Immersion of strong Brownian filtrations with honest time avoiding stopping times

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ABSTRACT: In this paper, we give a partial answer to the following question: if $\mathbb{F} \hookrightarrow \mathbb{G} \hookrightarrow \mathbb{H}$ (where the symbol ($\hookrightarrow$) indicates the immersion property), and if $\mathbb{F}$ and $\mathbb{H}$ are two strong Brownian filtrations, is $\mathbb{G}$ also a strong Brownian filtration ? For a long time, there has been no attempts to respond to this question, and to our knowledge, there is no response, neither partial nor complete. We are, therefore, the first to give a response to this question, and we prove that $\mathbb{G}$ is a weak Brownian in the case of a progressive enlargement of $\mathbb{F}$ with an honest time τ that avoids all stopping times.

Key Words: Weak and strong Brownian filtrations, immersion, predictable representation property, progressive enlargement of filtrations, honest times.

Contents

1. Introduction

The theory of enlargements of filtrations has been developed since the end of the seventies, starting with the works of Barlow [2], Brémaud and Yor [4], Dellacherie and Meyer [5] , and Jeulin and Yor [7]. In fact, three lecture-notes volumes have been devoted to this subject; Jeulin [6], Mansuy and Yor [8], Aksamit and Jeanblanc [1].

The case of progressive enlargements is investigated in Barlow [2], Dellacherie and Meyer [5], Jeulin and Yor [7,6], then later in Aksamit and Jeanblanc [1].

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An *honest time* is by definition the end of an optional set; it is a random time but not a stopping time, this makes its analysis more delicate. It plays an important role in the theory of enlargements of filtrations, see Barlow [2], Dellacherie and Meyer [5], Jeulin and Yor [7,6], and Aksamit and Jeanblanc [1], and in the characterizations of strong Brownian filtrations, see Barlow et al. [3], Mansuy and Yor [8].

The **(H)** hypothesis (or immersion property: that is when any martingale of initial filtration remains a martingale in enlarged filtration) is studied by Brémaud, Yor and Mansuy [4,8], Aksamit and Jeanblanc [1].

The strong and weak Brownian filtrations are studied in details in Mansuy and Yor [8].

The main goal of this paper is to give a partial answer to the following question: if $\mathbb{F} \hookrightarrow \mathbb{G} \hookrightarrow \mathbb{H}$, where $\mathbb{F}$ and $\mathbb{H}$ are two strong Brownian filtrations, is $\mathbb{G}$ also a strong Brownian filtration ?
We study the case when $\mathbb{G}$ is the progressive enlargement of the initial filtration $\mathbb{F}$ with an honest time τ that avoids all stopping times.

The paper is organized as follows: Section 2 introduces definitions and preliminaries which will be relevant for this paper, and Section 3 is devoted to the study of the progressive enlargement of the filtration $\mathbb{F}$ with an honest time τ under the hypotheses **(CA)**:

- **(C)** all $\mathbb{F}$-martingales are **c**ontinuous (e.g: the Brownian filtration);

- **(A)** τ **a**voids every $\mathbb{F}$-stopping times T, i.e. $\mathbb{P}(\tau = T) = 0$.

Then we end this case with our main result under the hypotheses **(H)** and **(CA)** in Section 4.

2. Preliminaries

We consider a complete filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ that satisfies the *usual hypotheses*: i.e., $\mathbb{F}$ is right continuous ($\forall t \geq 0, \mathcal{F}_t = \mathcal{F}_{t^+} := \cap_{s>t}\mathcal{F}_s$), and $\mathcal{F}_0$ contains all the $\mathbb{P}$ null sets of $\mathbb{F}$, therefore $\mathcal{F}_t$ contains all the $\mathbb{P}$ null sets of $\mathbb{F}$ as well, for any $t \geq 0$. The σ-field $\bigvee_{t\in\mathbb{R}_+} \mathcal{F}_t = \sigma(\bigcup_{t\in\mathbb{R}_+} \mathcal{F}_t)$ denoted by $\mathcal{F}_\infty$.

2.1. Predictable Representation Property

Definition 2.1. *A continuous local martingale X enjoys the **predictable representation property (PRP)** if for any $\mathbb{F}^X$-local martingale $(M_t, t \geq 0)$, there is a constant c and an $\mathbb{F}^X$-predictable process $(m_s, s \geq 0)$ which satisfies $\int_0^t m_s^2 d\langle X \rangle_s < \infty$, such that*

$$M_t = c + \int_0^t m_s dX_s, \ \ t \geq 0.$$

More generally, a continuous $\mathbb{F}$-local martingale X enjoys the $\mathbb{F}$-predictable representation property if any $\mathbb{F}$-adapted martingale M can be written as $M_t = c + \int_0^t m_s dX_s$, with $\int_0^t m_s^2 d\langle X \rangle_s < \infty$. We do not require in that last definition that $\mathbb{F}$ is the natural filtration of X.

Brownian Motion Case

Let B be a real-valued Brownian motion and $\mathbb{F}^B$ its natural filtration.

Definition 2.2. *Let $(M_t, t \geq 0)$ be a square integrable $\mathbb{F}^B$-martingale (i.e., $\sup_t \mathbb{E}(M_t^2) < \infty$). There exists a constant c and a unique predictable process $(m_s, s \geq 0)$ which satisfies $\mathbb{E}\left(\int_0^t m_s^2 ds\right) < \infty$, such that*

$$\forall t, \quad M_t = c + \int_0^t m_s dB_s.$$

If M is an $\mathbb{F}^B$-local martingale, there exists a unique predictable process $(m_s, s \geq 0)$ which satisfies $\int_0^t m_s^2 ds < \infty$, such that

$$\forall t, \quad M_t = c + \int_0^t m_s dB_s.$$

2.2. Strong and Weak Brownian Filtrations

Definition 2.3.   • *A filtration $\mathbb{F}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\mathcal{F}_0$ is $\mathbb{P}$ a.s trivial, is said to be **weak Brownian** if there exists an $\mathbb{F}$-Brownian motion B such that B has the predictable representation property with respect to $\mathbb{F}$.*

    • *A filtration $\mathbb{F}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\mathcal{F}_0$ is $\mathbb{P}$ a.s trivial, is said to be **strong Brownian** if there exists an $\mathbb{F}$-Brownian motion B such that $\mathcal{F}_t = \mathcal{F}_t^B$.*

Note that a strong Brownian filtration is weak Brownian since the Brownian motion enjoys the PRP.

2.3. Immersion of Filtrations

Definition 2.4. *Let $\mathbb{F}$ and $\mathbb{G}$ be two filtrations on the same probability space. We say that $\mathbb{F}$ is **immersed** in $\mathbb{G}$, and we write $\mathbb{F} \hookrightarrow \mathbb{G}$, if $\mathbb{F}$ is included in $\mathbb{G}$, i.e., $\mathcal{F}_t \subset \mathcal{G}_t$, for all t, and every $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. This is also referred to as the **(H)** hypothesis which is defined as :*

    (H) Every $\mathbb{F}$-square integrable martingale is a $\mathbb{G}$-square integrable martingale.

We first remark that **(H)** holds if, and only if, every bounded $\mathbb{F}$-martingale (respectively. every $\mathbb{F}$-local martingale) is a $\mathbb{G}$-martingale, (resp. a $\mathbb{G}$-local martingale).

Hypothesis **(H)** implies that $\mathbb{F}$ has a nice structure -among other properties-relatively to $\mathbb{G}$, as is shown in the following theorem:

Theorem 2.5. *([4]) Hypothesis **(H)** is equivalent to the following properties:*

(H1) $\forall t \geq 0$, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionnally independent given $\mathcal{F}_t$,

(H2) $\forall t \geq 0, \forall G_t \in L^1(\mathcal{G}_t), \mathbb{E}(G_t|\mathcal{F}_\infty) = \mathbb{E}(G_t|\mathcal{F}_t)$,

(H3) $\forall t \geq 0, \forall F \in L^1(\mathcal{F}_\infty), \mathbb{E}(F|\mathcal{G}_t) = \mathbb{E}(F|\mathcal{F}_t)$.

In particular, under **(H)**, if B is an $\mathbb{F}$-Brownian motion, then it is a $\mathbb{G}$-martingale with bracket t. Since such a bracket does not depend on the filtration, B is, hence, a $\mathbb{G}$-Brownian motion.

3. Progressive enlargement with an honest time that avoids all stopping times

We now recall some basic results on the progressive enlargement of filtrations. Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual hypotheses (right continuous and complete), $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a Brownian filtration generated by the Brownian motion B.

Definition 3.1. *A positive random variable τ defined on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ is called an **honest time** if, for all $t \in \mathbb{R}_+$, there exists an $\mathcal{F}_t$-measurable random variable τ_t such that $\tau = \tau_t$ holds on $\{\tau \leq t\}$.*

These times are also referred to as the end of an $\mathbb{F}$-predictable set.

Definition 3.2. *Let τ be the end of an $\mathbb{F}$-predictable set Γ, that is*

$$\tau = \sup\{t : (t, w) \in \Gamma\}.$$

*We, then, call τ an **honest time**.*

Now we enlarge $\mathbb{F}$ with the process $(\tau \wedge t)_{t\geq 0}$, so that the new enlarged filtration $\mathbb{G}$ is the smallest filtration (satisfying the usual assumptions) containing $\mathbb{F}$ and making τ a stopping time, that is

$$\mathcal{G}_t = \mathcal{G}_{t+} := \bigcap_{s>t} \mathcal{G}_s = \bigcap_{s>t}(\mathcal{F}_s \vee \sigma(\tau \wedge s)).$$

In the literature, $\mathbb{G}$ is called the **progressive enlargement** of $\mathbb{F}$ with τ.

A few processes play a crucial role in our discussion:

- the $\mathbb{F}$ supermartingale

$$Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t),$$

  chosen to be càdlàg, coincides with the $\mathbb{F}$ optional projection of $1_{\{t<\tau\}}$, called the **Azéma supermartingale** associated with τ.

- the $\mathbb{F}$ dual optional and predictable projections of the increasing process $N_t = 1_{\{\tau \leq t\}}$ denoted by a and A respectively.

- the càdlàg martingale

$$\mu_t = \mathbb{E}(a_\infty | \mathcal{F}_t) = a_t + Z_t.$$

- the Doob-Meyer (additive) decomposition of Z:

$$Z_t = \mathbb{E}(A_\infty | \mathcal{F}_t) - A_t = M_t - A_t,$$

where M is an $\mathbb{F}$-martingale.

We shall assume two important hypotheses;

- **(C)** all $\mathbb{F}$-martingales are continuous (e.g: the Brownian filtration),

- **(A)** τ avoids every $\mathbb{F}$-stopping times T, i.e. $\mathbb{P}(\tau = T) = 0$.

We note that the continuity assumption **(C)** implies that the optional and Doob-Meyer decompositions of Z are the same.

When one assumes that the honest time τ avoids $\mathbb{F}$-stopping times, then one has:

Lemma 3.3. *A random time τ is an honest time and avoids $\mathbb{F}$-stopping times if and only if $Z_\tau = 1$ a.s. on $\{\tau < \infty\}$.*

The following corollary indicates some consequences of the assumption **(A)**.

Corollary 3.4. *If τ avoids stopping times, then Z is continuous.*

In the next lemma, we present equivalent characterizations of hypothesis **(H)** in the progressive enlargement case,

Lemma 3.5. *([5]) In the progressive enlargement setting, **(H)** holds between $\mathbb{F}$ and $\mathbb{G}$ if and only if one of the following equivalent conditions holds:*

(i) $\forall (t, s), s \le t, \mathbb{P}(\tau \le s | \mathcal{F}_\infty) = \mathbb{P}(\tau \le s | \mathcal{F}_t).$

(ii) $\forall t, \mathbb{P}(\tau \le t | \mathcal{F}_\infty) = \mathbb{P}(\tau \le t | \mathcal{F}_t).$

Note that, if **(H)** holds, then (ii) implies that the process $F_t = 1 - Z_t = \mathbb{P}(\tau \le t | \mathcal{F}_t)$ is increasing.

Remark 3.6. *If the initial filtration $\mathbb{F}$ is immersed into the enlarged filtration $\mathbb{G}$, then Z has no martingale part, i.e., Z is a decreasing predictable process, and it is equal to A. Therefore, the $\mathbb{F}$-Brownian motion B remains a $\mathbb{G}$-Brownian motion, and the bracket $\langle B, Z \rangle$ is equal to zero.*

A key point of our main result is the following proposition which provides the canonical decomposition of any $\mathbb{F}$ local martingale as a semimartingale in $\mathbb{G}$.

Proposition 3.7. *([7]) Let τ be an honest time. We assume **(CA)**. Then, any $\mathbb{F}$-local martingale X is a $\mathbb{G}$ semi-martingale with decomposition*

$$X_t = \widetilde{X}_t + \int_0^{t\wedge\tau} \frac{d\langle X, M\rangle_s}{Z_s} - \int_\tau^t \frac{d\langle X, M\rangle_s}{1 - Z_s}$$

where $\widetilde{X}$ is a $\mathbb{G}$-local martingale.

Proof: see Aksamit and Jeanblanc [1] p.108-109.   □

4. Main result

We give our main result by using proposition 3.7 and remark 3.6.

Proposition 4.1. *If $\mathbb{F} \hookrightarrow \mathbb{G} \hookrightarrow \mathbb{H}$, where $\mathbb{F}$ and $\mathbb{H}$ are strong Brownian filtrations, and if $\mathbb{G}$ is the progressive enlargement of $\mathbb{F}$ with an honest time τ that avoids all stopping times, then $\mathbb{G}$ is a weak Brownian filtration.*

Proof: On the one hand, $\mathbb{F}$ is a strong Brownian filtration, then any $\mathbb{F}$-local martingale can be written as

$$X_t = c + \int_0^t \phi_s dB_s$$

for some $c \in \mathbb{R}$ and a predictable process ϕ which satisfies $\int_0^t \phi_s^2 ds < \infty$.

Using proposition 3.7, we get that any $\mathbb{G}$-local martingale $\widetilde{X}_t$ can be written as

$$
\begin{aligned}
\widetilde{X}_t &= X_t - \int_0^{t\wedge\tau} \frac{d\langle X, M\rangle_s}{Z_s} + \int_\tau^t \frac{d\langle X, M\rangle_s}{1 - Z_s} \\
&= c + \int_0^t \phi_s dB_s - \int_0^{t\wedge\tau} \frac{\phi_s}{Z_s} d\langle B, M\rangle_s + \int_\tau^t \frac{\phi_s}{1 - Z_s} d\langle B, M\rangle_s,
\end{aligned}
$$

where $c \in \mathbb{R}$ and ϕ a predictable process which satisfies $\int_0^t \phi_s^2 ds < \infty$.

Since $\mathbb{F} \hookrightarrow \mathbb{G}$, and by remark 3.6, we have $\langle B, Z\rangle = \langle B, M\rangle = 0$; that is, the two integral terms on the right-hand side of the last equality vanish.

Consequently, any $\mathbb{G}$-local martingale $\widetilde{X}_t$ can be written as

$$\widetilde{X}_t = c + \int_0^t \phi_s dB_s, \tag{4.1}$$

where $c \in \mathbb{R}$ and ϕ a predictable process which satisfies $\int_0^t \phi_s^2 ds < \infty$.

On the other hand, we have $\mathbb{F} \hookrightarrow \mathbb{H}$, i.e., the Brownian motion B remains a local martingale with respect to $\mathbb{H}$ generated by the Brownian motion $B^{'}$ which

has the PRP with respect to $\mathbb{H}$, then $dB_t = U_t dB'_t$, i.e., $dB'_t = U_t^{-1} dB_t$, with U an $\mathbb{F}$-predictable process taking values in $\{-1, +1\}$.

Since $\mathbb{G} \hookrightarrow \mathbb{H}$, then any $\mathbb{G}$-local martingale $\widetilde{X}_t$ can be written as

$$\widetilde{X}_t = a + \int_0^t K_s dB'_s = a + \int_0^t K_s U_s^{-1} dB_s = a + \int_0^t K'_s dB_s, \qquad (4.2)$$

where $a \in \mathbb{R}$, K and $K' \equiv KU^{-1}$ are $\mathbb{G}$-predictable processes which satisfie $\int_0^t K_s^2 ds < \infty$ and $\int_0^t K_s'^2 ds < \infty$.

Consequently, by (4.1) and (4.2), we have $a = c \in \mathbb{R}$ and $\phi \equiv K'$.

Moreover, the $\mathbb{F}$-Brownian motion B remains a $\mathbb{G}$-Brownian motion which has the PRP with respect to $\mathbb{G}$. Then $\mathbb{G}$ is a weak Brownian filtration. $\qquad \square$

Perspective: Using Brownian transformations, we will establish additional conditions to meet the issue of the immersion property and prove that $\mathbb{G}$ is a strong Brownian filtration.

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