral of the First Kind, the Beta function $B(p,q)$ is in important relationship in factorial calculus. Its solution not is only defined through the use of multiple Gamma Functions, but furthermore shares a form that is characteristically similar to the Fractional Differintegral of many functions, particularly polynomials of the form t^α and the Mittag-Leffler Function. The Beta Integral and its solution in terms of the Gamma function as given following:

$$B(p,q) = \int_0^1 (1-t)^{p-1}\, t^{q-1}\, dt = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} = B(q,p)\,, \quad (p,q \in \Re)\,. \tag{1.8}$$

2. Integrals involving ℵ-function with Algebraic function

In this section we calculate the ℵ-function with some algebraic functions.

$$\begin{aligned}
I_1 &= \int_0^1 y^{-\rho}\,(1-y)^{\rho-\sigma-1}\,\aleph^{m,n}_{p_k,q_k,\tau_k;\,r}\,[zy]\,dy \\
&= \frac{1}{2\pi i}\int_{\mathcal{L}} \Omega^{m,n}_{p_k,q_k,\tau_k;r}\,(s)\; z^{-s}\left\{\int_0^1 y^{-\rho-s+1-1}\,(1-y)^{\rho-\sigma-1}\,dy\right\} ds \\
&= \frac{1}{2\pi i}\int_{\mathcal{L}} \Omega^{m,n}_{p_k,q_k,\tau_k;r}\,(s)\; z^{-s}\left\{B\,(1-\rho-s,\rho-\sigma)\right\} ds \\
&= \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j+B_j s)\;.\;\prod_{j=1}^{n}\Gamma(1-a_j-A_j s)}{\sum_{k=1}^{r}\tau_k\prod_{j=m+1}^{q_k}\Gamma(1-b_{jk}-B_{jk}s)\;.\;\prod_{j=n+1}^{p_k}\Gamma(a_{jk}+A_{jk}s)} \\
&\times\frac{\Gamma(1-\rho-s)\,\Gamma(\rho-\sigma)}{\Gamma(1-\sigma-s)}\,z^{-s}\,ds \\
&= \Gamma(\rho-\sigma)\;\aleph^{m,n+1}_{p_k+1,q_k+1,\tau_k;\,r}\left[z\,\middle|\,\begin{matrix}(\rho,1)\,,(a_j,A_j)_{1,n}\,,\;[\tau_j\,(a_{jk},A_{jk})]_{n+1,p_k;r}\\ (b_j,B_j)_{1,m}\,,\;(\sigma,1)\,,[\tau_j\,(b_{jk},B_{jk})]_{m+1,q_k;r}\end{matrix}\right].
\end{aligned} \tag{2.1}$$

$$I_2 = \int_0^1 x^{\rho-1} (1-x)^{\sigma-1} \aleph_{p_k,q_k,\tau_k;r}^{m,n} [zx]\, dx$$
$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \Omega_{p_k,q_k,\tau_k;r}^{m,n} (s)\ z^{-s} \left\{ \int_0^1 x^{\rho-s-1} (1-x)^{\sigma-1} dx \right\} ds$$
$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \Omega_{p_k,q_k,\tau_k;r}^{m,n} (s)\ z^{-s} \{B(\rho-s,\sigma)\}\, ds$$
$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s)\ .\ \prod_{j=1}^{n} \Gamma(1-a_j-A_j s)}{\sum_{k=1}^{r} \tau_k \prod_{j=m+1}^{q_k} \Gamma(1-b_{jk}-B_{jk}s)\ .\ \prod_{j=n+1}^{p_k} \Gamma(a_{jk}+A_{jk}s)}$$
$$\times\ \frac{\Gamma(\rho-s)\,\Gamma(\sigma)}{\Gamma(\rho+\sigma-s)} z^{-s}\, ds$$
$$= \Gamma(\sigma)\ \aleph_{p_k+1,q_k+1,\tau_k;r}^{m,n+1} \left[z \middle| \begin{matrix} (1-\rho,1), (a_j,A_j)_{1,n}, [\tau_j(a_{jk},A_{jk})]_{n+1,p_k;r} \\ (b_j,B_j)_{1,m}, (1-\rho-\sigma,1), [\tau_j(b_{jk},B_{jk})]_{m+1,q_k;r} \end{matrix} \right]. \tag{2.2}$$

$$I_3 = \int_1^\infty x^{-\rho} (x-1)^{\sigma-1} \aleph_{p_k,q_k,\tau_k;r}^{m,n} [zx]\, dx$$
$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \Omega_{p_k,q_k,\tau_k;r}^{m,n} (s)\ z^{-s} \left\{ \int_1^\infty x^{-\rho-s} (x-1)^{\sigma-1} dx \right\} ds.$$

Putting $x = t+1 \Rightarrow dx = dt$, and using the following relation:

$$\begin{aligned} \Gamma(\alpha)\,\Gamma(\beta) &= \Gamma(\alpha+\beta) \int_0^\infty x^{\alpha-1} (1+x)^{-(\alpha+\beta)}\, dx \\ &= \Gamma(\alpha+\beta) \int_0^\infty x^{\beta-1} (1+x)^{-(\alpha+\beta)}\, dx, \end{aligned} \tag{2.3}$$

we get

$$I_3 = \frac{1}{2\pi i} \int_{\mathcal{L}} \Omega_{p_k,q_k,\tau_k;r}^{m,n} (s)\ z^{-s} \left\{ \int_0^\infty t^{\sigma-1} (1+t)^{-(\sigma+\rho+s-\sigma)} dt \right\} ds$$
$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s)\ .\ \prod_{j=1}^{n} \Gamma(1-a_j-A_j s)}{\sum_{k=1}^{r} \tau_k \prod_{j=m+1}^{q_k} \Gamma(1-b_{jk}-B_{jk}s)\ .\ \prod_{j=n+1}^{p_k} \Gamma(a_{jk}+A_{jk}s)}$$
$$\times\ \frac{\Gamma(\sigma)\,\Gamma(\rho+s-\sigma)}{\Gamma(\rho+s)} z^{-s}\, ds$$
$$= \Gamma(\sigma)\ \aleph_{p_k+1,q_k+1,\tau_k;r}^{m+1,n} \left[z \middle| \begin{matrix} (a_j,A_j)_{1,n}, (\rho,1), [\tau_j(a_{jk},A_{jk})]_{n+1,p_k;r} \\ (\rho-\sigma,1), (b_j,B_j)_{1,m}, [\tau_j(b_{jk},B_{jk})]_{m+1,q_k;r} \end{matrix} \right]. \tag{2.4}$$

$$I_4 = \int_0^{\infty} x^{\rho-1} (x+\beta)^{-\sigma} \aleph^{m,n}_{p_k,q_k,\tau_k;\,r} [zx] \, dx$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \Omega^{m,n}_{p_k,q_k,\tau_k;r} (s) \, z^{-s} \beta^{-\sigma} \left\{ \int_0^{\infty} x^{\rho-s-1} \left(\frac{x}{\beta} + 1 \right)^{-\sigma} dx \right\} ds.$$

Putting $x = t\beta \Rightarrow dx = \beta dt$, then we arrive at

$$I_4 = \frac{1}{2\pi i} \beta^{\rho-\sigma} \int_{\mathcal{L}} \Omega^{m,n}_{p_k,q_k,\tau_k;r} (s) \, (z\beta)^{-s} \left\{ \int_0^{\infty} t^{\rho-s-1} (1+t)^{-(\sigma-\rho+s+\rho-s)} dt \right\} ds$$

$$= \frac{1}{2\pi i} \beta^{-\sigma+\rho} \int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \, . \, \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)}{\sum_{k=1}^{r} \tau_k \prod_{j=m+1}^{q_k} \Gamma(1 - b_{jk} - B_{jk} s) \, . \, \prod_{j=n+1}^{p_k} \Gamma(a_{jk} + A_{jk} s)}$$

$$\times \frac{\Gamma(\rho - s) \Gamma(\sigma - \rho + s)}{\Gamma(\sigma)} (z\beta)^{-s} \, ds$$

$$= \frac{\beta^{\rho-\sigma}}{\Gamma(\sigma)} \aleph^{m+1,n+1}_{p_k+1,q_k+1,\tau_k;\,r} \left[z\beta \left| \begin{array}{c} (1-\rho, 1), (a_j, A_j)_{1,n}, [\tau_j (a_{jk}, A_{jk})]_{n+1,p_k;r} \\ (b_j, B_j)_{1,m}, (\sigma - \rho, 1), [\tau_j (b_{jk}, B_{jk})]_{m+1,q_k;r} \end{array} \right. \right]. \tag{2.5}$$

$$I_5 = \int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} \aleph^{m,n}_{p_k,q_k,\tau_k;\,r} [z(1-x)^{\mu}] \, dx$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \Omega^{m,n}_{p_k,q_k,\tau_k;r} (s) \, z^{-s} \left\{ \int_{-1}^{1} (1-x)^{1+\rho-\mu s-1} (1+x)^{1+\sigma-1} dx \right\} ds.$$

Next, we use the formula [10, p. 261]

$$\int_{-1}^{1} (1-x)^{n+\alpha} (1+x)^{n+\beta} dx = 2^{2n+\alpha+\beta+1} B(1+\alpha+n, 1+\beta+n), \tag{2.6}$$

hence, we arrive at

$$I_5 = \frac{1}{2\pi i} 2^{\rho+\sigma+1} \int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \, . \, \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)}{\sum_{k=1}^{r} \tau_k \prod_{j=m+1}^{q_k} \Gamma(1 - b_{jk} - B_{jk} s) \, . \, \prod_{j=n+1}^{p_k} \Gamma(a_{jk} + A_{jk} s)}$$

$$\times \frac{\Gamma(1+\rho-\mu s) \Gamma(1+\sigma)}{\Gamma(2+\rho-\mu s+\sigma)} (2^{\mu} z)^{-s} \, ds$$

$$= 2^{\sigma+\rho+1} \Gamma(1+\sigma) \aleph^{m,n+1}_{p_k+1,q_k+1,\tau_k;\,r}$$

$$\left[2^{\mu} z \left| \begin{array}{c} (-\rho, \mu), (a_j, A_j)_{1,n}, [\tau_j (a_{jk}, A_{jk})]_{n+1,p_k;r} \\ (b_j, B_j)_{1,m}, (-1-\sigma-\rho, \mu), [\tau_j (b_{jk}, B_{jk})]_{m+1,q_k;r} \end{array} \right. \right]. \tag{2.7}$$

3. Integrals involving ℵ-function with Jacobi Polynomials

The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ [10, p. 254] is defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_2F_1\left[\begin{matrix} -n,\, 1+\alpha+\beta+n; \\ 1+\alpha; \end{matrix}\; \frac{1-x}{2}\right], \tag{3.1}$$

where ${}_2F_1$ is the classical hypergeometric functions; when $\alpha = \beta = 0$, then the polynomial in (3.1) becomes the Legendre polynomial [10, p. 157].
We also have

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!}.$$

In this section we derive integral formulas involving Aleph function multiplied with Jacobi polynomials.

$$I_6 = \int_{-1}^{1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\mu} P_n^{(\alpha,\beta)}(x)\, \aleph_{p_k,q_k,\tau_k;r}^{m,n}\left[z(1+x)^{l}\right] dx$$
$$= \frac{1}{2\pi i}\int_{\mathcal{L}} \Omega_{p_k,q_k,\tau_k;r}^{m,n}(s)\, z^{-s}\left\{\int_{-1}^{1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\mu-ls} P_n^{(\alpha,\beta)}(x)\, dx\right\} ds.$$

Next we use the following formula:

$$\int_{-1}^{1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\mu} P_n^{(\alpha,\beta)}(x)\, dx$$
$$= (-1)^n \frac{2^{\alpha+\mu+1}\Gamma(\mu+1)\Gamma(n+\alpha+1)\Gamma(\mu+\beta+1)}{n!\,\Gamma(\mu+\beta+n+1)\Gamma(\mu+\alpha+n+2)}$$
$$\times\, {}_3F_2\left[\begin{matrix} -\lambda,\, \mu+\beta+1,\, \mu+1; \\ \mu+\beta+n+1,\, \mu+\alpha+n+2; \end{matrix}\; 1\right], \tag{3.2}$$

where $\alpha > -1$ and $\beta > -1$. Also, ${}_3F_2$ is the special case of generalized hypergeometric series.
Then we have

$$I_6 = \frac{1}{2\pi i}\int_{\mathcal{L}} \Omega_{p_k,q_k,\tau_k;r}^{m,n}(s)\, z^{-s}(-1)^n\, 2^{\alpha+\mu-ls+1}$$
$$\times \frac{\Gamma(\mu-ls+1)\Gamma(n+\alpha+1)\Gamma(\mu-ls+\beta+1)}{n!\Gamma(\mu-ls+\beta+n+1)\Gamma(\mu-ls+\alpha+n+2)}$$
$$\times\, {}_3F_2\left[\begin{matrix} -\lambda,\, \mu-ls+\beta+1,\, \mu-ls+1; \\ \mu-ls+\beta+n+1,\, \mu-ls+\alpha+n+2; \end{matrix}\; 1\right]$$
$$= \frac{(-1)^n\, 2^{\alpha+\mu+1}\Gamma(n+\alpha+1)}{n!}\sum_{k=0}^{\infty}\frac{(-\lambda)_k\,(1)^k}{k!}\frac{1}{2\pi i}$$
$$\times \int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\,.\,\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{k=1}^{r}\tau_k\prod_{j=m+1}^{q_k}\Gamma(1-b_{jk}-B_{jk}s)\,.\,\prod_{j=n+1}^{p_k}\Gamma(a_{jk}+A_{jk}s)}$$

$$\times \frac{\Gamma(\mu - ls + \beta + k + 1)\,\Gamma(\mu - ls + k + 1)}{\Gamma(\mu - ls + \beta + n + k + 1)\,\Gamma(\mu - ls + \alpha + n + k + 2)} \left(2^{l} z\right)^{-s} ds$$

$$= \frac{(-1)^{n}\, 2^{\alpha+\mu+1}\Gamma(\alpha + n + 1)}{n!} \sum_{k=0}^{\infty} \frac{(-\lambda)_{k}\,(1)^{k}}{k!}$$

$$\times\ \aleph^{m,n+2}_{p_k+2,q_k+2,\tau_k;r}\left[2^{l} z \left| \begin{matrix} (-\mu-\beta-k,l),(-\mu-k,l),(a_j,A_j)_{1,n},[\tau_j(a_{jk},A_{jk})]_{n+1,p_k;r} \\ (b_j,B_j)_{1,m},(-\beta-\mu-n-k,l),(-1-\mu-\alpha-n-k,l),[\tau_j(b_{jk},B_{jk})]_{m+1,q_k;r} \end{matrix} \right. \right], \tag{3.3}$$

which provided that $\alpha > -1, \beta > -1$, $\Re(\lambda) > -1$ and $|arg z| < \frac{1}{2}\pi\Omega$.

$$I_7 = \int_{-1}^{1} (1-x)^{\delta}(1+x)^{\nu} P_n^{(\mu,\nu)}(x)\, P_m^{(\rho,\sigma)}(x)\, \aleph^{m,n}_{p_k,q_k,\tau_k;r}\left[z(1-x)^{l}\right] dx$$

$$= \frac{1}{2\pi i}\int_{\mathcal{L}} \Omega^{m,n}_{p_k,q_k,\tau_k;r}(s)\; z^{-s} \left\{\int_{-1}^{1} (1-x)^{\delta-ls}(1+x)^{\nu} P_n^{(\mu,\nu)}(x)\, P_n^{(\rho,\sigma)}(x)\, dx\right\} ds.$$

Next we use the (3.1), the we have

$$= \frac{(1+\rho)_m}{m!} \sum_{k=0}^{\infty} \frac{(-m)_k\,(1+\rho+\sigma+m)_k}{(1+\rho)_k\; 2^{k}\, k!}$$

$$\times\ \frac{1}{2\pi i}\int_{\mathcal{L}} \Omega^{m,n}_{p_k,q_k,\tau_k;r}(s)\; z^{-s}\left\{\int_{-1}^{1} (1-x)^{\delta-ls+k}(1+x)^{\nu} P_n^{(\mu,\nu)}(x)\, dx\right\} ds. \tag{3.4}$$

Again using (3.1) in (3.4), we get

$$I_7 = \frac{\Gamma(1+\rho+m)\,\Gamma(1+\mu+n)}{m!\,n!\,\Gamma(1+\mu)}$$

$$\times \sum_{k=0}^{\infty} \frac{(-m)_k\,(-n)_k\,(1+\rho+\sigma+m)_k\,(1+\mu+\nu+n)_k}{2^{2k}\,(k!)^{2}\,\Gamma(1+\rho+k)\,\Gamma(1+\mu+k)}$$

$$\frac{1}{2\pi i}\int_{\mathcal{L}} \Omega^{m,n}_{p_k,q_k,\tau_k;r}(s)\; 2^{-ls} z^{-s}\left\{\int_{-1}^{1} (1-x)^{\delta-ls+2k}(1+x)^{\nu}\, dx\right\} ds. \tag{3.5}$$

By applying the formula (2.6), equation (3.5) becomes

$$I_7 = 2^{\delta+\nu+1}\frac{\Gamma(1+\rho+m)\,\Gamma(1+\mu+n)}{m!\,n!}$$

$$\sum_{k=0}^{\infty} \frac{(-m)_k\,(-n)_k\,(1+\rho+\sigma+m)_k\,(1+\mu+\nu+n)_k}{(k!)^{2}\,\Gamma(1+\rho+k)\,\Gamma(1+\mu+k)}$$

$$\times \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j+B_j s)\,.\,\prod_{j=1}^{n}\Gamma(1-a_j-A_j s)}{\sum_{k=1}^{r}\tau_k \prod_{j=m+1}^{q_k}\Gamma(1-b_{jk}-B_{jk}s)\,.\,\prod_{j=n+1}^{p_k}\Gamma(a_{jk}+A_{jk}s)}$$

$$\times \frac{\Gamma(1+\delta-ls+2k)\,\Gamma(1+\nu)}{\Gamma(2+\delta-ls+2k+\nu)}\left(2^{l} z\right)^{-s} ds$$

$$= \frac{2^{\nu+\delta+1}\Gamma(1+\rho+m)\Gamma(1+\mu+n)}{m!\,n!}$$
$$\times \sum_{k=0}^{\infty} \frac{(-m)_k(-n)_k(1+\rho+\sigma+m)_k(1+\mu+\nu+n)_k\,\Gamma(1+\nu)}{(k!)^2\,\Gamma(1+\rho+k)\Gamma(1+\mu+k)}$$
$$\times \aleph^{m,n+1}_{p_k+1,q_k+1,\tau_k;r}\left[2^l z \,\middle|\, \begin{matrix} (-\delta-2k,l),(a_j,A_j)_{1,n},\,[\tau_j(a_{jk},A_{jk})]_{n+1,p_k;r} \\ (b_j,B_j)_{1,m},\,(-1-\delta-\nu-2k,l),[\tau_j(b_{jk},B_{jk})]_{m+1,q_k;r} \end{matrix}\right], \tag{3.6}$$

which provided that $\delta > 0$, $\Re(\nu) > -1$ and $|argz| < \frac{1}{2}\pi\Omega$.

$$I_8 = \int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma} P_n^{(\mu,\nu)}(x)\,\aleph^{m,n}_{p_k,q_k,\tau_k;r}\left[z(1-x)^l(1+x)^h\right]dx$$
$$= \frac{1}{2\pi i}\int_{\mathcal{L}} \Omega^{m,n}_{p_k,q_k,\tau_k;r}(s)\,z^{-s}\left\{\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\mu,\nu)}(x)(1-x)^{-ls}(1+x)^{-hs}dx\right\}ds.$$
$$= \frac{1}{2\pi i}\int_{\mathcal{L}} \Omega^{m,n}_{p_k,q_k,\tau_k;r}(s)\,z^{-s}$$
$$\times\left\{\int_{-1}^{1}(1-x)^{\rho-ls}(1+x)^{\sigma-hs}\frac{(1+\mu)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n,\,1+\mu+\nu+n; \\ 1+\mu;\end{matrix}\ \frac{1-x}{2}\right]dx\right\}ds.$$

$$I_8 = \frac{(1+\mu)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\mu+\nu+n)_k}{2^k\,k!\,(1+\mu)_k}$$
$$\times \frac{1}{2\pi i}\int_{\mathcal{L}} \Omega^{m,n}_{p_k,q_k,\tau_k;r}(s)\,z^{-s}\left\{\int_{-1}^{1}(1-x)^{\rho-ls+k-1+1}(1+x)^{\sigma-hs-1+1}dx\right\}ds. \tag{3.7}$$

By using (2.6) in (3.7), then we arrive at

$$I_8 = 2^{\rho+\sigma+1}\frac{(1+\mu)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\mu+\nu+n)_k}{(1+\mu)_k\,k!}$$
$$\times\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\,.\,\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{k=1}^{r}\tau_k\prod_{j=m+1}^{q_k}\Gamma(1-b_{jk}-B_{jk}s)\,.\,\prod_{j=n+1}^{p_k}\Gamma(a_{jk}+A_{jk}s)}$$
$$\times\frac{\Gamma(1+\rho-ls+k)\Gamma(1+\sigma-hs)}{\Gamma(2+\rho+\sigma-ls-hs+k)}\left(2^{l+h}z\right)^{-s}ds$$

$$= \frac{2^{\rho+\sigma+1}(1+\mu)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\mu+\nu+n)_k}{(1+\mu)_k\,k!}$$
$$\times\aleph^{m,n+2}_{p_k+2,q_k+2,\tau_k;r}\left[2^{l+h}z\,\middle|\,\begin{matrix}(-\rho-k,l),(-\sigma,h),(a_j,A_j)_{1,n},\,[\tau_j(a_{jk},A_{jk})]_{n+1,p_k;r} \\ (b_j,B_j)_{1,m},(-1-\rho-\sigma-k,l+h),[\tau_j(b_{jk},B_{jk})]_{m+1,q_k;r}\end{matrix}\right], \tag{3.8}$$

which provided that $\Re(\nu) > -1,\ \Re(\mu) > -1$ and $|argz| < \frac{1}{2}\pi\Omega$.

$$I_9 = \int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} P_n^{(\alpha,\beta)}(x) \aleph_{p_k,q_k,\tau_k;r}^{m,n}\left[z(1+x)^{-l}\right] dx$$

$$= \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\alpha+\beta+n)_k}{2^k\, k!\, (1+\alpha)_k}$$
$$\times \frac{1}{2\pi i} \int_{\mathcal{L}} \Omega_{p_k,q_k,\tau_k;r}^{m,n}(s)\, z^{-s} \left\{ \int_{-1}^{1} (1-x)^{\rho+k-1+1} (1+x)^{\sigma+ls-1+1} dx \right\} ds. \quad (3.9)$$

By using (2.6) in (3.9), then we arrive at

$$I_9 = 2^{\rho+\sigma+1} \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\alpha+\beta+n)_k}{(1+\alpha)_k\ k!}$$
$$\times \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s)\ .\ \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)}{\sum_{k=1}^{r} \tau_k \prod_{j=m+1}^{q_k} \Gamma(1 - b_{jk} - B_{jk} s)\ .\ \prod_{j=n+1}^{p_k} \Gamma(a_{jk} + A_{jk} s)}$$
$$\times \frac{\Gamma(1+\rho+k)\, \Gamma(1+\sigma+ls)}{\Gamma(2+\rho+\sigma+ls+k)} \left(2^{-l} z\right)^{-s} ds$$

$$= \frac{2^{\rho+\sigma+1} (1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\alpha+\beta+n)_k\, \Gamma(1+\rho+k)}{(1+\alpha)_k\ k!}$$
$$\times \aleph_{p_k+1,q_k+1,\tau_k;r}^{m+1,n}\left[2^{-l}z \,\middle|\, \begin{matrix} (a_j, A_j)_{1,n}, (1+\rho, l), [\tau_j (a_{jk}, A_{jk})]_{n+1,p_k;r} \\ (2+\rho+\sigma+k, l), (b_j, B_j)_{1,m}, [\tau_j (b_{jk}, B_{jk})]_{m+1,q_k;r} \end{matrix}\right], \quad (3.10)$$

which provided following:
(i) $\Re(\alpha) > -1,\ \Re(\beta) > -1$ and $|argz| < \frac{1}{2}\pi\Omega$;
(ii) $\Re\left(\rho + l\, min\left(\frac{b_j}{\beta_j}\right)\right) > -1 \qquad \left(j = \overline{1,m}\right)$

$$I_{10} = \int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} P_n^{(\mu,\nu)}(x) \aleph_{p_k,q_k,\tau_k;r}^{m,n}\left[z(1-x)^{l}(1+x)^{-h}\right] dx$$
$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \Omega_{p_k,q_k,\tau_k;r}^{m,n}(s)\, z^{-s}$$
$$\times \left\{ \int_{-1}^{1} (1-x)^{\rho-ls} (1+x)^{\sigma+hs} \frac{(1+\mu)_n}{n!}\, {}_2F_1\left[\begin{matrix} -n,\ 1+\mu+\nu+n; \\ 1+\mu; \end{matrix}\ \frac{1-x}{2}\right] dx \right\} ds,$$

$$= \frac{(1+\mu)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k\,(1+\mu+\nu+n)_k}{2^k\,k!\,(1+\mu)_k}$$

$$\times\ \frac{1}{2\pi i} \int_{\mathcal{L}} \Omega_{p_k,q_k,\tau_k;r}^{m,n}\,(s)\ z^{-s} \left\{ \int_{-1}^{1} (1-x)^{\rho-ls+k}\,(1+x)^{\sigma+hs}\,dx \right\} ds. \tag{3.11}$$

By using (2.6) in (3.11), then we get

$$I_{10} = 2^{\rho+\sigma+1} \frac{(1+\mu)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k\,(1+\mu+\nu+n)_k}{(1+\mu)_k\ k!}$$

$$\times\ \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \Gamma(b_j+B_js)\ .\ \prod_{j=1}^{n} \Gamma(1-a_j-A_js)}{\sum_{k=1}^{r} \tau_k \prod_{j=m+1}^{q_k} \Gamma(1-b_{jk}-B_{jk}s)\ .\ \prod_{j=n+1}^{p_k} \Gamma(a_{jk}+A_{jk}s)}$$

$$\times\ \frac{\Gamma(1+\rho-ls+k)\,\Gamma(1+\sigma+hs)}{\Gamma(2+\rho+\sigma-ls+hs+k)}\ \left(2^{l-h}\,z\right)^{-s}\ ds,$$

$$= \frac{2^{\rho+\sigma+1}\,(1+\mu)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k\,(1+\mu+\nu+n)_k}{(1+\mu)_k\ k!}$$

$$\times \aleph_{p_k+1,q_k+2,\tau_k;r}^{m+1,n+1}\left[2^{l-h} z \,\middle|\, \begin{matrix} (-\rho-k,l),(a_j,A_j)_{1,n},[\tau_j(a_{jk},A_{jk})]_{n+1,p_k;r} \\ (b_j,B_j)_{1,m},(-1-\rho-\sigma-k,l-h),(1+\sigma,h),[\tau_j(b_{jk},B_{jk})]_{m+1,q_k;r} \end{matrix}\right], \tag{3.12}$$

which provided that
(i) $\Re\left[\rho + l\,min\left(\frac{b_j}{\beta_j}\right)\right] > -1,\ \Re\left[\sigma + h\,min\left(\frac{b_j}{\beta_j}\right)\right] > -1, \quad (j=\overline{1,m})$; and
(ii) $|argz| < \frac{1}{2}\pi\Omega$.

4. Integrals involving ℵ-function with Legendre function

The Legendre functions are the solution of Legendre's differential equation [1, sec. 3.1]

$$\left(1-z^2\right)\frac{d^2f}{dz^2} - 2z\frac{df}{dz} + \left[\nu(\nu+1) - \mu^2\left(1-z^2\right)^{-1}\right] f = 0, \tag{4.1}$$

where z, μ, ν are unrestricted.

If we substitute $f = \left(z^2-1\right)^{\frac{1}{2\mu}}\nu$, then (4.1) becomes

$$\left(1-z^2\right)\frac{d^2\nu}{dz^2} - 2(\mu+1)\,z\frac{d\nu}{dz} + [\nu(\mu-\nu)(\mu+\nu+1)] = 0, \tag{4.2}$$

and with $\delta = \frac{1}{2} - \frac{1}{2}z$ as the independent variable the above differential equation becomes as following:

$$\delta(1-\delta)\frac{d^2\nu}{d\delta^2} + (\mu+1)(1-2\delta)\frac{d\nu}{d\delta} + [\nu(\nu-\mu)(\mu+\nu+1)] = 0. \tag{4.3}$$

The solution of (4.1) in the form of Gauss hypergeometric type equation with $a=\mu-\nu$, $b=\mu+\nu+1$ and $c=\mu+1$, as follows.

$$f=P_{\nu}^{\mu}(z)=\frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2\mu}}F\left[-\nu,\ \nu+1;\ 1-\mu;\ \frac{1}{2}-\frac{1}{2}z\right],\quad |1-z|<2, \tag{4.4}$$

where $P_{\nu}^{\mu}(z)$ is known as the Legendre function of the first kind [1].

Next, we derive the integrals with Legendre function.

$$\begin{aligned}I_{11}&=\int_0^1 x^{\sigma-1}\left(1-x^2\right)^{\frac{\mu}{2}}P_{\nu}^{\mu}(x)\,\aleph_{p_k,q_k,\tau_k;r}^{m,n}\left[zx^{\rho}\right]dx\\&=\frac{1}{2\pi i}\int_{\mathcal{L}}\Omega_{p_k,q_k,\tau_k;r}^{m,n}(s)\,z^{-s}\left\{\int_0^1 x^{\sigma-\rho s-1}\left(1-x^2\right)^{\frac{\mu}{2}}P_{\nu}^{\mu}(x)\,dx\right\}ds.\end{aligned}\tag{4.5}$$

Next, using the formula [1, sec. 3.12] for $\Re(\sigma)>0,\ \ \mu\in\mathbb{N}$.

$$\int_0^1 x^{\sigma-1}\left(1-x^2\right)^{\frac{\mu}{2}}P_{\nu}^{\mu}(x)\,dx=\frac{(-1)^{\mu}\,2^{-\sigma-\mu}\,\pi^{\frac{1}{2}}\,\Gamma(\sigma)\,\Gamma(1+\mu+\nu)}{\Gamma(1-\mu+\nu)\,\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}+\frac{\mu}{2}-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma}{2}+\frac{\mu}{2}+\frac{\nu}{2}\right)}. \tag{4.6}$$

Then the integral (4.5) becomes

$$\begin{aligned}I_{11}&=2^{-\sigma-\mu}\,(-1)^{\mu}\,(\pi)^{\frac{1}{2}}\,\frac{\Gamma(1+\mu+\nu)}{\Gamma(1-\mu+\nu)}\\&\times\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\ .\ \prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{k=1}^{r}\tau_k\prod_{j=m+1}^{q_k}\Gamma(1-b_{jk}-B_{jk}s)\ .\ \prod_{j=n+1}^{p_k}\Gamma(a_{jk}+A_{jk}s)}\\&\quad\times\frac{\Gamma(\sigma-\rho s)}{\Gamma\left(\frac{1}{2}+\frac{\sigma-\rho s}{2}+\frac{\mu}{2}-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma-\rho s}{2}+\frac{\mu}{2}+\frac{\nu}{2}\right)}\left(2^{-\rho}z\right)^{-s}ds,\\&=2^{-\sigma-\rho}\,(-1)^{\mu}\,(\pi)^{\frac{1}{2}}\,\frac{\Gamma(1+\mu+\nu)}{\Gamma(1-\mu+\nu)}\times\aleph_{p_k+1,q_k+2,\tau_k;r}^{m,n+1}\\&\left[2^{-\rho}z\left|\begin{matrix}(1-\sigma,\rho),(a_j,A_j)_{1,n},\ [\tau_j(a_{jk},A_{jk})]_{n+1,p_k;r}\\(b_j,B_j)_{1,m},\ \left(\frac{1}{2}-\frac{\mu}{2}+\frac{\nu}{2}-\frac{\sigma}{2},\frac{\rho}{2}\right),\left(-\frac{\sigma}{2}-\frac{\mu}{2}-\frac{\nu}{2},\frac{\rho}{2}\right)\ [\tau_j(b_{jk},B_{jk})]_{m+1,q_k;r}\end{matrix}\right.\right],\end{aligned}\tag{4.7}$$

which provided $|arg\,(z)|<\frac{1}{2}\pi\Omega$, $\sigma>0$ and $\mu\in\mathbb{N}\cup\{0\}$.

$$\begin{aligned}I_{12}&=\int_0^1 x^{\sigma-1}\left(1-x^2\right)^{\frac{-\mu}{2}}P_{\nu}^{\mu}(x)\,\aleph_{p_k,q_k,\tau_k;r}^{m,n}\left[zx^{\rho}\right]dx\\&=\frac{1}{2\pi i}\int_{\mathcal{L}}\Omega_{p_k,q_k,\tau_k;r}^{m,n}(s)\,z^{-s}\left\{\int_0^1 x^{\sigma-\rho s-1}\left(1-x^2\right)^{\frac{-\mu}{2}}P_{\nu}^{\mu}(x)\,dx\right\}ds.\end{aligned}\tag{4.8}$$

Next, using the formula [1, sec. 3.12] for $\Re(\sigma) > 0,\ \mu \in \mathbb{N}$.

$$\int_0^1 x^{\sigma-1}\left(1-x^2\right)^{\frac{-\mu}{2}} P_\nu^\mu(x)\,dx = \frac{2^{-\sigma+\mu}\,\pi^{\frac{1}{2}}\,\Gamma(\sigma)}{\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}-\frac{\mu}{2}-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma}{2}-\frac{\mu}{2}-\frac{\nu}{2}\right)}. \tag{4.9}$$

Then the integral (4.8) becomes

$$I_{12} = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^m \Gamma(b_j+B_js)\ .\ \prod_{j=1}^n \Gamma(1-a_j-A_js)}{\sum_{k=1}^r \tau_k \prod_{j=m+1}^{q_k}\Gamma(1-b_{jk}-B_{jk}s)\ .\ \prod_{j=n+1}^{p_k}\Gamma(a_{jk}+A_{jk}s)}$$

$$\times \frac{2^{-\sigma-\mu}\,\pi^{\frac{1}{2}}\,\Gamma(\sigma-\rho s)}{\Gamma\left(\frac{1}{2}+\frac{\sigma-\rho s}{2}-\frac{\mu}{2}-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma-\rho s}{2}-\frac{\mu}{2}-\frac{\nu}{2}\right)}\left(2^{-\rho}z\right)^{-s} ds,$$

$$= 2^{-\sigma-\rho}\,(\pi)^{\frac{1}{2}} \times \aleph^{m,n+1}_{p_k+1,q_k+2,\tau_k;r}$$
$$\left[2^{-\rho}z\left|\begin{matrix}(1-\sigma,\rho),(a_j,A_j)_{1,n},\ [\tau_j(a_{jk},A_{jk})]_{n+1,p_k;r}\\ (b_j,B_j)_{1,m},\ \left(\frac{1}{2}+\frac{\mu}{2}+\frac{\nu}{2}-\frac{\sigma}{2},\frac{\rho}{2}\right),\left(-\frac{\sigma}{2}+\frac{\mu}{2}+\frac{\nu}{2},\frac{\rho}{2}\right),[\tau_j(b_{jk},B_{jk})]_{m+1,q_k;r}\end{matrix}\right.\right], \tag{4.10}$$

which provided $|arg(z)| < \frac{1}{2}\pi\Omega$, $\Re(\sigma) > 0$ and $\Re(\mu) \in \mathbb{N}\cup\{0\}$.

5. Integrals involving ℵ-function and Hypergeometric function

The hypergeometric function defined for $c > 0$ as [10].

$$F(a,b;c;z) = 1 + \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{5.1}$$

where $(a)_n, (b)_n$ and $(c)_n$ are the Pochhammer symbols which are defined as follows:

$$(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \left\{\begin{array}{ll} 1, & (n=0,\gamma\neq 0)\\ \gamma(\gamma+1)\ldots(\gamma+n-1), & (n\in\mathbb{N},\gamma\in\mathbb{C})\end{array}\right. . \tag{5.2}$$

$$I_{13} = \int_1^\infty x^{-\rho}(x-1)^{\sigma-1}\ {}_2F_1\left[\begin{matrix}\sigma+\nu-\rho,\ \lambda+\sigma-\rho;\\ \sigma;\end{matrix}\ (1-x)\right]\aleph^{m,n}_{p_k,q_k,\tau_k;r}(zx)\,dx$$
$$= \frac{1}{2\pi i}\int_{\mathcal{L}} \Omega^{m,n}_{p_k,q_k,\tau_k;r}(s)\,z^{-s}$$
$$\left\{\int_1^\infty x^{-\rho-s}(x-1)^{\sigma-1}\ {}_2F_1\left[\begin{matrix}\sigma+\nu-\rho,\ \lambda+\sigma-\rho;\\ \sigma;\end{matrix}\ (1-x)\right]dx\right\}ds.$$

by putting $x = t+1 \Rightarrow dx = dt$, then we get

$$I_{13} = \sum_{k=0}^{\infty}\frac{(-1)^k\ (\nu+\sigma-\rho)_k\ (\lambda+\sigma-\rho)_k}{(\sigma)_k\ k!}$$

$$\times \frac{1}{2\pi i}\int_{\mathfrak{L}} \frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_js\right)\,.\,\prod_{j=1}^{n}\Gamma\left(1-a_j-A_js\right)}{\sum_{k=1}^{r}\tau_k\prod_{j=m+1}^{q_k}\Gamma\left(1-b_{jk}-B_{jk}s\right)\,.\,\prod_{j=n+1}^{p_k}\Gamma\left(a_{jk}+A_{jk}s\right)}$$

$$\times \frac{\Gamma\left(\sigma+k\right)\Gamma\left(\rho+s-\sigma-k\right)}{\Gamma\left(\rho+s\right)} z^{-s}\,ds$$

$$= \Gamma\left(\sigma+k\right)\sum_{k=0}^{\infty}\frac{\left(-1\right)^k\,\left(\nu+\sigma-\rho\right)_k\,\left(\lambda+\sigma-\rho\right)_k}{\left(\sigma\right)_k\,k!}$$

$$\times\,\aleph_{p_k+1,q_k+1,\tau_k;\,r}^{\,m+1,n}\left[z\left|\begin{array}{c}\left(a_j,A_j\right)_{1,n},\,\left(\rho,1\right),\,\left[\tau_j\left(a_{jk},A_{jk}\right)\right]_{n+1,p_k;r}\\ \left(\rho-\sigma-k,1\right),\,\left(b_j,B_j\right)_{1,m},\,\left[\tau_j\left(b_{jk},B_{jk}\right)\right]_{m+1,q_k;r}\end{array}\right.\right], \tag{5.3}$$

which provided $|arg\,z| < \frac{1}{2}\pi\Omega$.

6. Integrals involving Aleph function and Bessel Maitland function

The Bessel Maitland function (also known as Wright generalized Bessel function) defined as following [2]:

$$J_{\nu}^{\mu}\left(z\right)=\phi\left(\mu,\nu+1:z\right)=\sum_{n=0}^{\infty}\frac{1}{\Gamma\left(\mu n+\nu+1\right)}\frac{\left(-z\right)^n}{n!}. \tag{6.1}$$

$$I_{14}=\int_0^{\infty}x^{\rho}\,J_{\nu}^{\mu}\left(x\right)\,\aleph_{p_k,q_k,\tau_k;\,r}^{\,m,n}\left(zx^{\sigma}\right)dx$$

$$=\frac{1}{2\pi i}\int_{\mathfrak{L}}\Omega_{p_k,q_k,\tau_k;r}^{\,m,n}\,\left(s\right)\,z^{-s}\left\{\int_0^{\infty}x^{\rho-\sigma s}J_{\nu}^{\mu}\left(x\right)dx\right\}ds.$$

Now using the following formula [19]

$$\int_0^{\infty}x^{\rho}\,J_{\nu}^{\mu}\left(x\right)\,dx=\frac{\Gamma\left(\rho+1\right)}{\Gamma\left(1+\nu-\mu-\mu\rho\right)}\qquad\left(\Re\left(\rho\right)>-1,0<\mu<1\right), \tag{6.2}$$

then we arrive at

$$\begin{aligned} I_{14} &= \frac{1}{2\pi i}\int_{\mathfrak{L}} \frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_js\right)\,.\,\prod_{j=1}^{n}\Gamma\left(1-a_j-A_js\right)}{\sum_{k=1}^{r}\tau_k\prod_{j=m+1}^{q_k}\Gamma\left(1-b_{jk}-B_{jk}s\right)\,.\,\prod_{j=n+1}^{p_k}\Gamma\left(a_{jk}+A_{jk}s\right)}\\ &\quad\times\frac{\Gamma\left(1+\rho-\sigma s\right)}{\Gamma\left(1+\nu-\mu-\mu\rho+\mu\sigma s\right)}z^{-s}\,ds\\ &= \aleph_{p_k+2,q_k,\tau_k;\,r}^{\,m,n+1}\\ &\quad\left[z\left|\begin{array}{c}\left(-\rho,\sigma\right),\,\left(1+\nu-\mu-\mu\rho,\mu\sigma\right),\,\left(a_j,A_j\right)_{1,n},\,\left[\tau_j\left(a_{jk},A_{jk}\right)\right]_{n+1,p_k;r}\\ \left(b_j,B_j\right)_{1,m},\,\left[\tau_j\left(b_{jk},B_{jk}\right)\right]_{m+1,q_k;r}\end{array}\right.\right], \end{aligned} \tag{6.3}$$

which provided $|arg\,z| < \frac{1}{2}\pi\Omega$, $\sigma-\mu\sigma>0$, $\sigma>0$, $0<\mu<1$ and $\Re\left(\rho+1\right)>0$.

7. Integrals involving Aleph function and general class of polynomials

The general class of polynomials $S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[x]$ introduced by Srivastava is defined and represented as follows [20, p. 185, Eqn. (7)]:

$$S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[x] = \sum_{l_1=0}^{[n_1/m_1]} \dots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i}\, x^{l_i}, \tag{7.1}$$

where $n_1,...,n_r = 0,1,2,...;\ m_1,...,m_r$ is an arbitrary positive integers, the coefficients $A_{n_i,l_i}\,(n_i,l_i \geq 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients A_{n_i,l_i}, $S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[x]$ yields a number of known polynomials as its special cases. These includes, among other, the Bessel Polynomials, the Laguerre Polynomials, the Hermite Polynomials, the Jacobi Polynomials, the Gould-Hopper Polynomials, the Brafman Polynomials and several others [21, p.158-161].

Next, we establish the following integral:

$$I_{15} = \int_{-1}^{1} (1-x)^{\rho-1} (1+x)^{\sigma-1} S_{n_1,\dots,n_{r'}}^{m_1,\dots,m_{r'}} \left[y\,(1-x)^{\mu}\,(1+x)^{\nu}\right]$$

$$\times\, \aleph_{p_k,q_k,\tau_k;\,r}^{m,n} \left[z\,(1-x)^{h}\,(1+x)^{k} \,\middle|\, \begin{matrix} (a_j,A_j)_{1,n}\,,\ [\tau_j\,(a_{jk},A_{jk})]_{n+1,p_k;r} \\ (b_j,B_j)_{1,m}\,,\ [\tau_j\,(b_{jk},B_{jk})]_{m+1,q_k;r} \end{matrix} \right] dx,$$

by using (1.1) and (7.1), and the interchange the order of summations and integration we easily arrive at the following integral after a little simplification:

$$I_{15} = 2^{\rho+\sigma-1} \sum_{l_1=0}^{[n_1/m_1]} \dots \sum_{l_{r'}=0}^{[n_{r'}/m_{r'}]} \prod_{i=1}^{r'} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i}\, y^{l_i}\, 2^{(\mu+\nu)l_i} \times \aleph_{p_k+2,q_k+1,\tau_k;\,r}^{m,n+2}$$

$$\left[z\,2^{(h+k)} \,\middle|\, \begin{matrix} (1-\rho-\mu l_i,h)\,,(1-\sigma-\nu l_i,k)\,,(a_j,A_j)_{1,n}\,,\ [\tau_j\,(a_{jk},A_{jk})]_{n+1,p_k;r} \\ (b_j,B_j)_{1,m}\,,(1-\rho-\sigma-(\mu+\nu)\,l_i,h+k)\,,[\tau_j\,(b_{jk},B_{jk})]_{m+1,q_k;r} \end{matrix} \right], \tag{7.2}$$

which converge under the following conditions:
(i) $|arg\, z| < \frac{1}{2}\pi\Omega$,
(ii) $\rho \geq 1, \sigma \geq 1, \mu \geq 0, \nu \geq 0, h \geq 0, k \geq 0$ (h and k are not both zero simultaneously),
(iii) $\Re(\rho) + h\ \min\left[\Re\left(\frac{b_j}{\beta_j}\right)\right] > 0$ and $\Re(\sigma) + k\ min\left[\Re\left(\frac{b_j}{\beta_j}\right)\right] > 0$.

8. Some Special Cases

(i). If we replace δ by $\eta - 1$ and put $\mu = \nu = \rho = \sigma = 0$, then the integral formula (3.6) transform to the following integral involving product of Legendre polynomial and Aleph function:

$$I_{16} = 2^{\eta} \sum_{k=0}^{\infty} \frac{(-m)_k (-n)_k (1+m)_k (1+n)_k}{(k!)^2 \Gamma(1+k) \Gamma(1+k)}$$
$$\times \aleph^{m,n+1}_{p_k+1,q_k+1,\tau_k;r} \left[2^l z \left| \begin{matrix} (1-\eta-2k,l), (a_j,A_j)_{1,n}, [\tau_j (a_{jk},A_{jk})]_{n+1,p_k;r} \\ (b_j,B_j)_{1,m}, (-\eta-2k,l), [\tau_j (b_{jk},B_{jk})]_{m+1,q_k;r} \end{matrix} \right. \right], \tag{8.1}$$

provided $|arg\, z| < \frac{1}{2}\pi\Omega$

(ii). If we replace ρ by $\rho-1$ and σ by $\sigma-1$, and put $\mu=\nu=0$, then the integral formula (3.8) transform to the following integral involving product of Legendre polynomial and Aleph function:

$$I_{17} = 2^{\rho+\sigma-1} \sum_{k=0}^{\infty} \frac{(-n)_k (1+n)_k}{k!} \times \aleph^{m,n+2}_{p_k+2,q_k+1,\tau_k;r}$$
$$\left[2^{l+h} z \left| \begin{matrix} (1-\rho-k,l), (1-\sigma,h), (a_j,A_j)_{1,n}, [\tau_j (a_{jk},A_{jk})]_{n+1,p_k;r} \\ (b_j,B_j)_{1,m}, (1-\rho-\sigma-k,l+h), [\tau_j (b_{jk},B_{jk})]_{m+1,q_k;r} \end{matrix} \right. \right], \tag{8.2}$$

provided $|argz| < \frac{1}{2}\pi\Omega$.

(iii). By replacing ρ by $\rho-1$ and σ by $\sigma-1$, and putting $\mu=\nu=0$, then the integral (3.12) takes the following form:

$$I_{18} = 2^{\rho+\sigma-1} \sum_{k=0}^{\infty} \frac{(-n)_k (1+n)_k}{(k!)^2} \times \aleph^{m+1,n+1}_{p_k+1,q_k+2,\tau_k;r}$$
$$\left[2^{l-h} z \left| \begin{matrix} (1-\rho-k,l), (a_j,A_j)_{1,n}, [\tau_j (a_{jk},A_{jk})]_{n+1,p_k;r} \\ (b_j,B_j)_{1,m}, (1-\rho-\sigma-k,l-h), (\sigma,h), [\tau_j (b_{jk},B_{jk})]_{m+1,q_k;r} \end{matrix} \right. \right], \tag{8.3}$$

which provided that
(i) $|argz| < \frac{1}{2}\pi\Omega$, and
(ii) $\Re\left[\rho + l\, min\left(\frac{b_j}{\beta_j}\right)\right] > -1$, $\Re\left[\sigma + h\, min\left(\frac{b_j}{\beta_j}\right)\right] > -1$, $\left(j=\overline{1,m}\right)$.

Remark 8.1. *If we set $\tau_j = 1 \left(j \in \overline{1,r}\right)$ and put $r=1, m=1, n=p_k=p, q_k=q+1, b_1=0, B_1=1, a_j=1-a_j, b_jk=1-b_j, B_jk=B_j$, then Aleph function reduces to Wright's generalized hypergeometric function ${}_p\psi_q$ and we can easily obtain derived integrals in the form of Wright's generalized hypergeometric function.*

Remark 8.2. *If we set $\tau_j = 1 \left(j \in \overline{1,r}\right)$ and put $r=1, m=1, n=p_1=p, q_k=q, b_1=0, B_1=1, a_j=1-a_j, b_jk=1-b_j, A_j=B_j=A_jk=B_jk=1$, then Aleph function reduces to generalized hypergeometric function ${}_pF_q$ and we can easily obtain derived integrals in the form of generalized hypergeometric function.*

9. Concluding Remarks

The results obtained here are basic in nature and are likely to find useful applications in the study of simple and multiple variable hypergeometric series which in turn are useful in statistical mechanics, electrical networks and probability theory. If we follow Remark 1.2 then all given results can be written in the form of I-function and H-function.

Acknowledgments

The first-named author would like to express his deep thanks to NBHM (National Board of Higher Mathematics), India, for granting a Post-Doctoral Fellowship (sanction no.2/40(37)/2014/R&D-II/14131).

References

1. A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, Higher Transcendental Functions, Vol. I, McGraw-Hill, New York-Toronto-London, 1953.
2. V.S. Kiryakova, Generalized fractional calculus and applications, *Pitamat Research Notes in Mathematics*, John Wiley and Sones, New York, **301**, 1994.
3. D. Kumar, P. Agarwal and S.D. Purohit, Generalized fractional integration of the $\bar{H}$-function involving general class of polynomials, *Walailak Journal of Science and Technology*, **11**(12) (2014), 1019–1030.
4. D. Kumar and J. Daiya, Generalized fractional differentiation of the H-function involving general class of polynomials, *Int. J. Pure Appl. Sci. Technol.*, **16**(1) (2013), 42–53.
5. D. Kumar, S. D. Purohit, A. Secer and A. Atangana, On generalized fractional kinetic equations involving generalized Bessel function of the first kind, *Math. Probl. Eng.* 2014, Article ID 289387, 1–7.
6. D. Kumar and R.K. Saxena, Generalized fractional calculus of the M-Series involving F_3 hypergeometric function, *Sohag J. Math.*, **2**(1), (2015), 17–22.
7. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, Vol. **204**, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.
8. A. M. Mathai and R. K. Saxena, *The H-function with Applications in Statistics and Other Disciplines*, Halsted Press (John Wiley & Sons), New York, London, Sydney,Toronto, 1978.
9. A. M. Mathai, R. K. Saxena and H. J. Haubold, *The H-function: Theory and Applications*, Springer, New York, 2010.
10. E.D. Rainville, Special functions, *The Macmillan Company*, New York, 1960.
11. J. Ram and D. Kumar, Generalized fractional integration involving Appell Hypergeometric of the product of two H -functions, *Vijanana Parishad Anusandhan Patrika*, **54**(3) (2011), 33–43.
12. J. Ram and D. Kumar, Generalized fractional integration of the $\aleph$- function, *J. Raj. Acad. Phy. Sci.* **10**(4) (2011), 373–382.
13. R.K. Saxena, J. Daiya and D. Kumar, Fractional integration of the $\bar{H}$-function and a general class of polynomials via pathway operator, *J. Indian Acad. Math.*, **35**(2) (2013), 261–274.
14. R. K. Saxena and T.K. Pogány, Mathieu-type series for the $\aleph$-function occurring in Fokker-Planck equation, *EJPAM* **3**(6) (2010), 980–988.

15. R. K. Saxena and T.K. Pogány, On fractional integration formulae for Aleph functions, *Appl. Math. Comput.* **218** (2011), 985–990.

16. R. K. Saxena, J. Ram and D. Kumar, Generalized fractional differentiation of the Aleph-Function associated with the Appell function F_3, *Acta Cienc. Indica Math.* **38 M**(4) (2012), 781–792.

17. R.K. Saxena, J. Ram and D. Kumar, Alternative Derivation of generalized fractional Kinetic Equations, *Journal of Fractional Calculus and Applications*, **4**(2) (2013), 322–334.

18. R. K. Saxena, J. Ram and D. Kumar, Generalized fractional integral of the product of two Aleph-functions, *Appl. Appl. Math.* **8**(2) (2013), 631–646.

19. V. P. Saxena, Formal solution of certain new pair of dual integral equations involving H-functions, *Proc. Nat. Acad. Sci. India Sect* **A51** (2001), 366–375.

20. H.M.Srivastava, A multilinear generating function for the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials, *Pacific J.Math.*, **117** (1985), 183–191.

21. H.M. Srivastava and N.P. Singh, The integration of certain products of the multivariable H-function with a general class of polynomials, *Rend. Circ. Mat. Palermo Ser. 2*, **32** (1983), 157–187.

22. N. Südland, B. Baumann and T. F. Nonnenmacher, Open problem: Who knows about the Aleph (ℵ)-functions ?, *Fract. Calc. Appl. Anal.* **1**(4) (1998), 401–402.

23. N. Südland, B. Baumann and T. F. Nonnenmacher, *Fractional driftless Fokker-Planck equation with power law diffusion coefficients*, in V.G. Gangha, E.W. Mayr, W.G. Vorozhtsov (Eds.), Computer Algebra in Scientific Computing (CASC Konstanz 2001), Springer, Berlin, 2001, 513–525.

Dinesh Kumar, R.K. Saxena and Jeta Ram
Department of Mathematics & Statistics
Jai Narain Vyas University, Jodhpur - 342005, India
E-mail address: dinesh_dino03@yahoo.com
E-mail address: ram.saxena@yahoo.com
E-mail address: bishnoi_jr@yahoo.com