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Constant Angle Spacelike Surface in de Sitter Space S_1^3

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ABSTRACT: In this paper; using the angle between unit normal vector field of surfaces and a fixed spacelike axis in R_1^4, we develop two class of spacelike surface which are called constant timelike angle surfaces with timelike and spacelike axis in de Sitter space S_1^3. Moreover we give constant timelike angle tangent surfaces which are examples constant angle surfaces in de Sitter space S_1^3.

Key Words: Constant angle surfaces, de Sitter space, Helix.

Contents

1. Introduction And Results

A constant angle curve whose tangents make constant angle with a fixed direction in ambient space is called a helix. A surface whose tangent planes makes a constant angle with a fixed vector field of ambient space is called constant angle surface. Constant angle surfaces have been studied for arbitrary dimension in Euclidean space $\mathbb{E}^n$ [13,14] and recently in product spaces $\mathbb{S}^2 \times \mathbb{R}$ [15] , $\mathbb{H}^2 \times \mathbb{R}$ [16] or different ambient spaces Nil_3 [17] . In [1] , Lopez and Munteanu studied constant hyperbolic angle surfaces whose unit normal timelike vector field makes a constant hyperbolic angle with a fixed timelike axis in Minkowski space $\mathbb{R}_1^4$. In the literature constant timelike and spacelike angle surface have not been investigated both in hyperbolic space H^3 and de sitter space S_1^3. A constant timelike and a spacelike angle surface in Hyperbolic space H^3 are developed in our paper [19]. In this paper we introduce constant timelike angle spacelike surfaces in de Sitter space S_1^3.

Let $x : M \longrightarrow \mathbb{R}_1^4$ be an immersion of a surface M into $\mathbb{R}_1^4$. We say that x is timelike (*resp.* spacelike, lightlike) if the induced metric on M via x is Lorentzian (*resp. Riemannian, degenerated*). If $\langle x, x\rangle = 1$, then x is an immersion of S_1^3.

Let $x : M \longrightarrow S_1^3$ be a immersion and let ξ be a timelike unit normal vector field to M . If there exists spacelike direction W such that timelike angle $\theta(\xi, U)$

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is constant on M , then M is called **constant timelike angle surfaces with spacelike axis**.

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2. Differantial Geometry of de Sitter Space S_1^3

In this section, Differential geometry of curves and surfaces are summarized in de Sitter space S_1^3. Let $\mathbb{R}_1^4$ be 4-dimensional vector space equipped with the scalar product $\langle , \rangle$ which is defined by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 \ .$$

From now on, the constant angle surface is proposed in Minkowskian ambient space $\mathbb{R}_1^4$. $\mathbb{R}_1^4$ is 4-dimensional vector space equipped with the scalar product $\langle , \rangle$, than $\mathbb{R}_1^4$ is called Lorentzian 4- space or 4-dimensional Minkowski space. The Lorentzian norm (length) of x is defined to be

$$\|x\| = |\langle x, x\rangle|^{\frac{1}{2}} \ .$$

If $\left(x_0^i, x_1^i, x_2^i, x_3^i\right)$ is the coordinate of x_i with respect to canonical basis $\{e_0, e_1, e_2, e_3\}$ of $\mathbb{R}_1^4$, then the lorentzian cross product $x_1 \times x_2 \times x_3$ is defined by the symbolic determinant

$$x_1 \times x_2 \times x_3 = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ x_0^1 & x_1^1 & x_2^1 & x_3^1 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{vmatrix} .$$

One can easly see that

$$\langle x_1 \times x_2 \times x_3, x_4\rangle = \det\left(x_1, x_2, x_3, x_4\right).$$

In [2], [3] and [5] Izimuya at all introduced and investigated differantial geometry of curves and surfaces Hyperbolic 3-space. If $\langle x, x\rangle > 0$, $\langle x, x\rangle = 0$ or $\langle x, x\rangle < 0$ for any non-zero $x \in \mathbb{R}_1^4$, then we call that x is spacelike, ligtlike or timelike ,respectively. In the rest of this section, we give background of context in [20].

Given a vector $v \in \mathbb{R}_1^4$ and a real number c, the hyperplane with pseudo normal v is defined by

$$HP\,(v, c) = \left\{x \in \mathbb{R}_1^4 \,|\langle x, v\rangle = c\right\}$$

We say that $HP\,(v, c)$ is a spacelike hyperplane, timelike hyperplane or lightlike hyperplane if v is timelike, spacelike or lightlike respectively ın [20]. We have following three types of pseudo-spheres in $\mathbb{R}_1^4$:

$$\text{Hyperbolic-3 space}: \ H^3\,(-1) = \left\{x \in \mathbb{R}_1^4 \,|\langle x, x\rangle = -1, x_0 \geq 1\right\},$$
$$\text{de Sitter 3- space:} S_1^3 = \left\{x \in \mathbb{R}_1^4 \,|\langle x, x\rangle = 1\right\},$$
$$\text{(open) lightcone:} LC^* = \left\{x \in \mathbb{R}_1^4 / \{0\} \,|\langle x, x\rangle = 0, x_0 > 0\right\}.$$

We also define the lightcone 3−sphere

$$S_+^3 = \{x = (x_0, x_1, x_2, x_3) \,|\, \langle x, x\rangle = 0, x_0 = 1\}.$$

A hypersurface given by the intersection of S_1^3 with a spacelike (resp.timelike) hyperplane is called an elliptic hyperquadric (resp. hyperbolic hyperquadric). If $c \neq 0$ and $HP(v, c)$ is lightlike , then $HP(v, c) \cap S_1^3$ is a de Sitter horosphere, [20].

Let $U \subset \mathbb{R}^2$ be open subset, and let $x : U \to S_1^3$ be an embedding. If the vector subspace $\tilde{U}$ whih generated by $\{x_{u_1}, x_{u_2}\}$ is spacelike, then x is called spacelike surface, if $\tilde{U}$ contain at least a timelike vector field, then x is called timelike surface in S_1^3.

In point of view Kasedou [20], we construct the extrinsic differential geometry on curves in S_1^3. Since S_1^3 is a Riemannian manifold, the regular curve $\gamma : I \to S_1^3$ is given by arclength parameter.

Theorem 2.1. *i) If $\gamma : I \to S_1^3$ is a spacelike curve with unit speed, then Frenet-Serre type formulae is obtained*

$$\begin{cases} \gamma'(s) &= t(s) \\ t'(s) &= \kappa_d(s)\, n(s) - \gamma(s) \\ n'(s) &= -\kappa_d(s)\, t(s) - \tau_d(s)\, e(s) \\ e'(s) &= -\tau_d(s)\, n(s) \end{cases}$$

where $\kappa_d(s) = \|t'(s) + \gamma(s)\|$ *and* $\tau_d(s) = -\dfrac{\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))}{(\kappa_d(s))^2}$.

ii) If $\gamma : I \to S_1^3$ is a timelike curve with unit speed, then Frenet-Serre type formulae is obtained

$$\begin{cases} \gamma'(s) &= t(s) \\ t'(s) &= \kappa_d(s)\, n(s) + \gamma(s) \\ n'(s) &= -\kappa_d(s)\, t(s) + \tau_d(s)\, e(s) \\ e'(s) &= -\tau_d(s)\, n(s) \end{cases}$$

where $\kappa_d(s) = \|t'(s) - \gamma(s)\|$ *and* $\tau_d(s) = -\dfrac{\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))}{(\kappa_d(s))^2}$.

It is easily see that $\kappa_d(s) = 0$ if and only if there exists a lightlike vector c such that $\gamma(s) - c$ is a geodesic.

Now we give extrinsic differential geometry on surfaces in S_1^3 due to Kasedou [20].

Let $U \subset \mathbb{R}^2$ is an open subset and $x : U \to S_1^3$ is a regular surface $M = x(U)$. If $e(u)$ is defined as follows

$$e(u) = \frac{x(u) \wedge x_{u_1}(u) \wedge x_{u_2}(u)}{\|x(u) \wedge x_{u_1}(u) \wedge x_{u_2}(u)\|}$$

then

$$\langle e, x\rangle \equiv \langle e, x_{u_i}\rangle \equiv 0, \langle e, e\rangle = -1$$

where $x_{u_i} = \dfrac{\partial x}{\partial u_i}$. Thus there is de Sitter Gauss image of x which is defined by mapping

$$E : U \to S_1^3,\ E(u) = e(u).$$

The lightcone Gauss image of x is defined by map

$$L^{\pm} : U \to LC^*,\ L^{\pm}(u) = x(u) \pm e(u).$$

Let $dx(u_0)$ and 1_{T_pM} be identify mapping on the tangent space T_pM. So derivate $dx(u_0)$ can be identified with T_pM relate to identification of U and M. That is

$$dL^{\pm}(u_0) = 1_{T_pM} \pm dE(u_0).$$

The linear transformation

$$S_p^{\pm} := -dL^{\pm}(u_0) : T_pM \to T_pM$$

and

$$A_p := -dE(u_0) : T_pM \to T_pM$$

is called the hyperbolic shape operator and de Sitter shape operator of M at $p = x(u_o)$.

Let $\bar{K}_i^{\pm}(p)$ and $K_i(p)$, $(i = 1, 2)$ be the eigenvalues of $S_p^{\pm}$ and A_p. Since

$$S_p^{\pm} = -1_{T_pM} \pm A_p,$$

$S_p^{\pm}$ and A_p have same eigenvectors and relations

$$\bar{K}_i^{\pm}(p) = -1 \pm K_i(p).$$

$\bar{K}_i^{\pm}(p)$ and $K_i(p)$, $(i = 1, 2)$ are called hyperbolic and de Sitter principal curvatures of M at $p = x(u_0)$.

Let $\gamma(s) = x(u_1(s), u_2(s))$ be a unit speed curve on M, with $p = \gamma(u_1(s_0), u_2(s_0))$. We consider the hyperbolic curvature vector $k(s) = t'(s) - \gamma(s)$ and the de Sitter normal curvature

$$K_n^{\pm}(s_0) = \left\langle k(s_0), L^{\pm}(u_1(s_0), u_2(s_0))\right\rangle = \left\langle t'(s_0), L^{\pm}(u_1(s_0), u_2(s_0))\right\rangle + 1$$

of $\gamma(s)$ at $p = \gamma(s_0)$. The de Sitter normal curvature depends only on the point p and the unit tangent vector of M at p analogous to the Euclidean case. Hyperbolic normal curvature of $\gamma(s)$ is defined to be

$$\bar{K}_n^{\pm}(s) = K_n^{\pm}(s) - 1.$$

The Hyperbolic Gauss curvature of $M = x(U)$ at $p = x(u_0)$ is defined to be

$$K_h^{\pm}(u_0) = \det S_p^{\pm} = \bar{K}_1^{\pm}(p)\,\bar{K}_2^{\pm}(p)\ .$$

The Hyperbolic mean curvature of $M = x(u)$ at $p = x(u_0)$ is defined to be

$$H_h^{\pm}(u_0) = \frac{1}{2} Trace S_p^{\pm} = \frac{\bar{K}_1^{\pm}(p) + \bar{K}_2^{\pm}(p)}{2} \ .$$

The extrinsic (de Sitter) Gauss curvature is defined to be

$$K_e(u_0) = \det A_p = K_1(p) K_2(p) \ ,$$

and the de Sitter mean curvature is

$$H_d(u_0) = \frac{1}{2} Trace A_p = \frac{K_1(p) + K_2(p)}{2} \ .$$

3. Constant Timelike Angle Spacelike Surfaces

Let us show the space of the tangent vector fields on M with $X(M)$ and denote the Levi-Civita connections of $\mathbb{R}_1^4, S_1^3$ and M by $\overline{\overline{D}}, \overline{D}$ and D. Then for each $X, Y \in X(M)$, we have

$$D_X Y = \left(\overline{\overline{D}}_X Y\right)^T \ , \widetilde{V}(X,Y) = \left(\overline{\overline{D}}_X Y\right)^{\perp}$$

and

$$\overline{\overline{D}}_X Y = \bar{D}_X Y - \langle X, Y \rangle x \ , \overline{\overline{D}}_X Y = D_X Y + \widetilde{V}(X,Y) \ , \tag{3.1}$$

where the superscript ^{T} and $^{\perp}$ denote the tangent and normal component of $\overline{\overline{D}}_X Y$. (3.1) equation is called the Gauss formula of S_1^3 and M.

If ξ is a normal vector field of M on S_1^3, then the Weingarten Endomorphism $A_\xi(X)$ and $B_x(X)$ are denoted by the tangent components of $-\overline{\overline{D}}_X \xi$ and $-\overline{\overline{D}}_X x$. So the Weingarten equations of the vector field ξ and x is like

$$\begin{cases} A_\xi(X) = -\overline{\overline{D}}_X \xi - \left\langle \overline{\overline{D}}_X x, \xi \right\rangle x \ , \\ B_x(X) = -\overline{\overline{D}}_X x - \left\langle \overline{\overline{D}}_X x, \xi \right\rangle \xi \ . \end{cases} \tag{3.2}$$

It is obvious that $A_\xi(X)$ and $B_x(X)$ are linear and self adjoint map for each $p \in M$. That is

$$\langle A_\xi(X), Y \rangle = \langle X, A_\xi(Y) \rangle \text{ and } \langle B_x(X), Y \rangle = \langle X, B_x(Y) \rangle \ .$$

The eigenvalues $K_i(p)$ and $\tilde{K}_i(p)$ of $(A_\xi)_p$ are called the principal curvature of M on S_1^3. The eigenvalues $\tilde{K}_i(p)$ of $(B_x)_p$ are called the principal curvature of M in $\mathbb{R}_1^4$. Also, for $X, Y \in X(M)$ we have

$$\langle A_\xi(X), Y \rangle = \left\langle \widetilde{V}(X,Y), \xi \right\rangle \quad , \quad \langle B_x(X), Y \rangle = \left\langle \widetilde{V}(X,Y), x \right\rangle \ .$$

Since $\widetilde{V}(X,Y)$ is second fundamental form of M on $\mathbb{R}_1^4$, so we can write as follows

$$\widetilde{V}(X,Y) = -\left\langle \widetilde{V}(X,Y), \xi \right\rangle \xi + \left\langle \widetilde{V}(X,Y), x \right\rangle x$$

and

$$\widetilde{V}(X,Y) = -\langle A_\xi(X), Y\rangle \xi + \langle B_x(X), Y\rangle x \ .$$

Let $\{v_1, v_2\}$ be a base of T_pM tangent plane and let us denote

$$a_{ij} = \left\langle \widetilde{V}(v_i, v_j), \xi \right\rangle = \langle A_\xi(v_i), v_j\rangle \tag{3.3}$$

$$b_{ij} = \left\langle \widetilde{V}(v_i, v_j), x \right\rangle = \langle B_x(v_i), v_j\rangle \tag{3.4}$$

Therefore

$$\overline{\overline{D}}_X Y = D_X Y + \widetilde{V}(X,Y)$$

and also since

$$\bar{D}_X Y = D_X Y - \langle A_\xi(X), Y\rangle \xi \text{ and } \overline{\overline{D}}_X Y = \bar{D}_X Y - \langle X, Y\rangle x \ ,$$

we obtain

$$\overline{\overline{D}}_X Y = D_X Y - \langle A_\xi(X), Y\rangle \xi - \langle X, Y\rangle x \ .$$

On the other hand for $\{v_1, v_2\}$ base , we get

$$\overline{\overline{D}}_{v_i} v_j = D_{v_i} v_j - a_{ij}\xi - \langle v_i, v_j\rangle x \ . \tag{3.5}$$

If this basis is orthonormal, then we have from (3.1) and (3.2)

$$\overline{\overline{D}}_{v_i} v_j = D_{v_i} v_j - a_{ij}\xi \ , \tag{3.6}$$

$$\overline{\overline{D}}_{v_i} \xi = -a_{i1}v_1 - a_{i_2}v_2 \ , \tag{3.7}$$

$$\overline{\overline{D}}_{v_i} x = -b_{i1}v_1 - b_{i2}v_2 \ . \tag{3.8}$$

3.1. Constant Timelike Angle Surfaces With Spacelike Axis

Definition 3.1. *Let $U \subset \mathbb{R}^2$ be open set,let $x : U \to S_1^3$ be an embedding where $M = x(U)$. Let $x : M \to S_1^3$ and ξ is timelike unit normal vector field on M, if there exist a constant spacelike vector W which has a constant timelike angle with ξ, then M is called* ***constant timelike angle surface with spacelike axis****.*

Since our surface is a spacelike surface, $\{x_u, x_v\}$ tangent vectors must be spacelike vectors. Let M be a spacelike surface with constant angle with spacelike axis and ξ is unit normal vector of M on S_1^3. Let us denote that the timelike angle between timelike vector ξ and spacelike vector W with θ . That is from [11]

$$\langle \xi, W\rangle = \sin h(-\theta) \, .$$

If timelike angle $\theta = 0$, then $\xi = W$. Throughout this section, without loss of generality we assume that $\theta \neq 0$. If W^T is the projection of W on the tangent plane of M, then we decompose W as

$$W = W^T + W^N.$$

So that we write

$$W = W^T + \lambda_1 \xi + \lambda_2 x \ .$$

If we take inner product of both sides of this inequality first with ξ, then with x

$$\lambda_1 = -\sin h\,(-\theta) \ \ , \lambda_2 = \langle W, x\rangle \ \ .$$

On the other hand, since W and x are two spacelike vector fields, then we can use define of the spacelike and timelike angle between W and x.

Theorem 3.2. *i) If φ is the spacelike angle between spacelike vectors W, x then we can write for [11]*

$$W = \sqrt{\sinh^2\theta + \sin^2\varphi}e_1 + (\sinh\theta)\,\xi + \cos\varphi x$$

and de Sitter projection W_d of W as follows

$$W_d = \sqrt{\sinh^2\theta + \sin^2\varphi}e_1 + (\sinh\theta)\,\xi \tag{3.9}$$

ii)If φ is timelike angle between spacelike vectors W and x, then we can write

$$W = \sqrt{\left|\cosh^2\theta - \cosh^2\varphi\right|}e_1 + (\sinh\theta)\,\xi - (\cosh\varphi)\,x \ .$$

and de sitter projection W_d of W as follows

$$W_d = \sqrt{\left|\cosh^2\theta - \cosh^2\varphi\right|}e_1 + (\sinh\theta)\,\xi \ . \tag{3.10}$$

Let $e_1 = \dfrac{W^T}{\|W^T\|}$ and let consider e_2 be a unit vector field on M orthogonal to e_1 . Then we have an oriented orthonormal basis $\{e_1, e_2, \xi, x\}$ for $\mathbb{R}^4_1$.Since W_d is constant vector field on S_1^3 and $\overline{\overline{D}}_{e_2} W_d = \overline{D}_{e_2} W_d = 0$, we have

$$\sqrt{\sinh^2\theta + \sin^2\varphi}\overline{\overline{D}}_{e_2} e_1 + (\sinh\theta)\,\overline{\overline{D}}_{e_2}\xi = 0. \tag{3.11}$$

By (3.11), we obtain

$$\sqrt{\sinh^2\theta + \sin^2\varphi}\left\langle \overline{\overline{D}}_{e_2} e_1, \xi\right\rangle + (\sinh\theta)\left\langle \overline{\overline{D}}_{e_2}\xi, \xi\right\rangle = 0,$$

or

$$\sqrt{\sinh^2\theta + \sin^2\varphi}a_{21} = 0.$$

Since $\sqrt{\sinh^2\theta + \sin^2\varphi} \neq 0$, we conclude $a_{21} = a_{12} = 0$. Using (3.7) in (3.11), we get

$$\overline{\overline{D}}_{e_2} e_1 = \frac{\sinh\theta}{\sqrt{\sinh^2\theta + \sin^2\varphi}} a_{22} e_2. \tag{3.12}$$

Similarly, since W_d is a constant vector field on S_1^3 , then we have

$$\overline{D}_{e_1}W_d = 0 \text{ and } \overline{\overline{D}}_{e_1}W_d = -\sqrt{\sinh^2\theta + \sin^2\varphi}x \ . \tag{3.13}$$

By (3.9), we obtain

$$\overline{\overline{D}}_{e_1}W_d = \sqrt{\sinh^2\theta + \sin^2\varphi}\overline{\overline{D}}_{e_1}e_1 + \sinh\theta\overline{\overline{D}}e_1\xi. \tag{3.14}$$

By (3.13) and (3.14), we conclude that

$$\sqrt{\sinh^2\theta + \sin^2\varphi}\overline{\overline{D}}_{e_1}e_1 + \sinh\theta\overline{\overline{D}}e_1\xi = -\sqrt{\sinh^2\theta + \sin^2\varphi}x \ . \tag{3.15}$$

By (3.15), we get

$$\sqrt{\sinh^2\theta + \sin^2\varphi}\left\langle\overline{\overline{D}}_{e_1}e_1, \xi\right\rangle = 0,$$

or

$$\sqrt{\sinh^2\theta + \sin^2\varphi}a_{11} = 0.$$

Since $\sqrt{\sinh^2\theta + \sin^2\varphi} \neq 0$, we conclude $a_{11} = 0$. Also ,using (3.7) in (3.15) , we obtain

$$\overline{\overline{D}}_{e_1}e_1 = -x \ . \tag{3.16}$$

Now we have proved the following theorem.

Theorem 3.3. *If D is Levi-Civita connection for a constant timelike angle with spacelike axis spacelike surface in S_1^3 is given by*

$$D_{e_1}e_1 = 0, \quad D_{e_2}e_1 = \frac{\sinh\theta}{\sqrt{\sinh^2\theta + \sin^2\varphi}}a_{22}e_2$$
$$D_{e_1}e_2 = 0, \quad D_{e_2}e_2 = \frac{-\sinh\theta}{\sqrt{\sinh^2\theta + \sin^2\varphi}}a_{22}e_1$$

Corollary 3.4. *Let M be a spacelike surface which is a constant timelike angle with spacelike axis on S_1^3. Then, there exist local coordinates u and v such that the metric on M writes as $\langle,\rangle = du^2 + \beta^2 dv^2$, where $\beta = \beta(u, v)$ is a smooth function on M, i.e. the coefficients of the first fundamental form are $E = 1$, $F = 0$, $G = \beta^2$.*

Now we find the $x = x(u, v)$ parametrization of the surface M with respect to the metric $\langle,\rangle = du^2 + \beta^2 dv^2$ on M. By the above parametrization $x(u, v)$ can obtain the following corollary.

Corollary 3.5. *There exist an equation system for constant timelike angle with spacelike axis spacelike surface on S_1^3 which is*

$$\begin{cases} x_{uu} = & -x \\ x_{uv} = & \dfrac{\beta_u}{\beta}x_v \\ x_{vv} = & -\beta\beta_u x_u + \dfrac{\beta_v}{\beta}x_v - \beta^2 a_{22}\xi - \beta^2 x. \end{cases} \tag{3.17}$$

Corollary 3.6. *Let ξ be unit normal vector of the constant timelike angle with spacelike axis spacelike surface M. Then the equation below hold*

$$\begin{cases} \xi_u = \overline{\overline{D}}_{x_u}\xi = 0 \\ \xi_v = \overline{\overline{D}}_{x_v}\xi = -a_{22}x_v. \end{cases} \tag{3.18}$$

Since $\xi_{uv} = \xi_{vu} = 0$, we have $\overline{\overline{D}}_{x_u}(-a_{22}x_v) = 0$. Using $a_{12} = 0$, $\overline{\overline{D}}_{x_u}x_v = \overline{\overline{D}}_{x_v}x_u$ and Theorem 2.1, we obtain

$$(a_{22})_u + \frac{\sinh\theta}{\sqrt{\sinh^2\theta + \sin^2\varphi}}(a_{22})^2 = 0 \tag{3.19}$$

So that

$$(a_{22})_u + \frac{\beta_u}{\beta}a_{22} = 0 \tag{3.20}$$

and than we get obtain

$$(\beta a_{22})_u = 0\ . \tag{3.21}$$

By (3.21), we see that there exist a smooth function $\psi = \psi(v)$ depending on v such that

$$\beta a_{22} = \psi(v)\,. \tag{3.22}$$

Proposition 3.7. *Let $x = x(u,v)$ be parametrization of a spacelike surface which is constant timelike angle with spacelike axis on S_1^3. If $a_{22} = 0$ on M , then the x describes an flat plane of de Sitter space S_1^3.*

Proof. If $a_{22} = 0$ on M,then by (3.18)

$$\begin{cases} \xi_u = 0 \\ \xi_v = 0 \end{cases}$$

This imply we have ξ is a constant vector field which normal vector is M surface. Thus $x = x(u,v)$ is de-Sitter plane in S_1^3. □

From now on, we are going to assume that $a_{22} \neq 0$. By solving equation (3.19), we obtain a function $\alpha = \alpha(v)$ such that

$$a_{22} = \frac{\sqrt{\sinh^2\theta + \sin^2\varphi}}{u\sinh\theta + \alpha(v)}\ .$$

Therefore by (3.22) ,we obtain

$$\beta(u,v) = \frac{\psi(v)}{\sqrt{\sinh^2\theta + \sin^2\varphi}}(u\sinh\theta + \alpha(v))\,.$$

Consequently,

$$
\begin{aligned}
x_{uu} &= -x, \\
x_{uv} &= \frac{\sinh\theta\psi(v)}{u\sinh\theta+\alpha(v)}x_v, \\
x_{vv} &= \left[\frac{-\psi^2(v)\sinh\theta\,(u\sinh\theta+\alpha(v))}{\sinh^2\theta+\sin^2\varphi}\right]x_u+\left[\frac{\psi'(v)}{\psi(v)}+\frac{\alpha'(v)}{u\sinh\theta+\alpha(v)}\right]x_v \\
&-\left[\frac{\psi^2(v)\,(u\sinh\theta+\alpha(v))}{\sqrt{\sinh^2\theta+\sin^2\varphi}}\right]\xi-\left[\frac{\psi^2(v)\,(u\sinh\theta+\alpha(v))^2}{\sinh^2\theta+\sin^2\varphi}\right]x.
\end{aligned}
$$

Here, if we spesificly choose $\psi(v)=e^v\sqrt{\sinh^2\theta+\sin^2\varphi}$ and $\alpha(v)=e^{-v}$, then this equation system becomes

$$
\begin{cases}
x_{uu} &= -x \\
x_{uv} &= \dfrac{e^v\sinh\theta}{1+ue^v\sinh\theta}x_v \\
x_{vv} &= -e^v\sinh\theta\,(ue^v\sinh\theta+1)\,x_u+\dfrac{ue^v\sinh\theta}{ue^v\sinh\theta+1}x_v- \\
& -e^v\sqrt{\sinh^2\theta+\sin^2\varphi}\,(ue^v\sinh\theta+1)\,\xi-(ue^v\sinh\theta+1)^2\,x\ .
\end{cases}
\tag{3.23}
$$

Now we have the following Theorem.

Theorem 3.8. *If M is satisfying (3.23), then there exist local coordinates u and v on M with having the parametrization*

$$
x_i(u,v)=\left(\frac{-c_{1i}(v)}{2e^v\sinh\theta\,(ue^v\sinh\theta+1)^2}+c_{2i}(v)\right)\quad,\quad i=1,2,3,4 \tag{3.24}
$$

Proof. From (3.23), the proof is clear. □

Example 3.9. *We can calculate Gauss and mean curvature of a spacelike surface with constant angle spacelike axis in de Sitter space S_1^3. Since*

$$\overline{\overline{D}}_X\xi=\bar{D}_X\xi$$

we can write

$$\overline{\overline{D}}_{v_i}\xi=\left\langle\overline{\overline{D}}_{v_i}\xi,v_1\right\rangle v_1+\left\langle\overline{\overline{D}}_{v_i}\xi,v_2\right\rangle v_2.$$

Thus from (3.7), we have

$$\overline{\overline{D}}_{v_i}\xi=-a_{i1}v_1-a_{i_2}v_2.$$

From $A_\xi(v_i)=-\overline{\overline{D}}_{v_i}\xi$ and $a_{21}=a_{12}=0$, $a_{11}=0$, we obtain

$$A_\xi=\begin{pmatrix}0&0\\0&a_{22}\end{pmatrix}.$$

Since eigenvalues of linear transformation $A_p:T_pM\to T_pM$ are principal curvatures of M at p, we obtain the following principal curvatures of M

$$K_1(p)=0 \text{ and } K_2(p)=a_{22}.$$

Hence Gauss and mean curvature of M at p are

$$K_e(p) = 0$$

$$H_d(p) = \frac{1}{2}a_{22},$$

where a_{22} is

$$a_{22} = \frac{\sqrt{\sinh^2\theta + \sin^2\varphi}}{e^{-v} + u\sinh\theta}.$$

Remark 3.10. *If we consider*

$$W_d = \sqrt{\cosh^2\theta - \cosh^2\varphi}e_1 + \sinh\theta\xi$$

the constant direction of spacelike surface with constant timelike angle in de Sitter space S_1^3, then we obtain similar results in Theorem 3.2-i.

Remark 3.11. *If the W_d constant direction of spacelike surface is chosen a timelike vector, then we obtain similar resultys in chapter 3.1*

4. Constant Timelike Angle Tangent Surfaces

4.1. Tangent Surface with Spacelike Axis

In this section we will focus on constant timelike angle spacelike tangent surfaces with spacelike axis in de Sitter space S_1^3.(see [2] and [6] for the Minkowski ambient space and Euclidean ambient space,respectively). Let $\alpha : I \to S_1^3 \subset \mathbb{R}_1^4$ be a regular spacelike curve given by arc-length. We define the tangent surface M ,which is generated by α , with

$$x(s,t) = \alpha(s)\cos t + \alpha'(s)\sin t\ ,\ \ (s,t) \in I \times \mathbb{R}\ . \tag{4.1}$$

The tangent plane at a point (s,t) of M is spanned by $\{x_s, x_t\}$, where

$$\left\{\begin{array}{ll} x_s & = \alpha'(s)\cos t + \alpha''(s)\sin t\ , \\ x_t & = -\alpha(s)\sin t + \alpha'(s)\cos t\ . \end{array}\right. . \tag{4.2}$$

By computing the coefficients of first fundamental form $\{E, F, G\}$ of M with respect to basis $\{x_s, x_t\}$, we get

$$\left\{\begin{array}{ll} E = & \langle x_s, x_s\rangle = 1 + \kappa_d^2(s)\sin^2 t\ , \\ F = & \langle x_s, x_t\rangle = 1\ , \\ G = & \langle x_t, x_t\rangle = 1\ . \end{array}\right.$$

Hence we have

$$EG - F^2 = \kappa_d^2(s)\sin^2 t\ .$$

Then, since $EG - F^2 > 0$, it is obvious that M is a spacelike surface. From Frenet-Serre type formulae , we obtain

$$\begin{cases} x(s,t) & = \alpha(s)\cos t + t(s)\sin t\ , \\ x_s(s,t) & = \alpha(s)\sin t + t(s)\cos t + n(s)\kappa_d(s)\sin t\ , \\ x_t(s,t) & = -\alpha(s)\sin t + t(s)\cos t\ . \end{cases} \tag{4.3}$$

Now let us calculate normal vector of M. As we already know the normal vector of M is

$$e = \frac{x \wedge x_s \wedge x_t}{\|x \wedge x_s \wedge x_t\|}\ . \tag{4.4}$$

Then, since

$$x \wedge x_s \wedge x_t = -(\alpha \wedge \alpha' \wedge \alpha'')\sin t\ ,$$

and

$$\|x \wedge x_s \wedge x_t\| = |\kappa_d \sin t| \quad , \quad \kappa_d \neq 0$$

we find

$$e = \pm\frac{\alpha \wedge \alpha' \wedge \alpha''}{|\kappa_d|} \tag{4.5}$$

Let us find W_d direction of constant timelike angle with spacelike axis surface M. Since (3.9) and

$$e_1 = \frac{x_s}{\|x_s\|} \text{ and } \|x_s\| = \sqrt{1 + \kappa_d^2 \sin^2 t}\ .$$

we get

$$\begin{aligned} W_d \quad & = \sin t\sqrt{\frac{\sinh^2\theta - \sinh^2\varphi}{1+\kappa_d^2\sin^2 t}}\alpha(s) + \cos t\sqrt{\frac{\sinh^2\theta - \sinh^2\varphi}{1+\kappa_d^2\sin^2 t}}t(s) + \\ & + \kappa_d(s)\sin t\sqrt{\frac{\sinh^2\theta - \sinh^2\varphi}{1+\kappa_d^2\sin^2 t}}n(s) + e(s)\cosh\theta\ . \end{aligned} \quad . \tag{4.6}$$

Theorem 4.1. *Let $\alpha : I \to S_1^3 \subset \mathbb{R}_1^4$ be a curve with $\kappa_d \neq 0$. If $x(s,t)$ tangent surface is constant timelike angle surface with spacelike axis, then α curve is planarly.*

Proof. Suppose that $x(s,t)$ tangent surface is constant timelike angle surface with spacelike axis such that α is a curve with $\kappa_d \neq 0$. Since

$$\xi = \frac{x \wedge x_s \wedge x_t}{\|x \wedge x_s \wedge x_t\|} = e\ ,$$

there exsist a $\theta > 0$ real number such that

$$\langle \xi, W_d \rangle = \langle e(s), W_d \rangle = \cosh\theta \quad .$$

If we differentiate the both sides of the last equation with respect to s then we get that

$$\langle e'(s), W_d \rangle = 0 \quad .$$

By the way we know that from Frenet-Serret equation system

$$e'(s) = \tau_d(s)\, n(s)\ .$$

Hence we get

$$\langle n(s), W_d\rangle = 0 \text{ or } \tau_d(s) = 0\ . \tag{4.7}$$

If in equation (4.7) $\langle n(s), W_d\rangle = 0$ then scalar producting of (4.6) equation with $n(s)$ that we have $t = 0$. This is contradict with definition of tangent surface. Therefore using equation (4.7) $\tau_d(s) = 0$ is obvious. It means that α is planarly line. □

Example 4.2. *Let $\alpha : I \to S_1^3 \subset \mathbb{R}_1^4$ be a regular curve given by arc-length*

$$\alpha(s) = \left(s \sinh(\arccos hs), s\cosh(\arccos hs), \sqrt{1-s^2}, 0\right).$$

Since the tangent surface M generated by α as the surface parametrized by

$$x(s,t) = \alpha(s)\cos t + \alpha'(s)\sin t\ ,\ \ (s,t) \in I \times \mathbb{R}\ .$$

The picture of the Stereographic projection of tangent surface appear in Figure 1

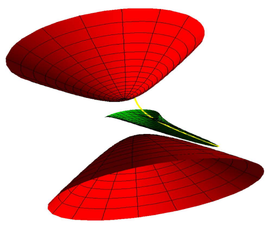

Figure 1:

Remark 4.3. *If we consider*

$$W_d = \sqrt{\left|\cosh^2\theta - \cosh^2\varphi\right|}e_1 + \sinh\theta\xi,$$

then we will get similar result.

Remark 4.4. *If the W_d constant direction of spacelike surface is chosen timelike, then we obtain similar results in chapter 4.1*

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