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A variation on arithmetic continuity

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abstract: A sequence (x_k) of points in $\mathbb{R}$, the set of real numbers, is called *arithmetically convergent* if for each $\varepsilon > 0$ there is an integer n such that for every integer m we have $|x_m - x_{<m,n>}| < \varepsilon$, where $k|n$ means that k divides n or n is a multiple of k, and the symbol $< m, n >$ denotes the greatest common divisor of the integers m and n. We prove that a subset of $\mathbb{R}$ is bounded if and only if it is arithmetically compact, where a subset E of $\mathbb{R}$ is arithmetically compact if any sequence of point in E has an arithmetically convergent subsequence. It turns out that the set of arithmetically continuous functions on an arithmetically compact subset of $\mathbb{R}$ coincides with the set of uniformly continuous functions where a function f defined on a subset E of $\mathbb{R}$ is arithmetically continuous if it preserves arithmetically convergent sequences, i.e., $(f(x_n))$ is arithmetically convergent whenever (x_n) is an arithmetic convergent sequence of points in E.

Key Words: arithmetical convergent sequences, boundedness, uniform continuity

Contents

1. Introduction

The concept of continuity and any concept involving continuity play a very important role not only in pure mathematics but also in other branches of sciences involving mathematics especially in computer science, information theory, biological science.

A real function f defined on $\mathbb{R}$ is continuous if and only if it preserves Cauchy sequences, i.e., $(f(x_n))$ is a Cauchy sequence whenever (x_n) is. Using the idea of continuity of a real function in terms of sequences in the sense that a function preserves a certain kind of sequences in the above manner, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: slowly oscillating continuity ([7]), quasi-slowly oscillating continuity ([18]), ward continuity ([15]), δ-ward continuity ([8]), statistical ward continuity ([10]), λ-statistical ward continuity ([24]), ρ-statistical ward continuity ([4]), lacunary ward continuity ([12], strongly lacunary ward continuity ([3]), and which enabled some authors to obtain conditions on the domain of a function for some

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characterizations of uniform continuity (see [39, Theorem 6], [2, Theorem 1 and Theorem 2], [18, Theorem 2.3], [24, Theorem 5]).

The purpose of this paper is to give a further investigation of the concept of arithmetic continuity of a real function given in [40], and prove interesting theorems.

2. Arithmetic continuity

A sequence $\mathbf{x} = (x_k)$ of points in $\mathbb{R}$ and $k, n \in \mathbb{N}$ the notation $\sum_{k|n} x_k$ means the finite sum of all the numbers x_k as k ranges over the integers that divide n including 1 and n. In general for integers k and n, we write $k|n$ to mean k divides n or n is a multiple of k. We use the symbol $< m, n >$ to denote the greatest common divisor of two integers m and n. In [37], W.H.Ruckle, introduced the notions arithmetically summable and arithmetical convergence as follows: A sequence $\mathbf{x} = (x_k)$ of points in $\mathbb{R}$ is called arithmetically summable if for each $\varepsilon > 0$ there is an integer n such that for every integer m we have $|\sum_{k|n} x_k - \sum_{k|<m,n>} x_k| < \varepsilon$, and a sequence $\mathbf{y} = (y_k)$ is called arithmetically convergent if for each $\varepsilon > 0$ there is an integer n such that for every integer m we have $|y_m - y_{<m,n>}| < \varepsilon$. A sequence (x_k) is arithmetically summable if and only if the sequence $\mathbf{y} = (y_n)$ defined by $y_n = \Sigma_{k|n} x_k$ is arithmetically convergent, but the sequence $\mathbf{y}$ is not convergent in the ordinary sense, and is in fact periodic. Recently, in [40], Yaying and Hazarika introduced the concepts of arithmetic continuity and arithmetic compactness in the senses that a function f defined on a subset E of $\mathbb{R}$ is arithmetically continuous if it preserves arithmetically convergent sequences, i.e., $(f(x_n)$ is arithmetically convergent whenever (x_n) is an arithmetic convergent sequence of points in E, and a subset E of $\mathbb{R}$ is arithmetically compact if any sequence of point in E has an arithmetically convergent subsequence. First, we note that any finite subset of $\mathbb{R}$ is arithmetically compact, the union of two arithmetically compact subsets of $\mathbb{R}$ is arithmetically compact and the intersection of any family of arithmetically compact subsets of $\mathbb{R}$ is arithmetically compact. Any G-sequentially compact subset of $\mathbb{R}$ is arithmetically compact for a regular subsequential method G (see [6], [9]). Furthermore any subset of an arithmetically compact set is arithmetically compact, any bounded subset of $\mathbb{R}$ is arithmetically compact, any slowly oscillating compact subset of $\mathbb{R}$ is arithmetically compact (see [7] for the definition of slowly oscillating compactness). These observations suggest to us the following.

Theorem 2.1. *A subset E of $\mathbb{R}$ is bounded if and only if it is arithmetically compact.*

Proof: If E is a bounded subset of $\mathbb{R}$, then any sequence of points in E has a convergent subsequence which is also arithmetically convergent. Conversely, suppose that E is not bounded. If it is not bounded below, then pick an element x_1 of E less than 0. Then we can choose an element x_2 of E such that $x_2 < -1 + x_1$. Similarly we can choose an element x_3 of E such that $x_3 < -2 + x_1 + x_2$. We can inductively choose x_{k+1} satisfying $x_{k+1} < -k + \sum_1^k x_i$ for each $k \in \mathbb{N}$. Hence $|x_n - x_m| \geq 1$ for each $n, m \in \mathbb{N}$. Then the sequence (x_k) does not have any arithmetically convergent subsequence. If E is unbounded above, then we can find a y_1 greater than 0. Then we can pick a y_2 such that $y_2 > 1 + y_1 + y_2$. We can successively find for each $k \in \mathbb{N}$ a y_{k+1} such that $y_{k+1} > k + \sum_1^k y_i$. Then $|y_m - y_n| > k$ for each $m, n \in \mathbb{N}$. Then the sequence (y_k) does not have any arithmetically convergent subsequence. Thus E is not arithmetically compact. This completes the proof. □

Corollary 2.2. *A subset of $\mathbb{R}$ is arithmetically compact if and only if it is ρ-statistically compact.*

Proof: The proof follows from [4, Theroem 1], and the preceding theorem, so is omitted. □

According to Theorem 2.1, we see that arithmetical compactness coincides with not only ρ-statistical ward compactness (([4]), but also ward compactness, p-ward compactness ([16]), statistical ward compactness ([10]), λ-statistical ward compactness ([24]), lacunary statistical ward compactness ([12]), strongly lacunary ward compactness ([3]), Abel ward compactness ([11]), and I-ward compactness for a nontrivial admissible ideal I ([20], [16]).

We also see that arithmetical compactness and closedness together coincide with not only compactness, but also statistical sequential compactness ([10]), λ-statistical sequential compactness ([24]), ρ-statistical ward compactness ([4], lacunary statistical sequential compactness ([12]), strongly lacunary sequential compactness ([3]), Abel sequential compactness ([11]), and I-sequential compactness for a nontrivial admissible ideal I ([20], [16]).

Yaying and Hazarika also introduced the concept of arithmetic continuity in the sense that a function defined on a subset E of $\mathbb{R}$ is arithmetically continuous if it preserves arithmetically convergent sequences, i.e. $(f(x_n))$ is an arithmetically convergent sequence whenever (x_n) is. We note that arithmetical continuity cannot be obtained by any sequential method G. Yaying and Hazarika proved that the composition of two arithmetically continuous functions is arithmetically continuous, the the sum of two arithmetically continuous functions is arithmetically continuous, and absolute value of an arithmetically continuous function is arithmetically continuous.

We note that it follows from [40, Theorem 2.6] that for functions defined on intervals, not only ward continuity, but also p-ward continuity, statistical ward continuity, λ-statistical ward continuity, ρ-statistical ward continuity imply arithmetical continuity.

In connection with arithmetical convergent sequences, and convergent sequences the problem arises to investigate the following types of ŞcontinuityŤ of functions on $\mathbb{R}$.

(AC) $(x_n) \in AC \Rightarrow (f(x_n)) \in AC$

(ACc) $(x_n) \in AC \Rightarrow (f(x_n)) \in c$

(c) $(x_n) \in c \Rightarrow (f(x_n)) \in c$

(cAC) $(x_n) \in c \Rightarrow (f(x_n)) \in AC$

We see that (AC) is the arithmetical continuity of f, and (c) is the ordinary continuity of f. It is easy to see that (ACc) implies (AC), and (AC) does not imply (ACc); and (AC) implies (cAC).

Now we give the implication (AC) implies (c), i.e. any arithmetically continuous function is continuous.

Theorem 2.3. *If f is arithmetically continuous on a subset E of $\mathbb{R}$, then it is continuous on E.*

Proof: Assume that f is an arithmetically continuous function on E. Let (x_n) be any convergent sequence with $\lim_{k\to\infty} x_k = \ell$. Then the sequence

$$(x_1, \ell, x_2, \ell, \ell, x_3, \ell, \ell, \ell, x_4, ..., x_{n-1}, \ell, \ell, \ell, ..., \ell, x_n, \ell, ...)$$

is convergent to ℓ. Hence it is arithmetically convergent. Since f is arithmetically continuous, the sequence

$$(f(x_1), f(\ell), f(x_2), f(\ell), f(\ell), f(x_3), f(\ell), f(\ell), f(\ell),$$

$$f(x_4), ..., f(x_{n-1}), f(\ell), f(\ell), f(\ell), ..., f(\ell), f(x_n), ...)$$

is arithmetically convergent. It follows from this that the sequence $(f(x_n))$ converges to $f(\ell)$. This completes the proof of the theorem. □

Related to G-continuity we have the following result.

Corollary 2.4. *If f is arithmetically continuous, then it is G-continuous for any regular subsequential method G.*

The preceding corollary ensures that arithmetical continuity implies either of the following continuities; ordinary continuity, statistical continuity, λ-statistical continuity ([24]), ρ-statistical continuity ([4]), lacunary statistical continuity ([5]), and I-sequential continuity for any non trivial admissible ideal I of $\mathbb{N}$ ([20]).

It is well known that any continuous function on a compact subset E of $\mathbb{R}$ is uniformly continuous on E. For arithmetically continuous functions we have the following result.

Theorem 2.5. *Let E be a arithmetically compact subset E of $\mathbb{R}$ and let $f : E \longrightarrow \mathbb{R}$ be an arithmetically continuous function on E. Then f is uniformly continuous on E.*

Proof: Suppose that f is not uniformly continuous on E so that there exists an $\varepsilon_0 > 0$ such that for any $\delta > 0$ $x, y \in E$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$. For each positive integer n, there are x_n and y_n such that $|x_n - y_n| < \frac{1}{n}$, and

$|f(x_n) - f(y_n)| \geq \varepsilon_0$. Since E is arithmetically compact, there exists an arithmetically convergent subsequence (x_{n_k}) of the sequence (x_n). It is clear that the corresponding subsequence (y_{n_k}) of the sequence (y_n) is also arithmetically convergent. Then the sequence $(x_{n_1}, y_{n_1}, x_{n_2}, y_{n_2}, x_{n_3}, y_{n_3}, ..., x_{n_k}, y_{n_k}, ...)$ is arithmetically convergent. But the transformed sequence

$$(f(x_{n_1}), f(y_{n_1}), f(x_{n_2}), f(y_{n_2}), f(x_{n_3}), f(y_{n_3}), ..., f(x_{n_k}), f(y_{n_k}), ...)$$

is not arithmetically convergent. Thus f does not preserve arithmetical convergent sequences. This contradiction completes the proof of the theorem. □

Corollary 2.6. *If a function f is arithmetically continuous on a bounded subset E of $\mathbb{R}$, then it is uniformly continuous on E.*

Proof: The proof easily follows from Theorem 2.5 and Lemma 2.1, so it is omitted. □

We note that arithmetical continuity on bounded subsets coincides with not only ward continuity, but also p-ward continuity ([17]), statistical ward continuity ([10]), λ-statistical ward continuity ([24]), ρ-statistical ward continuity ([4]), lacunary statistical ward continuity ([12]), strongly lacunary ward continuity ([3]), and I-ward continuity for a nontrivial admissible ideal I ([20], [16]).

Corollary 2.7. *Arithmetically continuous image of any bounded subset of $\mathbb{R}$ is arithmetically compact.*

The proof follows from [40, Theorem 4.2] and 2.1, so it is omitted.

Corollary 2.8. *Arithmetical continuous image of a G-sequentially compact subset of $\mathbb{R}$ is arithmetically compact for any subsequential regular method G.*

Proof: The proof follows from [6, Corollary 5] and Theorem 2.1, so it is omitted. □

Write $f \in (cAC)$, and say (cAC)-continuous if f transforms convergent sequences into arithmetical convergent sequences, i.e. $(x_n) \in c \Rightarrow (f(x_n)) \in AC$. It is a well known result that uniform limit of a sequence of continuous functions is continuous. This is also true in case of (cAC)-continuity, i.e. uniform limit of a sequence of functions of points in (cAC) is (cAC)-continuous.

Theorem 2.9. *If (f_n) is a sequence of functions defined on a subset E of $\mathbb{R}$ in (cAC) and (f_n) is uniformly convergent to a function f, then $f \in (cAC)$.*

Proof: Let ε be a positive real number and (x_k) be any convergent sequence of points in E. By uniform convergence of (f_n) there exists a positive integer N such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in E$ whenever $n \geq N$. As f_N is (cAC)-continuous, there exists an $n_0 \in \mathbb{N}$, greater than N such that

$$|f_N(x_m) - f_N(x_{<m,n>})| < \frac{\varepsilon}{3}$$

for every $m \geq n_0$. Now it follows that
$|f(x_m) - f(x_{<m,n>})| \leq$
$\leq |f(x_m) - f_N(x_m))| + |f_N(x_m)) - f_N(x_{<m,n>})| + |f_N(x_{<m,n>}) - f(x_{<m,n>})| <$
$\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$

This completes the proof of the theorem. □

Theorem 2.10. *The set of all* (cAC)*-continuous functions on a subset* E *of* $\mathbb{R}$ *is a closed subset of the set of all continuous functions on* E*, i.e.* $\overline{(cAC(E))} = (cAC(E))$ *where* $(cAC(E))$ *is the set of all* (cAC)*-continuous functions on* E*,* $\overline{(cAC(E))}$ *denotes the set of all cluster points of* $(cAC(E))$*.*

Proof: Let f be any element in the closure of $(cAC(E))$. Then there exists a sequence of points in $(cAC(E))$ such that $\lim_{k\to\infty} f_k = f$. To show that f is (cAC)-continuous, take any convergent sequence (x_k) of points in E. Let $\varepsilon > 0$. Since (f_k) converges to f, there exists an N such that for all $x \in E$ and for all $n \geq N$, $|f(x) - f_n(x)| < \frac{\varepsilon}{3}$. As f_N is (cAC)-continuous, we have
$|f_N(x_m) - f_N(x_{<m,n>})| < \frac{\varepsilon}{3}$. On the other hand,
$|f(x_m) - f(x_{<m,n>})|$
$\leq |f(x_m) - f_N(x_m)| + |f_N(x_m) - f_N(x_{<m,n>})| + |f_N(x_{<m,n>}) - f(x_{<m,n>})|$.
Now it follows that
$|f(x_m) - f(x_{<m,n>})| < \varepsilon$.

This completes the proof of the theorem. □

Corollary 2.11. *The set of all* $(cAC(E))$*-continuous functions on a subset* E *of* $\mathbb{R}$ *is a complete subspace of the space of all continuous functions on* E*.*

Proof: The proof easily follows from Theorem 2.10, so is omitted. □

3. Conclusion

In this paper we give a further investigation on arithmetical ward continuity introduced in [40], presenting theorems related to this kind of continuity, and some other kinds of continuities. One may expect this investigation to be a useful tool in the field of analysis in modeling various problems occurring in many areas of science, dynamical systems, computer science, information theory, and biological science. It turns out that the set of uniformly continuous functions coincides with the set of arithmetically continuous functions on bounded subsets of $\mathbb{R}$. On the other hand, we suggest to investigate arithmetically convergent sequences of fuzzy points or soft points (see [22], for the definitions and related concepts in fuzzy setting, and see [1] related concepts in soft setting). We also suggest to investigate arithmetically convergent double sequences (see for example [32], [35], [31], [36], [34], and [21] for the definitions and related concepts in the double sequences case). For another further study, we suggest to investigate arithmetically convergent sequences in a abstract metric space (see [25], [33], [23], and [38]). Yet another further study, our suggestion is to investigate the theory in 2-normed spaces (see [19], [26] for the related concepts).

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