Constant Angle Spacelike Surface in de Sitter Space $S^3_1$

Tuğba Mert and Baki Karlığa

ABSTRACT: In this paper, using the angle between unit normal vector field of surfaces and a fixed spacelike axis in $R^4_1$, we develop two class of spacelike surface which are called constant timelike angle surfaces with timelike and spacelike axis in de Sitter space $S^3_1$. Moreover we give constant timelike angle tangent surfaces which are examples constant angle surfaces in de Sitter space $S^3_1$.

Key Words: Constant angle surfaces, de Sitter space, Helix.

Contents

1 Introduction And Results 79
2 Differential Geometry of de Sitter Space $S^3_1$ 80
3 Constant Timelike Angle Spacelike Surfaces 83
   3.1 Constant Timelike Angle Surfaces With Spacelike Axis . . . . . . . . 84
4 Constant Timelike Angle Tangent Surfaces 89
   4.1 Tangent Surface with Spacelike Axis . . . . . . . . . . . . . . . . . . . 89

1. Introduction And Results

A constant angle curve whose tangents make constant angle with a fixed direction in ambient space is called a helix. A surface whose tangent planes makes a constant angle with a fixed vector field of ambient space is called constant angle surface. Constant angle surfaces have been studied for arbitrary dimension in Euclidean space $E^n$ [13,14] and recently in product spaces $S^2 \times R$ [15], $H^2 \times R$ [16] or different ambient spaces $Nil_3$ [17]. In [1], Lopez and Munteanu studied constant hyperbolic angle surfaces whose unit normal timelike vector field makes a constant hyperbolic angle with a fixed timelike axis in Minkowski space $R^4_1$. In the literature constant timelike and spacelike angle surface have not been investigated both in hyperbolic space $H^3$ and de sitter space $S^3_1$. A constant timelike and a spacelike angle surface in Hyperbolic space $H^3$ are developed in our paper [19]. In this paper we introduce constant timelike angle spacelike surfaces in de Sitter space $S^3_1$.

Let $x : M \rightarrow R^4_1$ be an immersion of a surface $M$ into $R^4_1$. We say that $x$ is timelike (resp. spacelike, lightlike) if the induced metric on $M$ via $x$ is Lorentzian (resp. Riemannian, degenerated). If $\langle x, x \rangle = 1$, then $x$ is an immersion of $S^3_1$.

Let $x : M \rightarrow S^3_1$ be a immersion and let $\xi$ be a timelike unit normal vector field to $M$. If there exists spacelike direction $W$ such that timelike angle $\theta(\xi, U)$

Submitted July 11, 2014. Published March 02, 2016

Typeset by \TeX style.
© Soc. Paran. de Mat.
is constant on $M$, then $M$ is called constant timelike angle surfaces with spacelike axis.

Let $x : M \rightarrow S^3$ be a immersion and let $\xi$ be a timelike unit normal vector field to $M$. If there exists timelike direction $W$ such that timelike angle $\theta(\xi, U)$ is constant on $M$, then $M$ is called constant timelike angle surfaces with timelike axis.

2. Differential Geometry of de Sitter Space $S^3_1$

In this section, Differential geometry of curves and surfaces are summarized in de Sitter space $S^3_1$. Let $\mathbb{R}^4_1$ be 4-dimensional vector space equipped with the scalar product $\langle \cdot, \cdot \rangle$ which is defined by

$$\langle x, y \rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$ From now on, the constant angle surface is proposed in Minkowskian ambient space $\mathbb{R}^4_1$. $\mathbb{R}^4_1$ is 4-dimensional vector space equipped with the scalar product $\langle \cdot, \cdot \rangle$, than $\mathbb{R}^4_1$ is called Lorentzian 4-space or 4-dimensional Minkowski space. The Lorentzian norm (length) of $x$ is defined to be

$$\|x\| = \left| \langle x, x \rangle \right|^{1/2}. $$

If $(x_0^i, x_1^i, x_2^i, x_3^i)$ is the coordinate of $x_i$ with respect to canonical basis $\{e_0, e_1, e_2, e_3\}$ of $\mathbb{R}^4_1$, then the lorentzian cross product $x_1 \times x_2 \times x_3$ is defined by the symbolic determinant

$$x_1 \times x_2 \times x_3 = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ x_0^1 & x_1^1 & x_2^1 & x_3^1 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{vmatrix}. $$

One can easily see that

$$\langle x_1 \times x_2 \times x_3, x_4 \rangle = \det (x_1, x_2, x_3, x_4). $$

In [2], [3] and [5] Izimuya at all introduced and investigated differential geometry of curves and surfaces Hyperbolic 3-space. If $\langle x, x \rangle > 0$, $\langle x, x \rangle = 0$ or $\langle x, x \rangle < 0$ for any non-zero $x \in \mathbb{R}^4_1$, then we call that $x$ is spacelike, lightlike or timelike, respectively. In the rest of this section, we give background of context in [20].

Given a vector $v \in \mathbb{R}^4_1$ and a real number $c$, the hyperplane with pseudo normal $v$ is defined by

$$HP(v, c) = \{ x \in \mathbb{R}^4_1 | \langle x, v \rangle = c \}. $$

We say that $HP(v, c)$ is a spacelike hyperplane, timelike hyperplane or lightlike hyperplane if $v$ is timelike, spacelike or lightlike respectively in [20]. We have following three types of pseudo-spheres in $\mathbb{R}^4_1$:

- **Hyperbolic-3 space**: $H^3(-1) = \{ x \in \mathbb{R}^4_1 | \langle x, x \rangle = -1, x_0 \geq 1 \},$
- **de Sitter 3-space**: $S^3_1 = \{ x \in \mathbb{R}^4_1 | \langle x, x \rangle = 1 \},$
- **(open) lightcone**: $LC^* = \{ x \in \mathbb{R}^4_1/ \{0\} | \langle x, x \rangle = 0, x_0 > 0 \}.$
We also define the lightcone $3$–sphere
\[ S^3_v = \{ x = (x_0, x_1, x_2, x_3) \mid \langle x, x \rangle = 0, x_0 = 1 \}. \]

A hypersurface given by the intersection of $S^3_v$ with a spacelike (resp. timelike) hyperplane is called an elliptic hyperquadric (resp. hyperbolic hyperquadric). If $c \neq 0$ and $HP(v, c)$ is lightlike, then $HP(v, c) \cap S^3_v$ is a de Sitter horosphere, [20].

Let $U \subset \mathbb{R}^2$ be open subset, and let $x : U \rightarrow S^3_v$ be an embedding. If the vector subspace $U$ which generated by \{ $x_u$, $x_v$ \} is spacelike, then $x$ is called space-like surface, if $U$ contain at least a timelike vector field, then $x$ is called timelike surface in $S^3_v$.

In point of view Kasedou [20], we construct the extrinsic differential geometry on curves in $S^3_v$. Since $S^3_v$ is a Riemannian manifold, the regular curve $\gamma : I \rightarrow S^3_v$ is given by arclength parameter.

**Theorem 2.1.** i) If $\gamma : I \rightarrow S^3_v$ is a spacelike curve with unit speed, then Frenet-Serre type formulae is obtained

\[
\begin{align*}
\gamma'(s) &= t(s) \\
t'(s) &= \kappa_d(s) n(s) - \gamma(s) \\
n'(s) &= -\kappa_d(s) t(s) - \tau_d(s) e(s) \\
e'(s) &= -\tau_d(s) n(s)
\end{align*}
\]

where $\kappa_d(s) = \|t'(s) + \gamma(s)\|$ and $\tau_d(s) = -\frac{\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))}{(\kappa_d(s))^2}$.

ii) If $\gamma : I \rightarrow S^3_v$ is a timelike curve with unit speed, then Frenet-Serre type formulae is obtained

\[
\begin{align*}
\gamma'(s) &= t(s) \\
t'(s) &= \kappa_d(s) n(s) + \gamma(s) \\
n'(s) &= -\kappa_d(s) t(s) + \tau_d(s) e(s) \\
e'(s) &= -\tau_d(s) n(s)
\end{align*}
\]

where $\kappa_d(s) = \|t'(s) - \gamma(s)\|$ and $\tau_d(s) = \frac{\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))}{(\kappa_d(s))^2}$.

It is easily see that $\kappa_d(s) = 0$ if and only if there exists a lightlike vector $c$ such that $\gamma(s) - c$ is a geodesic.

Now we give extrinsic differential geometry on surfaces in $S^3_v$ due to Kasedou [20].

Let $U \subset \mathbb{R}^2$ is an open subset and $x : U \rightarrow S^3_v$ is a regular surface $M = x(U)$. If $e(u)$ is defined as follows

\[ e(u) = \frac{x(u) \wedge x_{u_1}(u) \wedge x_{u_2}(u)}{\|x(u) \wedge x_{u_1}(u) \wedge x_{u_2}(u)\|} \]

then

\[ \langle e, x \rangle \equiv \langle e, x_{u_1} \rangle \equiv 0, \langle e, e \rangle = -1 \]
where \( x_{u_i} = \frac{\partial x}{\partial u_i} \). Thus there is de Sitter Gauss image of \( x \) which is defined by mapping
\[
E : U \to S^1, \ E ( u ) = e ( u ) .
\]
The lightcone Gauss image of \( x \) is defined by map
\[
L^\pm : U \to LC^*, \ L^\pm ( u ) = x ( u ) \pm e ( u ) .
\]
Let \( dx ( u_0 ) \) and \( 1_{T_p M} \) be identify mapping on the tangent space \( T_p M \). So derivate \( dx ( u_0 ) \) can be identified with \( T_p M \) relate to identification of \( U \) and \( M \).

The linear transformation
\[
S^\pm_p := - dL^\pm ( u_0 ) : T_p M \to T_p M
\]
and
\[
A_p := - dE ( u_0 ) : T_p M \to T_p M
\]
is called the hyperbolic shape operator and de Sitter shape operator of \( M \) at \( p = x ( u_0 ) \).

Let \( \bar{K}^\pm_i ( p ) \) and \( K_i ( p ) \), \( ( i = 1, 2 ) \) be the eigenvalues of \( S^\pm_p \) and \( A_p \). Since
\[
S^\pm_p = - 1_{T_p M} \pm A_p ,
\]
\( S^\pm_p \) and \( A_p \) have same eigenvectors and relations
\[
\bar{K}^\pm_i ( p ) = - 1 \pm K_i ( p ) .
\]
\( \bar{K}^\pm_i ( p ) \) and \( K_i ( p ) \), \( ( i = 1, 2 ) \) are called hyperbolic and de Sitter principal curvatures of \( M \) at \( p = x ( u_0 ) \).

Let \( K^\pm_n ( s_0 ) \) be a unit speed curve on \( M \), with \( p = \gamma ( u_1 ( s_0 ) , u_2 ( s_0 ) ) \). We consider the hyperbolic curvature vector \( k ( s ) = t' ( s ) - \gamma ( s ) \) and the de Sitter normal curvature
\[
K^\pm_n ( s_0 ) = \langle k ( s_0 ) , L^\pm ( u_1 ( s_0 ) , u_2 ( s_0 ) ) \rangle = \langle t' ( s_0 ) , L^\pm ( u_1 ( s_0 ) , u_2 ( s_0 ) ) \rangle + 1
\]
of \( \gamma ( s ) \) at \( p = \gamma ( s_0 ) \). The de Sitter normal curvature depends only on the point \( p \) and the unit tangent vector of \( M \) at \( p \) analogous to the Euclidean case. Hyperbolic normal curvature of \( \gamma ( s ) \) is defined to be
\[
\bar{K}^\pm_n ( s ) = K^\pm_n ( s ) - 1 .
\]
The Hyperbolic Gauss curvature of \( M = x ( U ) \) at \( p = x ( u_0 ) \) is defined to be
\[
K^\pm_n ( u_0 ) = \det S^\pm_p = \bar{K}^\pm_1 ( p ) \bar{K}^\pm_2 ( p ) .
\]
The Hyperbolic mean curvature of $M = x(u)$ at $p = x(u_0)$ is defined to be
\[ H^\pm_{h}(u_0) = \frac{1}{2} \text{Trace} S^\pm_p = \frac{K^\pm_1(p) + K^\pm_2(p)}{2}. \]

The extrinsic (de Sitter) Gauss curvature is defined to be
\[ K_e(u_0) = \det A_p = K_1(p) K_2(p), \]
and the de Sitter mean curvature is
\[ H_d(u_0) = \frac{1}{2} \text{Trace} A_p = \frac{K_1(p) + K_2(p)}{2}. \]

3. Constant Timelike Angle Spacelike Surfaces

Let us show the space of the tangent vector fields on $M$ with $X(M)$ and denote the Levi-Civita connections of $\mathbb{R}^4_1$, $S^3_1$ and $M$ by $\nabla, \bar{\nabla}$ and $\bar{D}$. Then for each $X, Y \in X(M)$, we have
\[ D_X Y = \left( \bar{\nabla}_X Y \right)^T, \bar{V}(X, Y) = \left( \bar{\nabla}_X Y \right)^\perp \]
and
\[ \bar{D}_X Y = \bar{D}_X Y - \left( X, Y \right)_x, \bar{D}_X Y = D_X Y + \bar{V}(X, Y), \quad (3.1) \]
where the superscript $^T$ and $^\perp$ denote the tangent and normal component of $\bar{\nabla}_X Y$.

(3.1) equation is called the Gauss formula of $S^3_1$ and $M$.

If $\xi$ is a normal vector field of $M$ on $S^3_1$, then the Weingarten Endomorphism $A_\xi(X)$ and $B_x(X)$ are denoted by the tangent components of $-\bar{D}_X \xi$ and $-\bar{D}_X x$. So the Weingarten equations of the vector field $\xi$ and $x$ is like
\[ \begin{cases} 
A_\xi(X) = -\bar{D}_X \xi - \left( \bar{D}_X x, \xi \right)_x, \\
B_x(X) = -\bar{D}_X x - \left( \bar{D}_X x, \xi \right)_\xi.
\end{cases} \quad (3.2) \]

It is obvious that $A_\xi(X)$ and $B_x(X)$ are linear and self adjoint map for each $p \in M$. That is
\[ \langle A_\xi(X), Y \rangle = \langle X, A_\xi(Y) \rangle \text{ and } \langle B_x(X), Y \rangle = \langle X, B_x(Y) \rangle. \]

The eigenvalues $K_i(p)$ and $\bar{K}_i(p)$ of $(A_\xi)_p$ are called the principal curvature of $M$ on $S^3_1$. The eigenvalues $\bar{K}_i(p)$ of $(B_x)_p$ are called the principal curvature of $M$ in $\mathbb{R}^4_1$. Also, for $X, Y \in X(M)$ we have
\[ \langle A_\xi(X), Y \rangle = \left( \bar{V}(X, Y), \xi \right), \quad \langle B_x(X), Y \rangle = \left( \bar{V}(X, Y), x \right). \]

Since $\bar{V}(X, Y)$ is second fundamental form of $M$ on $\mathbb{R}^4_1$, so we can write as follows
\[ \bar{V}(X, Y) = -\left( \bar{V}(X, Y), \xi \right)_\xi + \left( \bar{V}(X, Y), x \right)_x \]
\[ \hat{V}(X,Y) = -\langle A_\xi(X),Y \rangle \xi + \langle B_x(X),Y \rangle x. \]

Let \( \{v_1, v_2\} \) be a base of \( T_pM \) tangent plane and let us denote
\[ a_{ij} = \langle \hat{V}(v_i,v_j), \xi \rangle = \langle A_\xi(v_i), v_j \rangle \]
\[ b_{ij} = \langle \hat{V}(v_i,v_j), x \rangle = \langle B_x(v_i), v_j \rangle \]

Therefore
\[ \overline{D}_XY = D_XY + \hat{V}(X,Y) \]

and also since
\[ \bar{D}_XY = D_XY - \langle A_\xi(X),Y \rangle \xi \]

we obtain
\[ \overline{D}_XY = D_XY - \langle A_\xi(X),Y \rangle \xi - \langle X,Y \rangle x. \]

On the other hand for \( \{v_1, v_2\} \) base, we get
\[ \overline{D}_{vi}v_j = D_{vi}v_j - a_{ij} \xi - \langle v_i, v_j \rangle x. \]

If this basis is orthonormal, then we have from (3.1) and (3.2)
\[ \overline{D}_{vi}v_j = D_{vi}v_j - a_{ij} \xi, \]
\[ \overline{D}_{vi} \xi = -a_{i1}v_1 - a_{i2}v_2, \]
\[ \overline{D}_{vi}x = -b_{i1}v_1 - b_{i2}v_2. \]

### 3.1. Constant Timelike Angle Surfaces With Spacelike Axis

**Definition 3.1.** Let \( U \subset \mathbb{R}^2 \) be open set, let \( x : U \rightarrow S^3_1 \) be an embedding where \( M = x(U) \). Let \( x : M \rightarrow S^3_1 \) and \( \xi \) is timelike unit normal vector field on \( M \), if there exist a constant spacelike vector \( W \) which has a constant timelike angle with \( \xi \), then \( M \) is called \textit{constant timelike angle surface with spacelike axis}.

Since our surface is a spacelike surface, \( \{x_u, x_v\} \) tangent vectors must be spacelike vectors. Let \( M \) be a spacelike surface with constant angle with spacelike axis and \( \xi \) is unit normal vector of \( M \) on \( S^3_1 \). Let us denote that the timelike angle between timelike vector \( \xi \) and spacelike vector \( W \) with \( \theta \). That is from [11]
\[ \langle \xi, W \rangle = \sin h(-\theta). \]

If timelike angle \( \theta = 0 \), then \( \xi = W \). Throughout this section, without loss of generality we assume that \( \theta \neq 0 \). If \( W_T \) is the projection of \( W \) on the tangent plane of \( M \), then we decompose \( W \) as
\[ W = W_T + W_N. \]
So that we write 
\[ W = W^T + \lambda_1 \xi + \lambda_2 x. \]

If we take inner product of both sides of this inequality first with \( \xi \), then with \( x \) 
\[ \lambda_1 = -\sin h (-\theta), \quad \lambda_2 = \langle W, x \rangle. \]

On the other hand, since \( W \) and \( x \) are two spacelike vector fields, then we can use define of the spacelike and timelike angle between \( W \) and \( x \).

**Theorem 3.2.** i) If \( \varphi \) is the spacelike angle between spacelike vectors \( W, x \) then we can write for \[11\]
\[ W = \sqrt{\sinh^2 \theta + \sin^2 \varphi} e_1 + (\sinh \theta) \xi + \cos \varphi x \]
and de Sitter projection \( W_d \) of \( W \) as follows
\[ W_d = \sqrt{\sinh^2 \theta + \sin^2 \varphi} e_1 + (\sinh \theta) \xi. \]

ii) If \( \varphi \) is timelike angle between spacelike vectors \( W \) and \( x \), then we can write 
\[ W = \sqrt{\cosh^2 \theta - \cosh^2 \varphi} e_1 + (\sinh \theta) \xi - (\cosh \varphi) x. \]
and de sitter projection \( W_d \) of \( W \) as follows
\[ W_d = \sqrt{\cosh^2 \theta - \cosh^2 \varphi} e_1 + (\sinh \theta) \xi. \]

Let \( e_1 = \frac{W^T}{\|W^T\|} \) and let consider \( e_2 \) be a unit vector field on \( M \) orthogonal to \( e_1 \). Then we have an oriented orthonormal basis \( \{e_1, e_2, \xi, x\} \) for \( \mathbb{R}^4 \). Since \( W_d \) is constant vector field on \( S^3 \) and \( D_{e_2} W_d = D_{e_2} W_d = 0 \), we have
\[ \sqrt{\sinh^2 \theta + \sin^2 \varphi} D_{e_2} e_1 + (\sinh \theta) D_{e_2} \xi = 0. \]

By (3.11), we obtain
\[ \sqrt{\sinh^2 \theta + \sin^2 \varphi} \langle D_{e_2} e_1, \xi \rangle + (\sinh \theta) \langle D_{e_2} \xi, \xi \rangle = 0, \]
or
\[ \sqrt{\sinh^2 \theta + \sin^2 \varphi} u_{21} = 0. \]

Since \( \sqrt{\sinh^2 \theta + \sin^2 \varphi} \neq 0 \), we conclude \( u_{21} = u_{12} = 0 \). Using (3.7) in (3.11), we get
\[ D_{e_2} e_1 = \frac{\sinh \theta}{\sqrt{\sinh^2 \theta + \sin^2 \varphi}} a_{23} e_2. \]
Similarly, since $W_d$ is a constant vector field on $S^3_1$, then we have

$$D_{e_1}W_d = 0 \text{ and } D_{e_1}W_d = -\sqrt{\sinh^2 \theta + \sin^2 \varphi}x.$$ \hspace{1cm} (3.13)

By (3.9), we obtain

$$D_{e_1}W_d = \sqrt{\sinh^2 \theta + \sin^2 \varphi}e_1 + \sinh \theta D_{e_1}\xi.$$ \hspace{1cm} (3.14)

By (3.13) and (3.14), we conclude that

$$\sqrt{\sinh^2 \theta + \sin^2 \varphi}\langle D_{e_1}e_1, \xi \rangle = -\sqrt{\sinh^2 \theta + \sin^2 \varphi}x.$$ \hspace{1cm} (3.15)

By (3.15), we get

$$\sqrt{\sinh^2 \theta + \sin^2 \varphi}\langle D_{e_1}e_1, \xi \rangle = 0,$$

or

$$\sqrt{\sinh^2 \theta + \sin^2 \varphi}a_{11} = 0.$$

Since $\sqrt{\sinh^2 \theta + \sin^2 \varphi} \neq 0$, we conclude $a_{11} = 0$. Also, using (3.7) in (3.15), we obtain

$$D_{e_1}e_1 = -x.$$ \hspace{1cm} (3.16)

Now we have proved the following theorem.

**Theorem 3.3.** If $D$ is Levi-Civita connection for a constant timelike angle with spacelike axis spacelike surface in $S^3_1$ is given by

$$D_{e_1}e_1 = 0, \quad D_{e_2}e_1 = \frac{\sinh \theta}{\sqrt{\sinh^2 \theta + \sin^2 \varphi}}a_{22}e_2,$$

$$D_{e_1}e_2 = 0, \quad D_{e_2}e_2 = \frac{-\sinh \theta}{\sqrt{\sinh^2 \theta + \sin^2 \varphi}}a_{22}e_1.$$

**Corollary 3.4.** Let $M$ be a spacelike surface which is a constant timelike angle with spacelike axis on $S^3_1$. Then, there exist local coordinates $u$ and $v$ such that the metric on $M$ writes as $\langle \cdot, \cdot \rangle = du^2 + \beta^2 dv^2$, where $\beta = \beta (u, v)$ is a smooth function on $M$, i.e. the coefficients of the first fundamental form are $E = 1$, $F = 0$, $G = \beta^2$.

Now we find the $x = x(u, v)$ parametrization of the surface $M$ with respect to the metric $\langle \cdot, \cdot \rangle = du^2 + \beta^2 dv^2$ on $M$. By the above parametrization $x(u, v)$ can obtain the following corollary.

**Corollary 3.5.** There exist an equation system for constant timelike angle with spacelike axis spacelike surface on $S^3_1$ which is

$$\begin{align*}
x_{uu} &= -x \\
x_{uv} &= \frac{\beta}{\beta}x_v \\
x_{vv} &= -\beta \beta x_u + \frac{\beta}{\beta}x_v - \beta^2 a_{22}\xi - \beta^2 x.
\end{align*}$$ \hspace{1cm} (3.17)
Corollary 3.6. Let $\xi$ be unit normal vector of the constant timelike angle with spacelike axis spacelike surface $M$. Then the equation below hold

$$\begin{aligned}
\xi_u &= \overrightarrow{D}_{x_u} \xi = 0 \\
\xi_v &= \overrightarrow{D}_{x_v} \xi = -a_{22} x_v.
\end{aligned}$$

(3.18)

Since $\xi_{uv} = \xi_{vu} = 0$, we have $\overrightarrow{D}_{x_v}(-a_{22} x_u) = 0$. Using $a_{12} = 0$, $\overrightarrow{D}_{x_v} x_v = \overrightarrow{D}_{x_u} x_u$ and Theorem 2.1, we obtain

$$\left(a_{22}\right)_u + \frac{\sinh \theta}{\sqrt{\sinh^2 \theta + \sin^2 \varphi}} (a_{22})^2 = 0$$

(3.19)

So that

$$\left(a_{22}\right)_u + \frac{\beta}{\beta} a_{22} = 0$$

(3.20)

and then we get obtain

$$\left(\beta a_{22}\right)_u = 0.$$  

(3.21)

By (3.21), we see that there exist a smooth function $\psi = \psi(v)$ depending on $v$ such that

$$\beta a_{22} = \psi(v).$$

(3.22)

Proposition 3.7. Let $x = x(u, v)$ be parametrization of a spacelike surface which is constant timelike angle with spacelike axis on $S^3_1$. If $a_{22} = 0$ on $M$, then the $x$ describes an flat plane of de Sitter space $S^3_1$.

Proof. If $a_{22} = 0$ on $M$,then by (3.18)

$$\begin{aligned}
\xi_u &= 0 \\
\xi_v &= 0
\end{aligned}$$

This imply we have $\xi$ is a constant vector field which normal vector is $M$ surface. 

Thus $x = x(u, v)$ is de-Sitter plane in $S^3_1$.

From now on, we are going to assume that $a_{22} \neq 0$. By solving equation (3.19), we obtain a function $\alpha = \alpha(v)$ such that

$$a_{22} = \frac{\sqrt{\sinh^2 \theta + \sin^2 \varphi}}{u \sinh \theta + \alpha(v)}.$$ 

Therefore by (3.22), we obtain

$$\beta(u, v) = \frac{\psi(v)}{\sqrt{\sinh^2 \theta + \sin^2 \varphi}} (u \sinh \theta + \alpha(v)).$$

Consequently,
\begin{align*}
x_{uu} &= -x, \\
x_{uv} &= \frac{\sinh \theta \psi (v)}{u \sinh \theta + \alpha (v)} x_v, \\
x_{vv} &= -\psi^2 (v) \sinh \theta (u \sinh \theta + \alpha (v)) x_u + \left[ \frac{\psi' (v)}{\psi (v)} + \frac{\alpha' (v)}{u \sinh \theta + \alpha (v)} \right] x_v \\
&= \left[ \frac{\psi^2 (v) (u \sinh \theta + \alpha (v))}{\sqrt{\sinh^2 \theta + \sin^2 \varphi}} \right] \xi - \left[ \frac{\psi^2 (v) (u \sinh \theta + \alpha (v))^2}{\sinh^2 \theta + \sin^2 \varphi} \right] x_v.
\end{align*}

Here, if we specifically choose \( \psi (v) = e^v \sqrt{\sinh^2 \theta + \sin^2 \varphi} \) and \( \alpha (v) = e^{-v} \), then this equation system becomes

\begin{align*}
x_{uu} &= -x, \\
x_{uv} &= \frac{e^v \sinh \theta}{1 + ue^v \sinh \theta} x_v, \\
x_{vv} &= -e^v \sinh \theta (ue^v \sinh \theta + 1) x_u + \frac{ue^v \sinh \theta}{ue^v \sinh \theta + 1} x_v - e^v \sqrt{\sinh^2 \theta + \sin^2 \varphi} (ue^v \sinh \theta + 1) \xi - (ue^v \sinh \theta + 1)^2 x_v.
\end{align*}

Now we have the following Theorem.

**Theorem 3.8.** If \( M \) is satisfying (3.23), then there exist local coordinates \( u \) and \( v \) on \( M \) with having the parametrization

\begin{align*}
x_i (u, v) = \begin{pmatrix} -c_{1i} (v) \\ 2e^v \sinh \theta (ue^v \sinh \theta + 1)^2 + c_{2i} (v) \end{pmatrix}, \quad i = 1, 2, 3, 4 \tag{3.24}
\end{align*}

**Proof.** From (3.23), the proof is clear. \( \square \)

**Example 3.9.** We can calculate Gauss and mean curvature of a spacelike surface with constant angle spacelike axis in de Sitter space \( S^3_1 \). Since

\[ \overline{D}_X \xi = D_X \xi \]

we can write

\[ \overline{D}_v \xi = (\overline{D}_v \xi, v_1) v_1 + (\overline{D}_v \xi, v_2) v_2. \]

Thus from (3.7), we have

\[ \overline{D}_v \xi = -a_{11} v_1 - a_{12} v_2. \]

From \( A_\xi (v_i) = -\overline{D}_v \xi \) and \( a_{21} = a_{12} = 0 \), \( a_{11} = 0 \), we obtain

\[ A_\xi = \begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}. \]

Since eigenvalues of linear transformation \( A_p : T_p M \rightarrow T_p M \) are principal curvatures of \( M \) at \( p \), we obtain the following principal curvatures of \( M \)

\[ K_1 (p) = 0 \] and \( K_2 (p) = a_{22} \).
Hence Gauss and mean curvature of $M$ at $p$ are

$$K_e(p) = 0$$

$$H_d(p) = \frac{1}{2} a_{22},$$

where $a_{22}$ is

$$a_{22} = \sqrt{\sinh^2 \theta + \sin^2 \varphi} / (c - v + u \sinh \theta).$$

**Remark 3.10.** If we consider

$$W_d = \sqrt{cosh^2 \theta - cosh^2 \varphi e_1 + sinh \theta \xi}$$

the constant direction of spacelike surface with constant timelike angle in de Sitter space $S^3_1$, then we obtain similar results in Theorem 3.2-i.

**Remark 3.11.** If the $W_d$ constant direction of spacelike surface is chosen a timelike vector, then we obtain similar results in chapter 3.1

### 4. Constant Timelike Angle Tangent Surfaces

#### 4.1. Tangent Surface with Spacelike Axis

In this section we will focus on constant timelike angle spacelike tangent surfaces with spacelike axis in de Sitter space $S^3_1$. (see [2] and [6] for the Minkowski ambient space and Euclidean ambient space, respectively). Let $\alpha : I \to S^3_1 \subset \mathbb{R}^4_1$ be a regular spacelike curve given by arc-length. We define the tangent surface $M$, which is generated by $\alpha$, with

$$x(s, t) = \alpha(s) \cos t + \alpha'(s) \sin t, \quad (s, t) \in I \times \mathbb{R}.$$

The tangent plane at a point $(s, t)$ of $M$ is spanned by $\{x_s, x_t\}$, where

$$\begin{align*}
    x_s &= \alpha'(s) \cos t + \alpha''(s) \sin t, \\
    x_t &= -\alpha(s) \sin t + \alpha'(s) \cos t.
\end{align*}$$

By computing the coefficients of first fundamental form $\{E, F, G\}$ of $M$ with respect to basis $\{x_s, x_t\}$, we get

$$\begin{align*}
    E &= \langle x_s, x_s \rangle = 1 + \kappa_d^2(s) \sin^2 t, \\
    F &= \langle x_s, x_t \rangle = 1, \\
    G &= \langle x_t, x_t \rangle = 1.
\end{align*}$$

Hence we have

$$EG - F^2 = \kappa_d^2(s) \sin^2 t.$$
Then, since $EG - F^2 > 0$, it is obvious that $M$ is a spacelike surface. From Frenet-Serre type formulae, we obtain
\[
\begin{align*}
x(s, t) &= \alpha(s) \cos t + t(s) \sin t, \\
x_s(s, t) &= \alpha(s) \sin t + t(s) \cos t + n(s) \kappa_d(s) \sin t, \\
x_t(s, t) &= -\alpha(s) \sin t + t(s) \cos t.
\end{align*}
\] (4.3)

Now let us calculate normal vector of $M$. As we already know the normal vector of $M$ is
\[
e = \frac{x \wedge x_s \wedge x_t}{\|x \wedge x_s \wedge x_t\|}. \quad (4.4)
\]

Then, since
\[
x \wedge x_s \wedge x_t = -(\alpha \wedge \alpha' \wedge \alpha'') \sin t,
\]
and
\[
\|x \wedge x_s \wedge x_t\| = |\kappa_d \sin t|, \quad \kappa_d \neq 0
\]
we find
\[
e = \pm \frac{\alpha \wedge \alpha' \wedge \alpha''}{|\kappa_d|}. \quad (4.5)
\]

Let us find $W_d$ direction of constant timelike angle with spacelike axis surface $M$. Since (3.9) and
\[
e_1 = \frac{x_s}{\|x_s\|} \quad \text{and} \quad \|x_s\| = \sqrt{1 + \kappa_d^2 \sin^2 t}.
\]
we get
\[
W_d = \sin t \sqrt{\frac{\sinh^2 \theta - \sinh^2 \varphi}{1 + \kappa_d^2 \sin^2 t}} \alpha(s) + \cos t \sqrt{\frac{\sinh^2 \theta - \sinh^2 \varphi}{1 + \kappa_d^2 \sin^2 t}} \tau(s) + \\
+ \kappa_d(s) \sin t \sqrt{\frac{\sinh^2 \theta - \sinh^2 \varphi}{1 + \kappa_d^2 \sin^2 t}} n(s) + e(s) \cosh \theta. \quad (4.6)
\]

**Theorem 4.1.** Let $\alpha : I \to S^3_1 \subset \mathbb{R}^4_1$ be a curve with $\kappa_d \neq 0$. If $x(s, t)$ tangent surface is constant timelike angle surface with spacelike axis, then $\alpha$ curve is planarly.

**Proof.** Suppose that $x(s, t)$ tangent surface is constant timelike angle surface with spacelike axis such that $\alpha$ is a curve with $\kappa_d \neq 0$. Since
\[
\xi = \frac{x \wedge x_s \wedge x_t}{\|x \wedge x_s \wedge x_t\|} = e,
\]
there exist a $\theta > 0$ real number such that
\[
\langle \xi, W_d \rangle = \langle e(s), W_d \rangle = \cosh \theta.
\]

If we differentiate the both sides of the last equation with respect to $s$ then we get that
\[
\langle e'(s), W_d \rangle = 0.
\]
By the way we know that from Frenet-Serret equation system

\[ e'(s) = \tau_d(s)n(s). \]

Hence we get

\[ \langle n(s), W_d \rangle = 0 \text{ or } \tau_d(s) = 0. \tag{4.7} \]

If in equation (4.7) \( \langle n(s), W_d \rangle = 0 \) then scalar producting of (4.6) equation with \( n(s) \) that we have \( t = 0 \). This is contradict with definition of tangent surface. Therefore using equation (4.7) \( \tau_d(s) = 0 \) is obvious. It means that \( \alpha \) is planar line.

\[ \square \]

**Example 4.2.** Let \( \alpha : I \to S^3_1 \subset \mathbb{R}^4_1 \) be a regular curve given by arc-length

\[ \alpha(s) = \left( s \sinh(\arccosh s), s \cosh(\arccosh s), \sqrt{1-s^2}, 0 \right). \]

Since the tangent surface \( M \) generated by \( \alpha \) as the surface parametrized by

\[ x(s, t) = \alpha(s) \cos t + \alpha'(s) \sin t, (s, t) \in I \times \mathbb{R}. \]

The picture of the Stereographic projection of tangent surface appear in Figure 1

![Figure 1:](image)

**Remark 4.3.** If we consider

\[ W_d = \sqrt{\cosh^2 \theta - \cosh^2 \varphi} |e_1 + \sinh \theta \xi, \]

then we will get similar result.

**Remark 4.4.** If the \( W_d \) constant direction of spacelike surface is chosen timelike, then we obtain similar results in chapter 4.1.
References


Tuğba Mert
Cumhuriyet University
E-mail address: tmert@cumhuriyet.edu.tr

and

Baki Karliga
Gazi University
E-mail address: karliga@gazi.edu.tr