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Generalized Quasi-Uniformity in terms of Covers

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abstract: Recently g-quasi uniformity has been introduced, and in the literature there already is the notion of strong quasi-uniform cover for quasi-uniform spaces. Here in this paper we generalize the notion of strong quasi-uniform cover to study g-quasi uniformity in terms of it. As applications, we formulate some g-topological concepts via this new type of cover.

Key Words: g-quasi uniformity, strong g-quasi uniform cover, g-topological properties.

Contents

1. Introduction and Preliminaries

In 2002, Á. Császár introduced the concept of generalized topology. Actually the study was initiated in 1997 with the paper [1] by Császár himself. Though the latter paper was about monotonic mappings and the term generalized topology was never mentioned there, it paved the way for generalized topology. This study was further carried on in [2]. Later in [3], generalized topology came as a particular case of this notion. Following [3], till now generalized topology has been studied thoroughly by a large number of mathematicians (e.g. see [6], [7], [8], [9], [12]). Now we recall some basic facts about generalized topology from the existing literature.

Let X be a non-empty set and $\mu \subseteq \exp(X)$. Then μ is called a generalized topology or in short, a GT, if $\emptyset \in \mu$ and $U_\alpha \in \mu$ $(\forall \alpha \in \Lambda)$ implies $\bigcup\limits_{\alpha\in\Lambda} U_\alpha \in \mu$. Here (X, μ) is called a generalized topological space or simply, a GTS. The members of a GT are called generalized open sets. As in the case of topological space, here also a subset is called generalized closed if it is the complement of a generalized open set. For a subset A of X, the μ-closure of A, denoted by $c_\mu(A)$, is defined to be the least μ-closed set containing A i.e., the intersection of all μ-closed sets containing A and it is given by the set $\{x \in X : A \cap G \neq \emptyset, \forall G \in \mu \text{ with } x \in G\}$. So A is μ-closed

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if $A = c_\mu(A)$ and vice-versa. The μ-interior of A, written as $i_\mu(A)$, is defined as the greatest μ-open set contained in A i.e., the union of all μ-open sets contained in A and it is given by the formula $i_\mu(A) = \{x \in X : \exists G \in \mu \text{ such that } x \in G \subseteq A\}$. So A is μ-open if and only if $A = i_\mu(A)$. A subcollection μ' of $\exp(X)$ forms a base for some GT μ on X if $\emptyset \in \mu'$ and the collection consisting of arbitrary unions of members of μ' is the generalized topology μ, and in this case the GT μ is said to be generated by μ'. A GT is called a strong GT if X is also a member of the GT. Here we may take a note that in some different context, the notion of strong generalized topology was introduced in [11], with the terminology, supratopology. If (X', μ') is another GTS then a function $f : X \to X'$ is said to be (μ, μ')-continuous (or simply g-continuous, where μ and μ' are known) if $G' \in \mu' \implies f^{-1}(G') \in \mu$. Here μ' can be replaced by any base of μ'. This notion of generalized continuity i.e., g-continuity has an equivalent form in terms of the points of the space. f will be (μ, μ')-continuous if for each $x \in X$ and $G' \in \mu'$ with $f(x) \in G'$, $\exists G \in \mu$ such that $x \in G$ and $f(G) \subseteq G'$.

Recently g-quasi uniformity has been introduced in [10] as a straightforward generalization of quasi-uniformity. In the same paper it has also been shown that a g-quasi uniformity induces a strong GT on a set and vice-versa. This is also analogous to the corresponding result of topology concerning quasi-uniformity. Again the notion of strong quasi-uniform cover in a topological space had been introduced and quasi-uniformity had been characterized axiomatically by such covers in [13], while some topological properties are also characterized in [15] in terms of such covers. Now the natural question arises as to whether such a characterization is possible for the generalized premise of a GTS. To answer it we introduce the notion of strong g-quasi uniform cover. With this the desired characterization is achieved. Now we list some basic definitions and results about generalized quasi-uniformity recalled from [10].

Definition 1.1. *Let X be a non-empty set. A non-empty collection $\mathcal{U} \subseteq \exp(X \times X)$ is said to form a generalized quasi uniformity, or simply a g-quasi uniformity on X if*

1. *$\Delta \subseteq U$, $\forall U \in \mathcal{U}$, where $\Delta = \{(x, x) : x \in X\}$.*
2. *$U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$ implies $V \in \mathcal{U}$.*
3. *$U \in \mathcal{U}$ implies $\exists V \in \mathcal{U}$ such that $V \circ V \subseteq U$, where $W \circ W' = \{(x, z) : \exists y \in X$ such that $(x, y) \in W, (y, z) \in W'\}$, for $W, W' \in \mathcal{U}$.*

The pair $(X, \mathcal{U})$ is called a g-quasi uniform space.

Definition 1.2. *Let $(X, \mathcal{U})$ be a g-quasi uniform space. Then $\mathcal{B} \subseteq \mathcal{U}$ is said to be a base for $\mathcal{U}$ if for any $U \in \mathcal{U}, \exists B \in \mathcal{B}$ such that $B \subseteq U$.*

In this case the g-quasi uniformity $\mathcal{U}$ is called the g-quasi uniformity generated by $\mathcal{B}$.

Definition 1.3. *Let X be a non-empty set. A non-empty family $\mathcal{B} \subseteq \exp(X \times X)$ forms a base for a g-quasi uniformity on X if*

1. $\Delta \subseteq B, \forall B \in \mathcal{B}$.

2. $B \in \mathcal{B}$ *implies* $\exists W \in \mathcal{B}$ *such that* $W \circ W \subseteq B$.

Lemma 1.4. *Let* (X, μ) *be a strong* GTS*. Then* $\mathcal{B} = \{B_G : G \in \mu\}$ *forms a base for some g-quasi uniformity on* X *such that* $\mu(\mathcal{B}) = \mu$*, where* $B_G = (G \times G) \cup ((X \setminus G) \times X)$*, for* $G \in \mu$.

The induced g-quasi uniformity as obtained in the above Lemma will be called 'Pervin g-quasi uniformity', remembering the analogous construction by Pervin [14] for topological spaces.

Theorem 1.5. *Let* $(X, \mathcal{U})$ *be a g-quasi uniform space. Then the collection* $\{G \subseteq X : g \in G \Rightarrow \exists U \in \mathcal{U}$ *such that* $g \in U(g) \subseteq G\}$ *forms a strong* GT *on* X*, where* $U(g) = \{x \in X : (g, x) \in U\}$.

This GT *is called the* GT *induced by* $\mathcal{U}$ *on* X *and is denoted by* $\mu(\mathcal{U})$.

Definition 1.6. *Let* $(X, \mathcal{U}_X)$ *and* $(Y, \mathcal{U}_Y)$ *be two g-quasi uniform spaces. Then a function* $f : X \to Y$ *is called g-quasi uniformly continuous if for each* $V \in \mathcal{U}_Y, \exists W \in \mathcal{U}_X$ *such that* $(x_1, x_2) \in W \implies (f(x_1), f(x_2)) \in V$.

Lastly we set the following definitions :

Definition 1.7. *A base* $\mathcal{B}$ *for a g-quasi uniformity is called transitive if each* $B \in \mathcal{B}$ *is a transitive relation i.e.,* $B \circ B = B$.

Definition 1.8. *A g-quasi uniformity with a transitive base is called a transitive g-quasi uniformity.*

Example 1.9. *For any given strong* GTS*, the Pervin g-quasi uniformity is transitive.*

Proof: It is a routine check. □

2. Strong g-Quasi Uniform Covers

Let X be a non-empty set and $\mathcal{A} \subseteq \exp(X)$. For each $x \in X$, we shall denote by $\mathcal{A}_x$ the collection $\{A \in \mathcal{A} : x \in A\}$ and the set $\{(x, y) : x \in X \text{ and } y \in \bigcap \mathcal{A}_x\}$ will be denoted by $U_{\mathcal{A}}$. It can be easily shown that $U_{\mathcal{A}}$ is a reflexive and transitive relation.

A typical cover of a g-quasi uniform space, required for our purpose here, is introduced in the following manner.

Definition 2.1. *A cover* $\mathcal{A}$ *of a g-quasi uniform space* $(X, \mathcal{U})$ *is said to be a strong g-quasi uniform cover of* X *if* $\exists U \in \mathcal{U}$ *such that for each* $x \in X, x \in U(x) \subseteq \bigcap \mathcal{A}_x$.

Example 2.2. *Let* $(X, \mathcal{U})$ *be a g-quasi uniform space. Then clearly* $\{X\}$ *is a strong g-quasi uniform cover of* X.

Example of a non-trivial strong g-quasi uniform cover is given by the following result.

Lemma 2.3. *Let $(X, \mathcal{U})$ be a g-quasi uniform space and B be a transitive member of $\mathcal{U}$. Then $\{B(x) : x \in X\}$ is a strong g-quasi uniform cover of X.*

Proof: Let B be a transitive member of $\mathcal{U}$ and $x \in X$. Also let $x \in B(z)$ and $y \in B(x)$, for some y, $z \in X$. So, (z, x), $(x, y) \in B$. Then by transitivity of B we get, $(z, y) \in B$ i.e., $y \in B(z)$. Thus we have, $x \in B(x) \subseteq \bigcap\{B(z) : z \in X$ and $x \in B(z)\}$. Hence $\{B(x) : x \in X\}$ is a strong g-quasi uniform cover of X. □

Theorem 2.4. *Let $(X, \mathcal{U})$ be a g-quasi uniform space. Then each strong g-quasi uniform cover is a $\mu(\mathcal{U})$-open cover.*

Proof: Let $\mathcal{A}$ be a strong g-quasi uniform cover of X. Then $\exists U \in \mathcal{U}$ such that for each $x \in X, x \in U(x) \subseteq \bigcap \mathcal{A}_x$. Let $A \in \mathcal{A}$ and $x \in A$. Then we have $x \in U(x) \subseteq \bigcap \mathcal{A}_x \subseteq A$. Then by Theorem 1.5, A is $\mu(\mathcal{U})$-open. □

Theorem 2.5. *Let (X, μ) be a strong GTS with a compatible g-quasi uniformity, having a transitive base $\mathcal{B}$. Then $\{B(x) : x \in X, B \in \mathcal{B}\}$ is a base for μ.*

Proof: From Theorem 2.4 it follows that $B(x)$ is μ-open for each $B \in \mathcal{B}$ and $x \in X$. Now let, $G \in \mu$ and $x \in G$. As $\mathcal{B}$ is compatible with μ, $\exists B \in \mathcal{B}$ such that $x \in B(x) \subseteq G$. Hence $\{B(x) : x \in X, B \in \mathcal{B}\}$ is a base for μ. □

Now we characterize g-quasi uniform space in terms of strong g-quasi uniform covers. It is worth mentioning in this connection, that we are able to achieve the characterization with one condition instead of two in the corresponding result in [13].

Theorem 2.6. *Let $(X, \mathcal{U})$ be a g- quasi uniform space and $\mathfrak{A}$ be the collection of all strong g- quasi uniform covers of X. Then if $\mathcal{A} \in \mathfrak{A}$ and $\mathcal{D}$ is a cover of X such that for each $x \in X, \bigcap \mathcal{A}_x \subseteq \bigcap \mathcal{D}_x$, then $\mathcal{D} \in \mathfrak{A}$.*

Conversely, let X be a non-empty set and $\mathfrak{A}$ be a collection of covers of X, satisfying the aforesaid condition. Then $\{U_\mathcal{A} : \mathcal{A} \in \mathfrak{A}\}$ is a transitive base for a g-quasi uniformity, say $\mathcal{U}_\mathfrak{A}$, on X, with respect to which $\mathfrak{A}$ is precisely the collection of all strong quasi-uniform covers of X.

Proof: Let $\mathfrak{A}$ be the collection of all strong g-quasi uniform covers of a g- quasi uniform space $(X, \mathcal{U})$. Now as $\mathcal{A} \in \mathfrak{A}$, there exists $U \in \mathcal{U}$ such that for each $x \in X$, $x \in U(x) \subseteq \bigcap \mathcal{A}_x$ holds. Again $\mathcal{D}$ is a cover of X such that for each $x \in X$, $\bigcap \mathcal{A}_x \subseteq \bigcap \mathcal{D}_x$. So for each $x \in X$, $x \in U(x) \subseteq \bigcap \mathcal{D}_x$. Hence $\mathcal{D} \in \mathfrak{A}$.

Conversely, suppose that $\mathfrak{A}$ is a collection of covers of X, satisfying the given condition. Now clearly $\Delta \subseteq U_\mathcal{A}$, $\forall \mathcal{A} \in \mathfrak{A}$. Next let $\mathcal{A} \in \mathfrak{A}$ and $(x, y) \in U_\mathcal{A} \circ U_\mathcal{A}$. Then $\exists z \in X$ such that $(x, z), (z, y) \in U_\mathcal{A}$. So $z \in \bigcap \mathcal{A}_x$ and $y \in \bigcap \mathcal{A}_z$. Thus, $y \in \bigcap \mathcal{A}_x$ i.e., $(x, y) \in U_\mathcal{A}$. It now follows that $U_\mathcal{A} \circ U_\mathcal{A} = U_\mathcal{A}$. Hence $U_\mathcal{A}$ is transitive $\forall \mathcal{A} \in \mathfrak{A}$. Thus $\{U_{\mathcal{A}} : \mathcal{A} \in \mathfrak{A}\}$ is a transitive base for a g-quasi uniformity $\mathcal{U}_{\mathfrak{A}}$ (say) on X.

For the rest of the proof we have to show that the set of strong g-quasi uniform covers of the space $(X, \mathcal{U}_{\mathfrak{A}})$ coincides with $\mathfrak{A}$. So let $\mathcal{D} \in \mathfrak{A}$. Now for each $x \in X$, $U_{\mathcal{D}}(x) = \bigcap \mathcal{D}_x$ i.e., $x \in U_{\mathcal{D}}(x) \subseteq \bigcap \mathcal{D}_x$, for each $x \in X$. Thus $\mathcal{D}$ is a strong g-quasi uniform cover of $(X, \mathcal{U}_{\mathfrak{A}})$. Next let $\mathcal{C}$ be a strong g-quasi uniform cover of $(X, \mathcal{U}_{\mathfrak{A}})$. Then $\exists \mathcal{A} \in \mathfrak{A}$ such that $x \in U_{\mathcal{A}}(x) \subseteq \bigcap \mathcal{C}_x$, for each $x \in X$ i.e., for each $x \in X$, $\bigcap \mathcal{A}_x \subseteq \bigcap \mathcal{C}_x$. So by the assumed condition $\mathcal{C} \in \mathfrak{A}$. Hence we are through. □

Now, let $\mathcal{U}$ be a compatible g-quasi uniformity for a strong GTS (X, μ), and let $\mathfrak{A}$ be the collection of all strong g-quasi uniform covers of $(X, \mathcal{U})$. Then by Theorem 2.6, $\mathfrak{A}$ induces a transitive g-quasi uniformity $\mathcal{U}_{\mathfrak{A}}$ on X. The question arises as to whether $\mathcal{U}$ and $\mathcal{U}_{\mathfrak{A}}$ are same. We answer the query in the following result.

Theorem 2.7. *Let (X, μ) be a strong GTS and $\mathcal{U}$ be a g-quasi uniformity on X such that $\mu = \mu(\mathcal{U})$. Let $\mathfrak{A}$ be the collection of all strong g-quasi uniform covers of $(X, \mathcal{U})$. Then the transitive g-quasi uniformity $\mathcal{U}_{\mathfrak{A}}$, induced by $\mathfrak{A}$ on X, is a subcollection of $\mathcal{U}$ and if $\mathcal{U}$ is transitive then $\mathcal{U}$ and $\mathcal{U}_{\mathfrak{A}}$ are same and hence $\mu(\mathcal{U}) = \mu = \mu(\mathcal{U}_{\mathfrak{A}})$.*

Proof: Let $\mathcal{A} \in \mathfrak{A}$. Then $\exists U \in \mathcal{U}$ such that $U(x) \subseteq \bigcap \mathcal{A}_x, \forall x \in X$. Now, $(x, y) \in U \implies y \in U(x) \subseteq \bigcap \mathcal{A}_x \implies (x, y) \in U_{\mathcal{A}}$. Thus $U \subseteq U_{\mathcal{A}}$. So $U_{\mathcal{A}} \in \mathcal{U}$. Thus $\mathcal{U}_{\mathfrak{A}} \subseteq \mathcal{U}$ i.e., $\mathcal{U}_{\mathfrak{A}}$ is a subcollection of $\mathcal{U}$. Also, if $\mathcal{U}$ is not transitive, then $\mathcal{U}_{\mathfrak{A}} \neq \mathcal{U}$.

Now let $\mathcal{U}$ be transitive with a transitive base $\mathcal{B}$ and $B \in \mathcal{B}$. Then $\mathcal{A} = \{B(x) : x \in X\} \in \mathfrak{A}$ (by Lemma 2.3). Now, $(x, y) \in U_{\mathcal{A}} \implies y \in \bigcap \mathcal{A}_x = \bigcap\{B(z) : x \in B(z), z \in X\} \subseteq B(x)$, as $x \in B(x)$. So $(x, y) \in B$. Thus $U_{\mathcal{A}} \subseteq B$ and hence $\mathcal{U} \subseteq \mathcal{U}_{\mathfrak{A}}$. Then the result follows. □

Remark 2.8. *It follows from the above proof that the collection of all strong g-quasi uniform covers with respect to a g-quasi uniformity $\mathcal{U}$ on a strong GTS X may coincide with that for a strictly smaller g-quasi uniformity on X.*

It has already been noted that every strong GTS has a transitive g-quasi uniformity, namely the Pervin g-quasi uniformity, compatible with the GT of the space. Now let $\mathcal{U}$ be a compatible transitive g-quasi uniformity for a strong GTS (X, μ), and $\mathfrak{A}$ be the collection of all strong g-quasi uniform covers of X. Then by Theorem 2.6 and 2.7, $\{U_{\mathcal{A}} : \mathcal{A} \in \mathfrak{A}\}$ forms a transitive base for $\mathcal{U}$. Then by Theorem 2.5 we have :

Theorem 2.9. *Let $\mathcal{U}$ denote a compatible transitive g-quasi uniformity for a strong GTS (X, μ), and $\mathfrak{A}$ be the collection of all strong g-quasi uniform covers of X. Then $\{U_{\mathcal{A}}(x) : x \in X, \mathcal{A} \in \mathfrak{A}\}$ i.e.,$\{\bigcap \mathcal{A}_x : x \in X, \mathcal{A} \in \mathfrak{A}\}$ is a base for $(X, \mu(\mathcal{U}))$.*

Theorem 2.10. *Let $(X, \mathcal{U}_X)$ and $(Y, \mathcal{U}_Y)$ be two g-quasi uniform spaces, where $\mathcal{U}_Y$ has a transitive base $\mathcal{B}_Y$. Then a function $f : X \to Y$ is g-quasi uniformly continuous if and only if for any strong g-quasi uniform cover $\mathcal{A}$ of $(Y, \mathcal{U}_Y)$, $f^{-1}(\mathcal{A}) = \{f^{-1}(A) : A \in \mathcal{A}\}$ is a strong g-quasi uniform cover of $(X, \mathcal{U}_X)$.*

Proof: Let $f : X \to Y$ be a g-quasi uniformly continuous function and $\mathcal{A}$ be a strong g-quasi uniform cover of $(Y, \mathcal{U}_Y)$. Then $\exists V \in \mathcal{U}_Y$ such that $y \in V(y) \subseteq \bigcap\{A : A \in \mathcal{A}_y\}, \forall y \in Y$. Now let $x \in X$. Then $f(x) \in V(f(x)) \subseteq \bigcap\{A : A \in \mathcal{A}_{f(x)}\}$. Now f being g-quasi uniformly continuous, $\exists W \in \mathcal{U}_X$ such that $(x_1, x_2) \in W \implies (f(x_1), f(x_2)) \in V$. Let $z \in W(x)$. So, $(x, z) \in W \implies (f(x), f(z)) \in V \implies f(z) \in V(f(x)) \subseteq \bigcap\{A : A \in \mathcal{A}_{f(x)}\} \implies f(z) \in A, \forall A \in \mathcal{A}$ such that $f(x) \in A$. So $z \in f^{-1}(A), \forall A \in \mathcal{A}$ with $x \in f^{-1}(A)$ i.e., $x \in W(x) \subseteq \bigcap\{f^{-1}(A) : f^{-1}(A) \in (f^{-1}(\mathcal{A}))_x\}$. Thus $f^{-1}(\mathcal{A})$ is a strong g-quasi uniform cover of $(X, \mathcal{U}_X)$.

Conversely suppose that the condition holds. Let $V \in \mathcal{U}_Y$. Then $\exists B \in \mathcal{B}_Y$ such that $B \subseteq V$. Now by Lemma 2.3, $\{B(y) : y \in Y\}$ is a strong g-quasi uniform cover of $(Y, \mathcal{U}_Y)$. So by the given condition, $\{f^{-1}(B(y)) : y \in Y\}$ is a strong g-quasi uniform cover of $(X, \mathcal{U}_X)$. Then $\exists W \in \mathcal{U}_X$ such that $x \in W(x) \subseteq \bigcap\{f^{-1}(B(y)) : x \in f^{-1}(B(y))\}, \forall x \in X$. So, $(x, z) \in W \implies z \in W(x) \implies z \in f^{-1}(B(y))$, if $x \in f^{-1}(B(y))$. Now, $f(x) \in B(f(x))$ i.e., $x \in f^{-1}(B(f(x)))$. So, $z \in f^{-1}(B(f(x))) \implies f(z) \in B(f(x)) \implies (f(x), f(z)) \in B \subseteq V$. Thus, f is g-quasi uniformly continuous. □

Note 2.11. *It should be noted that in the above result transitivity of the range g-quasi uniform space was not required for the forward part.*

3. g-Topological Properties in Terms of Strong g-Quasi Uniform Covers

In this section several generalized topological properties i.e., g-topological properties will be characterized in terms of strong g-quasi uniform covers. For characterizations of the corresponding topological properties in terms of strong quasi-uniform covers we refer to [15].

Theorem 3.1. *Let $(X, \mu(\mathcal{B}_X))$ and $(Y, \mu(\mathcal{B}_Y))$ be two strong GTSs, where $\mathcal{B}_X$ and $\mathcal{B}_Y$ are transitive bases for some g-quasi uniformities on X and Y respectively. Then a function $f : X \to Y$ is g-continuous if and only if for each $x \in X$ and each strong g-quasi uniform cover $\mathcal{A}_Y$ of Y, there exists a strong g-quasi uniform cover $\mathcal{A}_X$ of X such that $f(\bigcap(\mathcal{A}_X)_x) \subseteq \bigcap(\mathcal{A}_Y)_{f(x)}$.*

Proof: Let $f : X \to Y$ be g-continuous, $x \in X$ and $\mathcal{A}_Y$ be a strong g-quasi uniform cover of Y. Then $f(x) \in \bigcap(\mathcal{A}_Y)_{f(x)} \in \mu(\mathcal{B}_Y)$, by Theorem 2.9. Now for g-continuity of f, $\exists\ G \in \mu(\mathcal{B}_X)$ such that $x \in G$ and $f(G) \subseteq \bigcap(\mathcal{A}_Y)_{f(x)}$. Again by Theorem 2.9, there exists a strong g-quasi uniform cover $\mathcal{A}_X$ of X such that $x \in \bigcap(\mathcal{A}_X)_x \subseteq G$ and hence $f(\bigcap(\mathcal{A}_X)_x) \subseteq \bigcap(\mathcal{A}_Y)_{f(x)}$.

Conversely suppose that the condition holds. Let $x \in X$ and $f(x) \in G \in \mu(\mathcal{B}_Y)$. Then there exists a strong g-quasi uniform cover $\mathcal{A}_Y$ of Y such that $f(x) \in \bigcap(\mathcal{A}_Y)_{f(x)} \subseteq G$. Now by the given condition, there exists a strong g-quasi uniform cover $\mathcal{A}_X$ of X such that $f(\bigcap(\mathcal{A}_X)_x) \subseteq \bigcap(\mathcal{A}_Y)_{f(x)}$. Now $x \in \bigcap(\mathcal{A}_X)_x \in \mu(\mathcal{B}_X)$. Hence f is g-continuous. □

Definition 3.2. *A GTS (X,μ) is said to be μ-first countable at a point $x \in X$, if there exist a countable local base for μ at x.*

If the space is μ-first countable at each of its points then it is called μ-first countable.

Theorem 3.3. *A strong GTS (X,μ) with $\mu = \mu(\mathcal{B})$, $\mathcal{B}$ being a transitive base for a g-quasi uniformity on X, is μ-first countable if and only if for each $x \in X, \exists$ a countable family $\{\mathcal{A}_n : n \in \mathbb{N}\}$ of strong g-quasi uniform covers of X such that $\{\bigcap(\mathcal{A}_n)_x : n \in \mathbb{N}\}$ is a local base for μ at x.*

Proof: Let (X,μ) be g-first countable and $x \in X$. Again let $\{C_n : n \in \mathbb{N}\}$ be a local base for μ at x. Now for each $n \in \mathbb{N}, x \in C_n \in \mu(\mathcal{B})$. So for each $n \in \mathbb{N}$, there exists a strong g-quasi uniform cover $\mathcal{A}_n$ of X such that $x \in \bigcap(\mathcal{A}_n)_x \subseteq C_n$. Thus $\{\bigcap(\mathcal{A}_n)_x : n \in \mathbb{N}\}$ is a local base for μ at x.

The converse part follows trivially in view of Theorem 2.9. □

Definition 3.4. *A GTS (X,μ) is said to be μ-separable if there exists a countable set $S \subseteq X$ such that $c_\mu(S) = X$.*

Lemma 3.5. *Let (X,μ) be a strong GTS with $\mu = \mu(\mathcal{B})$, $\mathcal{B}$ being a transitive base for a g-quasi uniformity on X. Then for $C \subseteq X$,*

$$c_\mu(C) = \{x \in X : (\bigcap \mathcal{A}_x) \bigcap C \neq \phi, \forall \mathcal{A} \in \mathfrak{A}\},$$

where $\mathfrak{A}$ is the collection of all strong g-quasi uniform covers of X.

Proof: Let $x \in c_\mu(C)$ and $\mathcal{A} \in \mathfrak{A}$. Now $x \in \bigcap \mathcal{A}_x \in \mu$. Then as $x \in c_\mu(C)$, $(\bigcap \mathcal{A}_x) \bigcap C \neq \phi$.

Again, let $x \in X$ such that $(\bigcap \mathcal{A}_x) \bigcap C \neq \phi, \forall \mathcal{A} \in \mathfrak{A}$. Now $x \in G \in \mu$ implies that $\exists \mathcal{A} \in \mathfrak{A}$ such that $x \in \bigcap \mathcal{A}_x \subseteq G \implies (\bigcap \mathcal{A}_x) \bigcap C \subseteq C \bigcap G$. So $C \bigcap G \neq \emptyset$ and hence $x \in c_\mu(C)$. □

Theorem 3.6. *A strong GTS (X,μ) with $\mu = \mu(\mathcal{B})$, $\mathcal{B}$ being a transitive base for a g-quasi uniformity on X, is μ-separable if and only if $\exists$ a countable set $C \subseteq X$ such that for any strong g-quasi uniform cover $\mathcal{A}$ of X,*

$$(\cap \mathcal{D}) \cap C \neq \phi, \forall \mathcal{D} \subseteq \mathcal{A} \text{ with } \cap \mathcal{D} \neq \phi.$$

Proof: Let (X,μ) be μ-separable. Then there exists a countable set $C \subseteq X$ such that $c_\mu(C) = X$. Now, let $\mathcal{A}$ be a strong g-quasi uniform cover of X and $\mathcal{D} \subseteq \mathcal{A}$ such that $\cap \mathcal{D} \neq \phi$. Again, let $x \in \cap \mathcal{D}$. Now as $x \in c_\mu(C)$, $(\bigcap \mathcal{A}_x) \bigcap C \neq \phi$, by Lemma 3.5. Now $\bigcap \mathcal{A}_x \subseteq \bigcap \mathcal{D}$. So $(\cap \mathcal{D}) \cap C \neq \phi$.

Conversely suppose that the condition holds. Let $x \in X$ and $x \in G \in \mu$. Then there exists a strong g-quasi uniform cover $\mathcal{A}$ of X such that $x \in \bigcap \mathcal{A}_x \subseteq G$. Now by the given condition, $(\bigcap \mathcal{A}_x) \bigcap C \neq \phi \implies C \cap G \neq \emptyset \implies x \in c_\mu(C)$. Hence the result follows. □

Lemma 3.7. *Let (X, μ) be a strong GTS with $\mu = \mu(\mathcal{B})$, $\mathcal{B}$ being a transitive base for a g-quasi uniformity on X. Then for $C \subseteq X$,*

$$i_\mu(C) = \{x \in X : \exists \mathcal{A} \in \mathfrak{A} \text{ such that } \bigcap \mathcal{A}_x \subseteq C\},$$

where $\mathfrak{A}$ is the collection of all strong g-quasi uniform covers of X.

Proof: Let $x \in i_\mu(C)$. Then $\exists\ G \in \mu$ such that $x \in G \subseteq C$. So by Theorem 2.9, $\exists\ \mathcal{A} \in \mathfrak{A}$ such that $x \in \bigcap \mathcal{A}_x \subseteq G \subseteq C$.

Again, let $x \in X$ such that $\bigcap \mathcal{A}_x \subseteq C$ for some $\mathcal{A} \in \mathfrak{A}$. Now as $\bigcap \mathcal{A}_x \in \mu$, $x \in i_\mu(C)$. □

Now the notion of μ-connectedness as discussed in [4] yields the following definition.

Definition 3.8. *A strong GTS (X, μ) is called μ-connected if it cannot be expressed as disjoint union of two non-empty μ-open sets.*

Lemma 3.9 ([4])**.** *A g-topological space is (X, μ) μ-connected if and only if no non-empty proper subset of it is both μ-open and μ-closed.*

Theorem 3.10. *A strong GTS (X, μ), with $\mu = \mu(\mathcal{B})$, $\mathcal{B}$ being a transitive base for a g-quasi uniformity on X, is μ-connected if and only for any non-empty set $C \subsetneqq X$, $\exists\ x \in X$ such that for each strong g-quasi uniform cover $\mathcal{A}$ of X, $\bigcap \mathcal{A}_x$ intersects both C and $(X \setminus C)$ non-trivially.*

Proof: Let (X, μ) be μ-connected and C be a non-empty proper subset of X. So by Lemma 3.9, C is not both μ-open and μ-closed i.e., $i_\mu(C) \subsetneqq c_\mu(C)$. Take $x \in c_\mu(C) \setminus i_\mu(C)$ and $\mathcal{A}$ be an arbitrary strong g-quasi uniform cover of X. Then by Lemma 3.5, $(\bigcap \mathcal{A}_x) \bigcap C \neq \phi$, as $x \in c_\mu(C)$. Again $x \notin i_\mu(C)$. So by Lemma 3.7, $\bigcap \mathcal{A}_x \nsubseteq C$ i.e., $(\bigcap \mathcal{A}_x) \bigcap (X \setminus C) \neq \phi$.Thus $\bigcap \mathcal{A}_x$ intersects both C and $(X \setminus C)$ non-trivially.

Conversely suppose that the condition holds and C is a non-empty proper subset of X. Then by the given condition, $\exists x \in X$ such that for each strong g-quasi uniform cover $\mathcal{A}$ of X, $\bigcap \mathcal{A}_x$ intersects both C and $(X \setminus C)$ non-trivially. So by Lemma 3.5 and Lemma 3.7, $x \in c_\mu(C) \setminus i_\mu(C)$ i.e., C is not both μ-open and μ-closed. Thus (X, μ) is μ-connected. □

Now let us recall a few notions from the existing literature of generalized topology.

Definition 3.11 ([5])**.** *Let (X, μ) be a strong GTS and $x_1, x_2 \in X$ be two arbitrarily chosen distinct members of X. Then (X, μ) is said to be*

1. *μ-T_0, if $\exists G \in \mu$ such that G contains exactly one of x_1 and x_2.*
2. *μ-T_1, if $\exists G_1, G_2 \in \mu$ such that $x_1 \in G_1$, $x_2 \in G_2$, $x_1 \notin G_2$ and $x_2 \notin G_1$.*
3. *μ-T_2, if $\exists$ disjoint $G_1, G_2 \in \mu$ such that $x_1 \in G_1$ and $x_2 \in G_2$.*

Definition 3.12. *A GTS (X, μ) is said to be*

1. *[12] μ-regular, if for any μ-closed set F of X and a point $x \notin F$, $\exists$ disjoint $G_1, G_2 \in \mu$ such that $F \subseteq G_1$ and $x \in G_2$.*

2. *[8] μ-normal, if for any two disjoint μ-closed sets F_1 and F_2, $\exists$ disjoint $G_1, G_2 \in \mu$ such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$.*

Theorem 3.13. *A strong GTS (X, μ), with $\mu = \mu(\mathcal{B})$, $\mathcal{B}$ being a transitive base for a g-quasi uniformity on X, is μ-T_0 if and only if for distinct $x, y \in X$, there exists a strong g-quasi uniform cover $\mathcal{A}$ of X such that either $y \notin \bigcap \mathcal{A}_x$ or $x \notin \bigcap \mathcal{A}_y$.*

Proof: Let (X, μ) be μ-T_0 and $x, y \in X$ with $x \neq y$. Then $\exists G \in \mu$ such that either $x \in G, y \notin G$ or $x \notin G, y \in G$. Then by Theorem 2.9, there exists a strong g-quasi uniform cover $\mathcal{A}$ of X such that either $x \in \bigcap \mathcal{A}_x \subseteq G, y \notin \bigcap \mathcal{A}_x$ or $y \in \bigcap \mathcal{A}_y \subseteq G, x \notin \bigcap \mathcal{A}_y$ i.e., either $y \notin \bigcap \mathcal{A}_x$ or $x \notin \bigcap \mathcal{A}_y$.

Conversely suppose that the condition holds. Now by Theorem 2.9, $\bigcap \mathcal{A}_x$, $\bigcap \mathcal{A}_y \in \mu(\mathcal{B}) = \mu$. Then the condition follows immediately. □

Theorem 3.14. *A strong GTS (X, μ), with $\mu = \mu(\mathcal{B})$, $\mathcal{B}$ being a transitive base for a g-quasi uniformity on X, is μ-T_1 if and only if for $x, y \in X$ with $x \neq y$, there exist strong g-quasi uniform covers $\mathcal{A}_1$ and $\mathcal{A}_2$ of X such that $y \notin \bigcap(\mathcal{A}_1)_x$ and $x \notin \bigcap(\mathcal{A}_2)_y$.*

Proof: Let (X, μ) be μ-T_1 and $x, y \in X$ with $x \neq y$. Then $\exists G_x, G_y \in \mu$ such that $x \in G_x, y \notin G_x$ and $y \in G_y, x \notin G_y$. Then by Theorem 2.9, there exist strong g-quasi uniform covers $\mathcal{A}_1$, $\mathcal{A}_2$ of X such that $x \in \bigcap(\mathcal{A}_1)_x \subseteq G_x$ and $y \in \bigcap(\mathcal{A}_2)_y \subseteq G_y$. Thus $y \notin \bigcap(\mathcal{A}_1)_x$ and $x \notin \bigcap(\mathcal{A}_2)_y$.

Conversely suppose that the condition holds for two given distinct points x, y of X. Now $x \in \bigcap(\mathcal{A}_1)_x$, $y \in \bigcap(\mathcal{A}_2)_y$. Again $\bigcap(\mathcal{A}_1)_x$, $\bigcap(\mathcal{A}_2)_y \in \mu(\mathcal{B}) = \mu$. Thus X is μ-T_1. □

Theorem 3.15. *A strong GTS (X, μ), with $\mu = \mu(\mathcal{B})$, $\mathcal{B}$ being a transitive base for a g-quasi uniformity on X, is μ-T_2 if and only if for two distinct points x and y of X, there exist strong g-quasi uniform covers $\mathcal{A}_1$ and $\mathcal{A}_2$ of X such that $(\bigcap(\mathcal{A}_1)_x) \bigcap (\bigcap(\mathcal{A}_2)_y) = \emptyset$.*

Proof: Let (X, μ) be μ-T_2 and $x, y \in X$ with $x \neq y$. Then $\exists G_x, G_y \in \mu$ such that $x \in G_x$ and $y \in G_y$ and $G_x \cap G_y = \emptyset$. Then by Theorem 2.9, there exist strong g-quasi uniform covers $\mathcal{A}_1$, $\mathcal{A}_2$ of X such that $x \in \bigcap(\mathcal{A}_1)_x \subseteq G_x$ and $y \in \bigcap(\mathcal{A}_2)_y \subseteq G_y$. Hence $(\bigcap(\mathcal{A}_1)_x) \bigcap (\bigcap(\mathcal{A}_2)_y) = \emptyset$.

Conversely suppose that the condition holds for two given distinct points x and y of X. Now $x \in \bigcap(\mathcal{A}_1)_x$, $y \in \bigcap(\mathcal{A}_2)_y$. Again $\bigcap(\mathcal{A}_1)_x$, $\bigcap(\mathcal{A}_2)_y \in \mu(\mathcal{B}) = \mu$. Thus X is μ-T_2. □

Theorem 3.16. *A strong GTS (X,μ), with $\mu=\mu(\mathcal{B})$, $\mathcal{B}$ being a transitive base for a g-quasi uniformity on X, is μ-regular if and only if for any given μ-closed set $C\subseteq X$ and $x\in X\setminus C$, there exist strong g-quasi uniform covers $\mathcal{A}$, $\mathcal{A}^c$ $(c\in C)$ of X such that $(\bigcap(\mathcal{A}^x)_x)\bigcap(\bigcap(\mathcal{A}^c)_c)=\emptyset, \forall c\in C$.*

Proof: Let (X,μ) be μ-regular, C a given μ-closed set of X and $x\in X\setminus C$. Then $\exists G_x, G_C\in\mu$ such that $x\in G_x$, $C\subseteq G_C$ and $G_x\cap G_C=\emptyset$. Then there exist strong g-quasi uniform covers $\mathcal{A}$, $\mathcal{A}^c$ $(c\in C)$ of X such that $x\in\bigcap(\mathcal{A}^x)_x\subseteq G_x$ and $c\in\bigcap(\mathcal{A}^c)_c\subseteq G_C$, $\forall c\in C$. Hence $(\bigcap(\mathcal{A}^x)_x)\bigcap(\bigcap(\mathcal{A}^c)_c)=\emptyset$, $\forall c\in C$.

Conversely suppose that the condition holds and C is a μ-closed set in X and $x\in X$ such that $x\notin C$. Now for each $c\in C$, $c\in\bigcap(\mathcal{A}^c)_c$ and $x\in\bigcap(\mathcal{A}^x)_x$. Again $\bigcap(\mathcal{A}^x)_x$, $\bigcap(\mathcal{A}^c)_c\in\mu(\mathcal{B})=\mu$, $\forall c\in C$. Put $\bigcup\limits_{c\in C}(\bigcap(\mathcal{A}^c)_c)=G_C$ and $\bigcap(\mathcal{A}^x)_x=G_x$. Then by the given condition, $G_C\cap G_x=\emptyset$ and $C\subseteq G_C, x\in G_x$. Thus X is μ-regular. □

Theorem 3.17. *A strong GTS (X,μ), with $\mu=\mu(\mathcal{B})$, $\mathcal{B}$ being a transitive base for a g-quasi uniformity on X, is μ-normal if and only if for any two μ-closed sets $C,D\subseteq X$ with $C\cap D=\emptyset$, there exist strong g-quasi-uniform covers $\mathcal{A}^x$ of X, $\forall x\in C\cup D$, such that $(\bigcap(\mathcal{A}^c)_c)\bigcap(\bigcap(\mathcal{A}^d)_d)=\emptyset$, $\forall c\in C$ and $\forall d\in D$.*

Proof: Let (X,μ) be μ-normal, and C,D be two given disjoint μ-closed sets of X. Then $\exists G_C, G_D\in\mu$ such that $C\subseteq G_C, D\subseteq G_D$ and $G_C\cap G_D=\emptyset$. Then for each $x\in C\cup D$, there exists a strong g-quasi uniform cover $\mathcal{A}^x$ of X such that $c\in\bigcap(\mathcal{A}^c)_c\subseteq G_C$, $\forall c\in C$ and $d\in\bigcap(\mathcal{A}^d)_d\subseteq G_D$, $\forall d\in D$. Hence $(\bigcap(\mathcal{A}^c)_c)\bigcap(\bigcap(\mathcal{A}^d)_d)=\emptyset$, $\forall c\in C$ and $\forall d\in D$.

Conversely suppose that the condition holds and C,D are two disjoint μ-closed sets in X. Now for each $x\in C\cup D$, $x\in\bigcap(\mathcal{A}^x)_x$. Again $\bigcap(\mathcal{A}^x)_x\in\mu(\mathcal{B})=\mu$, $\forall x\in C\cup D$. Put $\bigcup\limits_{c\in C}(\bigcap(\mathcal{A}^c)_c)=G_C$ and $\bigcup\limits_{d\in CD}(\bigcap(\mathcal{A}^d)_d)=G_D$.Then by the given condition, $G_C\cap G_D=\emptyset$ and $C\subseteq G_C$, $D\subseteq G_D$. Thus X is μ-normal. □

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