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On multiplicative difference sequence spaces and related dual properties

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ABSTRACT: The main purpose of the present article is to introduce the multiplicative difference sequence spaces of order m by defining the multiplicative difference operator $\Delta_*^m(x_k) = x_k \; x_{k+1}^{-m} \; x_{k+2}^{\binom{m}{2}} \; x_{k+3}^{-\binom{m}{3}} \; x_{k+4}^{\binom{m}{4}} \ldots x_{k+m}^{(-1)^m}$ for all $m, k \in \mathbb{N}$. By using the concept of multiplicative linearity some topological properties are investigated and determined their dual spaces via multiplicative infinite matrices.

Key Words: Multiplicaitve difference sequence spaces, multiplicative difference operator Δ_m^*, multiplicative linear spaces, β-duals.

Contents

1. Introduction

The concept of difference sequence space was initially introduced by Kızmaz [1]. Also Et and Çolak [2] generalized difference sequence spaces by defining $\lambda(\Delta^m) = \{x = (x_k) : \Delta^m(x) \in \lambda\}$, where $\lambda \in \{\ell_\infty, c, c_0\}$, $m \in \mathbb{N}$, the set of non-negative integers, $\Delta^0 x = (x_k), \Delta^m x = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and $\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}$. These are Banach spaces with the norm defined by $\|x\|_{\Delta^m} = \sum_{i=1}^{m} |x_i| + \sup_k |\Delta^m x_k|$. Also this type sequence spaces were studied by many authors the reader may refer to [3,4,5,6,7].

Let ω^* be the space of all positive real valued sequences defined by

$$\omega^* = \{x = (x_k) \mid x : \mathbb{N} \to \mathbb{R}^+, \; k \to x_k\},$$

and any subspace of ω^* is called a sequence space over the real field $\mathbb{R}^+$.

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Quite recently Çakmak and Başar [8] have defined the multiplicative sets ℓ_∞^*, c^*, c_0^* and ℓ_1^* of sequences as follows:

$$\begin{aligned}
\ell_\infty^* &:= \left\{x=(x_k)\in\omega^* : \sup_{k\in\mathbb{N}}|x_k|^* < \infty\right\},\\
c^* &:= \left\{x=(x_k)\in\omega^* : \exists l\in\mathbb{R}^+ \ni \ {}^*\lim_{k\to\infty}|x_k/l|^* = 1\right\},\\
c_1^* &:= \left\{x=(x_k)\in\omega^* : \ {}^*\lim_{k\to\infty}|x_k|^* = 1\right\},\\
\ell_1^* &:= \left\{x=(x_k)\in\omega^* : \prod_k |x_k|^* < \infty\right\}.
\end{aligned}$$

In particular the sets ℓ_∞^*, c^* and c_0^* of sequences over $\mathbb{R}^+$, normed by $\|\cdot\|_\infty = \sup_{k\in\mathbb{N}}|x_k|^*$, it is proved that these are all Banach spaces. Also Kadak [9] have examined the spaces bs^*, cs^* and cs_1^* defined by

$$\begin{aligned}
bs^* &:= \left\{x=(x_k)\in\omega^* : \left(\prod_{j=0}^{k} x_j\right)\in\ell_\infty^*\right\},\\
cs^* &:= \left\{x=(x_k)\in\omega^* : \left(\prod_{j=0}^{k} x_j\right)\in c^*\right\},\\
cs_1^* &:= \left\{x=(x_k)\in\omega^* : \left(\prod_{j=0}^{k} x_j\right)\in c_1^*\right\}.
\end{aligned}$$

Now, we define the sets

$$\begin{aligned}
\ell_\infty(\Delta_*^m) &= \{x=(x_k)\in\omega^* : \Delta_*^m x\in\ell_\infty^*\}\\
c(\Delta_*^m) &= \{x=(x_k)\in\omega^* : \Delta_*^m x\in c^*\}\\
c_1(\Delta_*^m) &= \{x=(x_k)\in\omega^* : \Delta_*^m x\in c_1^*\}
\end{aligned}$$

where $m\in\mathbb{Z}$, $\Delta_*^0 x=(x_k)$, $\Delta_* x=(x_k/x_{k+1})$, $\Delta_*^m x=(\Delta_*^{m-1}x_k/\Delta_*^{m-1}x_{k+1})$ and so that

$$\Delta_*^m(x_k) = x_k\, x_{k+1}^{-\binom{m}{1}}\, x_{k+2}^{\binom{m}{2}}\, x_{k+3}^{-\binom{m}{3}}\, x_{k+4}^{\binom{m}{4}}\ldots x_{k+m}^{(-1)^m\binom{m}{m}}. \tag{1.1}$$

Also the inverse operator can be interpreted as:

$$\Delta_*^{-m}(x_k) = x_k\, x_{k+1}^{m}\, x_{k+2}^{\frac{m(m+1)}{2!}}\, x_{k+3}^{\frac{m(m+1)(m+2)}{3!}}\ldots x_{k+m}^{m+1}. \tag{1.2}$$

For instance,

- $\Delta_*^2(x_k) = \left(\frac{x_k x_{k+2}}{x_{k+1}^2}\right) = \left(\frac{x_0x_2}{x_1^2}, \frac{x_1x_3}{x_2^2}, \cdots, \frac{x_kx_{k+2}}{x_{k+1}^2}, \cdots\right)$

- $\Delta_*^3(x_k) = \left(\frac{x_k x_{k+2}^3}{x_{k+1}^3 x_{k+3}}\right) = \left(\frac{x_0 x_2^3}{x_1^3 x_3}, \frac{x_1 x_3^3}{x_2^3 x_4}, \cdots, \frac{x_k x_{k+2}^3}{x_{k+1}^3 x_{k+3}}, \cdots\right)$
- $\Delta_*^{-2}(x_k) = \left(x_k x_{k+1}^2 x_{k+2}^3\right) = \left(x_0 x_1^2 x_2^3, x_1 x_2^2 x_3^3, \ldots, x_k x_{k+1}^2 x_{k+2}^3, \ldots\right)$.

The main focus of this paper is to extend the difference sequence spaces of order m defined earlier to the multiplicative form of these spaces. Moreover, by using multiplicative difference operator Δ_*^m, the duals of these spaces are investigated with respect to the multiplicative infinite matrices.

2. Preliminaries and definitions

In the period from 1967 till 1972, Grossman and Katz [10] introduced the non-Newtonian calculus consisting of the branches of geometric, bigeometric, quadratic and biquadratic calculus etc. Also Grossman extended this notion to the other fields in [11,12]. Many authors have extensively developed the notion of multiplicative calculus. The complete mathematical description of multiplicative calculus, was given by Bashirov et al. [13]. Also some authors have also worked on the classical sequence spaces and related topics by using this type calculus [14,15]. Further Kadak et al. [16,17,18] have examined matrix transformations between certain sequence spaces and have generalized Runge-Kutta numerical method.

Definition 2.1. *[13] Let X be a nonempty set. A multiplicative metric (*metric) is a mapping $d^* : X \times X \to \mathbb{R}^+$ satisfying the following conditions:*

(i) $d^*(x,y) \geq 1$ for all $x, y \in X$ and $d^*(x,y) = 1$ if and only if $x = y$;

(ii) $d^*(x,y) = d^*(y,x)$ for all $x, y \in X$;

(iii) $d^*(x,y) \leq d^*(x,z) \cdot d^*(y,z)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

The multiplicative absolute value of $x \in \mathbb{R}^+$ is defined by

$$|x|^* = \begin{cases} x & , \quad (x \geq 1), \\ 1/x & , \quad (x < 1). \end{cases} \tag{2.1}$$

On the base of this, one can define multiplicative metric spaces as alternative to the ordinary metric spaces.

Definition 2.2. *[8] Let $X = (X, d^*)$ be a *metric space. Then, the basic notions are given as follows:*

(a) *A sequence (x_n) in $X = (X, d^*)$ is said to be multiplicative convergent (*convergent) if for every given $\varepsilon > 1$, there exist an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $x \in X$ such that $d^*(x_n, x) < \varepsilon$ for all $n > n_0$ and, is denoted by ${}^*\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow{*} x$, as $n \to \infty$.*

(b) *A sequence $(x_n) \in (X, d^*)$ is said to be multiplicative bounded (*bounded) if there exists a number M such that $|x_n|^* \leq M$ for every natural number n.*

(c) *A sequence (x_n) in $X = (X, d^*)$ is said to be multiplicative Cauchy (*Cauchy) if for every $\varepsilon > 1$ there is an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $d^*(x_n, x_m) < \varepsilon$ for all $m, n > n_0$.*

(d) *A *complete metric space is a *metric space in which every *Cauchy sequence is *convergent.*

Now, we give the basic concepts *open and *closed sets.

Definition 2.3. *Given any point $x_0 \in X$. Then, for a real number $r > 0$,*

$$B(x_0; r) = \{x \in X \mid d^*(x, x_0) < r\}$$

*is a *neighborhood (or *open ball) of centre x_0 and radius r and*

$$B[x_0; r] = \{x \in X \mid d^*(x, x_0) \leq r\}$$

*is a *closed ball of centre x_0 and radius r.*

Definition 2.4. *Let (X, d^*) be a *metric space. Then $G \subset X$ is called *open set if and only if every point of G has a *neighborhood contained in G. Similarly $G \subset X$ is called *closed set if and only if its complement is *open.*

Definition 2.5. *[17] A multiplicative vector space (*vector space) over the field $\mathbb{R}^+$ is a set $V \subset \mathbb{R}^2$ on which two operations are defined, called *addition and scalar *multiplication, and denoted $(\oplus)$ and $(\odot)$ by*

$$\begin{aligned} \oplus : V \times V &\longrightarrow V \\ (u, v) &\longrightarrow u \oplus v = (u_1 v_1, u_2 v_2), \\ \odot : \mathbb{R}^+ \times V &\longrightarrow V \\ (\lambda, u) &\longrightarrow \lambda \odot u = (u_1^\lambda, u_2^\lambda) \end{aligned}$$

*where the *vectors $u = (u_1, u_2)$, $v = (v_1, v_2) \in V$ and the scalar $\lambda \in \mathbb{R}^+$. Then the operations must satisfy the following conditions:*

(a) For all $\lambda \in \mathbb{R}^+$ and all $u, v \in V$, $u \oplus v$ and $\lambda \odot v$ are uniquely defined and belong to V.

(b) For all $\xi, \eta \in \mathbb{R}^+$ and all $u, v, w \in V$, $u \oplus (v \oplus w) = (u \oplus v) \oplus w$ and $\xi \odot (\eta \odot v) = (\xi \odot \eta) \odot v$.

(c) For all $u, v \in V$, $u \oplus v = v \oplus u$.

(d) The set V contains an additive identity element, denoted by $\theta_A = (1, 1)$, such that for all $u \in V$, $u \oplus \theta_A = u$.

(e) The set V contains an additive inverse element, denoted by $u_+^{-1} = (1/u_1, 1/u_2) \in V$, such that for all $u \in V$, $u \oplus u_+^{-1} = u_+^{-1} \oplus u = \theta_A$.

(f) *For all* $\xi, \eta \in \mathbb{R}^+$ *and all* $u, v \in V$, $\xi \odot (u \oplus v) = (\xi \odot u) \oplus (\xi \odot v)$ *and* $(\xi \oplus \eta) \odot u = (\xi \odot u) \oplus (\eta \odot u)$.

(g) *The set* V *contains an element* 1_* *such that* $1_* \odot u = u$ *for all* $u \in V$.

Definition 2.6. *[17] Let* X *be a *vector space over the field* $\mathbb{R}^+$ *and* $\|\cdot\|_*$ *be a function from* X *to* $\mathbb{R}^+$ *satisfying the following axioms: For* $x, y \in X$ *and* $\lambda \in \mathbb{R}^+$,

(N1) $\|x\|_* = 1 \Leftrightarrow x = \theta^*, \ \theta^* = (1, 1, \dots)$

(N2) $\|\lambda \odot x\|_* = \lambda \odot \|x\|_* \ (= \|x\|_*^{\lambda})$

(N3) $\|x \oplus y\|_* \le \|x\|_* \oplus \|y\|_* \ (= \|x\|_* \cdot \|y\|_*)$.

Then, $(X, \|\cdot\|_*)$ *is said a *normed space. It is trivial that a *norm on* X *defines a *metric* d^* *on* X *which is given by* $d^*(x, y) = \|x/y\|_*$; $(x, y \in X)$ *and is called the *metric induced by the *norm.*

Definition 2.7. *(Multiplicative linearity) Let* V *and* W *be two *linear spaces. An operator* $T : V \to W$ *is said to be multiplicative linear (*linear) if*

$$T(v_1 \oplus v_2) = T(v_1) \oplus T(v_2) \text{ and } T(\lambda \odot v) = \lambda \odot T(v)$$

for all $v_1, v_2 \in V$ *and* $\lambda \in \mathbb{R}^+$, *or equivalently,*

$$T(v_1 \cdot v_2) = T(v_1) \cdot T(v_2) \text{ and } T(v^{\lambda}) = (T(v))^{\lambda}.$$

The important point to note here is the notion of *linearity have not the same meaning than in the standard case since the *linear space has NOT an ordinary linear structure with the usual operations.

Definition 2.8. **(i)** *The *limit of a function* f, *denoted by* ${}^*\lim\limits_{x \to a} f(x) = b$, *at an element* a *in* $\mathbb{R}^+$ *is, if it exists, the unique number* b *in* $\mathbb{R}^+$ *such that*

$$\forall \varepsilon > 1, \exists \delta > 1 \ni |f(x)/b|^* < \varepsilon \text{ for all } x, \varepsilon \in \mathbb{R}^+, |x/a|^* < \delta \text{ for } \delta \in \mathbb{R}^+.$$

A function f *is *continuous at a point* a *in* $\mathbb{R}^+$ *if and only if* a *is an argument of* f *and* ${}^*\lim\limits_{x \to a} f(x) = f(a)$.

(ii) *A topological *vector(linear) space* X *is a *vector space over the topological field that endowed with a topology such that *vector addition and scalar multiplication are *continuos functions.*

(iii) *A topological *vector space is called *normable if the topology of the space can be induced by a *norm.*

Definition 2.9. *A sequence space* λ *with a *linear topology is called a *K-space provided each of the maps* $p_i : \lambda \to \mathbb{R}^+$ *defined by* $p_i(x) = x_i$ *is *continuous for all* $i \in \mathbb{N}$. *A *K-space is called a *FK-space provided* λ *is a complete linear *metric space. A *FK-space whose topology is *normable is called a *BK-space.*

3. Multiplicative difference sequence spaces of order m

In fact, the development of the different sequence spaces has been taken place due to the introduction of several new modern techniques in functional analysis involving topological structures, dual spaces, matrix mappings etc. Now, in this section we investigate and discuss some interesting properties of the difference operator Δ_*^m.

Theorem 3.1.

(i) *The operator $\Delta_*^m : \omega^* \to \omega^*$ is *linear.*

(ii) *If X is a *linear space, then $X(\Delta_*^m)$ is also a *linear space.*

Proof: The proofs are straightforward, hence omitted. □

Theorem 3.2. *Let $m, n \in \mathbb{N}$ and $(x_k) \in \omega^*$. The followings hold:*

(i) $\Delta_*^m(\Delta_*^n(x_k)) = \Delta_*^n(\Delta_*^m(x_k)) = \Delta_*^{m+n}(x_k)$,

(ii) $\Delta_*^m(\Delta_*^{-m}(x_k)) = \Delta_*^{-m}(\Delta_*^m(x_k)) = x_k$.

Proof:

(i) By taking into account Theorem 3.1, we have

$$\begin{aligned}
\Delta_*^m(\Delta_*^n(x_k)) &= \Delta_*^m\big(x_k\, x_{k+1}^{-\binom{n}{1}}\, x_{k+2}^{\binom{n}{2}}\, x_{k+3}^{-\binom{n}{3}}\, x_{k+4}^{\binom{n}{4}} \dots x_{k+n}^{(-1)^n}\big) \\
&= \big(x_k\, x_{k+1}^{-\binom{n}{1}}\, x_{k+2}^{\binom{n}{2}}\, x_{k+3}^{-\binom{n}{3}} \dots x_{k+n}^{(-1)^n}\big) \\
&\quad \big(x_{k+1}\, x_{k+2}^{-\binom{n}{1}}\, x_{k+3}^{\binom{n}{2}}\, x_{k+4}^{-\binom{n}{3}} \dots x_{k+n+1}^{(-1)^n}\big)^{-\binom{m}{1}} \\
&\quad \big(x_{k+2}\, x_{k+3}^{-\binom{n}{1}}\, x_{k+4}^{\binom{n}{2}}\, x_{k+5}^{-\binom{n}{3}} \dots x_{k+n+2}^{(-1)^n}\big)^{\binom{m}{2}} \dots \Big(x_{k+m}^{(-1)^n}\Big)^{(-1)^m} \\
&= x_k\, x_{k+1}^{-n-m}\, x_{k+2}^{\binom{n}{2}+mn+\binom{m}{2}}\, x_{k+3}^{-\binom{n}{3}-m\binom{n}{2}-n\binom{m}{2}-\binom{m}{3}} \dots x_{k+m}^{(-1)^{m+n}} \\
&= x_k\, x_{k+1}^{-(m+n)}\, x_{k+2}^{\binom{m+n}{2}}\, x_{k+3}^{-\binom{m+n}{3}}\, x_{k+4}^{\binom{m+n}{4}} \dots x_{k+m}^{(-1)^{m+n}} \\
&= \Delta_*^{m+n}(x_k).
\end{aligned}$$

(ii) The proof may be obtained by using similar technique with case (i).

□

Theorem 3.3. *If X is a Banach space with respect to *norm $\|\cdot\|_\infty$, then $X(\Delta_*^m)$ is also a Banach space with the *norm $\|\cdot\|_{\Delta_*}$ defined by*

$$||x||_{\Delta_*} = \left(\prod_{i=1}^{m} |x_i|^*\right) \cdot \|\Delta_*^m x\|_\infty \tag{3.1}$$

where $\|\Delta_*^m x\|_\infty = sup_k|\Delta_*^m x_k|^*$ for all $k \in \mathbb{N}$.

Proof: By using the *linearity of Δ_*^m, it is clear that $X(\Delta_*^m)$ is a *normed space with the *norm in (3.1). We must show that $X(\Delta_*^m)$ is *complete. Suppose (x^n) is any *Cauchy sequence in $X(\Delta_*^m)$, where $x^n = (x_j^n) \in X(\Delta_*^m)$. Assume $\|x^n/x^k\|_{\Delta_*} \xrightarrow{*} 1$ as $k, n \to \infty$ for all $k, n \geq n_0(\epsilon)$, that is $\|x_j^n/x_j^k\|_{\Delta_*} \xrightarrow{*} 1$ as $k, n \to \infty$. Thus,

$$\|x_j^n/x_j^k\|_{\Delta_*} = \left(\prod_{i=1}^{m} |x_i^n/x_i^k|^*\right) \cdot \|\Delta_*^m(x_j^n/x_j^k)\|_\infty \xrightarrow{*} 1$$

where

$$\|\Delta_*^m(x_j^n/x_j^k)\|_\infty = \sup_j \left|(x_j^n/x_j^k)(x_{j+1}^n/x_{j+1}^k)^{-m}(x_{j+2}^n/x_{j+2}^k)^{\frac{m(m-1)}{2}} \cdots (x_{j+m}^n/x_{j+m}^k)^{(-1)^m}\right|^*.$$

Therefore $(x_i^1, x_i^2, \cdots)$ and $(\Delta_*^m(x_j^1), \Delta_*^m(x_j^2), \cdots)$ are *Cauchy sequences in real field $\mathbb{R}$ and X^*, respectively. By using the completeness of $\mathbb{R}$ and X^*, we have that they are *convergent and suppose that $x_i^n \xrightarrow{*} x_i$ in $\mathbb{R}$ and $(\Delta_*^m(x^n)) \xrightarrow{*} y_j$ in X^* as $n \to \infty$. Let $y_j = \Delta_*^m(x_j)$ so that

$$\begin{aligned} x_j &= \prod_{p=1}^{j-m} y_p^{(-1)^m \binom{j-p-1}{m-1}} \\ &= \prod_{p=1}^{j} y_{p-m}^{(-1)^m \binom{j+m-p-1}{m-1}}, \quad (y_{1-m} = y_{2-m} = \cdots = y_0 = 1) \end{aligned}$$

for sufficiently large j, i.e. $j > 2m$. Then $(\Delta_*^m(x^n)) = (\Delta_*^m(x_j^1), \Delta_*^m(x_j^2), \cdots)$ *converges to $(\Delta_*^m(x_j))$ in X^*. Thus $\|x^n/x\|_{\Delta_*} \xrightarrow{*} 1$, as $k \to \infty$. This shows that $X(\Delta_*^m)$ is a Banach space. □

Theorem 3.4. *The sequence spaces $X(\Delta_*^m)$ for $X = \{\ell_\infty, c_1, c\}$ are *linearly isomorphic to the spaces ℓ_∞^*, c_1^* and c^*, respectively.*

Proof: We prove this theorem for the space $\ell_\infty(\Delta_*^m)$. To prove this, we should show the existence of a *norm preserving *linear bijection between the spaces $\ell_\infty(\Delta_*^m)$ and ℓ_∞^*. Consider the transformation T defined from $\ell_\infty(\Delta_*^m)$ to ℓ_∞^* by $u \mapsto v = Tu = \prod_{j=0}^{k} \Delta_*^m(u_j)$. The *linearity of T is trivial. Further, it is clear that $u = \theta^*$ whenever $Tu = \theta^*$ and hence T is injective.

Let $v = (v_k) \in \ell_\infty^*$ and define the sequence $u = (u_k) \in \omega^*$ by $u_k = \Delta_*^{-m}(v_k/v_{k-1})$ for each $k \in \mathbb{N}$ and $v_{-1} = 1$. By taking into account Theorem 3.2(ii), we observe that

$$\begin{aligned} \prod_{j=0}^{k} \Delta_*^m(u_j) &= \prod_{j=0}^{k} \Delta_*^m(\Delta_*^{-m}(v_j/v_{j-1})) \\ &= \prod_{j=0}^{k} \frac{v_j}{v_{j-1}} = v_k \end{aligned}$$

Since $\sup_{k\in\mathbb{N}} \left|\prod_{j=0}^{k} \Delta_*^m(u_j)\right|^* = \sup_{k\in\mathbb{N}} |v_k|^* < \infty$ then $u \in \ell_\infty(\Delta_*^m)$. Hence T is surjective and *norm preserving. □

Lemma 3.5. *If $X^* \subset Y^*$, then $X(\Delta_*^m) \subset Y(\Delta_*^m)$.*

Proof: Let $X^* \subset Y^*$ and $x = (x_k) \in X(\Delta_*^m)$. It is trivial that $(\Delta_*^m x_k) \in X^*$ implies that $(\Delta_*^m x_k) \in Y^*$. Hence $x \in Y(\Delta_*^m)$ and $X(\Delta_*^m) \subset Y(\Delta_*^m)$. □

Theorem 3.6. *Let X be a Banach space and K, a *closed subset of X. Then $K(\Delta_*^m)$ is also *closed subset of $X(\Delta_*^m)$.*

Proof: From Lemma 3.5 it is clear that $K(\Delta_*^m) \subset X(\Delta_*^m)$. Now we must show that $\overline{K(\Delta_*^m)} = \overline{K}(\Delta_*^m)$. Let $x \in \overline{K(\Delta_*^m)}$, then there exists a sequence $(x^n) \in \overline{K(\Delta_*^m)}$ such that $\|x^n/x\|_{\Delta_*} \xrightarrow{*} 1$ as $n \to \infty$. Hence

$$\|x^n/x\|_{\Delta_*} = \left(\prod_{i=1}^{m} |x_i^n/x_i|^*\right) \cdot \|\Delta_*^m(x_j^n/x_j)\|_\infty \xrightarrow{*} 1$$

as $n \to \infty$ in K and implies that $x \in \overline{K}(\Delta_*^m)$.

Conversely, let $x \in \overline{K}(\Delta_*^m)$, then $x \in K(\Delta_*^m)$. Since K is *closed $\overline{K(\Delta_*^m)} = K(\Delta_*^m)$. Hence $K(\Delta_*^m)$ is *closed subset of $X(\Delta_*^m)$. □

Theorem 3.7. *If X is a *BK-space with the *norm $\|\cdot\|_\infty$, then $X(\Delta_*^m)$ is also *BK-space with the *norm given in (3.1).*

Proof: It is obvious that $X(\Delta_*^m)$ is Banach space (see Thm. 3.3). Suppose that $\|x_j^n/x_j\|_{\Delta_*} \xrightarrow{*} 1$ which implies that $\|\Delta_*^m(x_j^n/x_j)\|_\infty \xrightarrow{*} 1$ as $n \to \infty$ for each $j \in \mathbb{N}$. Thus

$$\sup_j \left|(x_j^n/x_j)(x_{j+1}^n/x_{j+1})^{-m}(x_{j+2}^n/x_{j+2})^{\frac{m(m-1)}{2}} \cdots (x_{j+m}^n/x_{j+m})^{(-1)^m}\right|^* \xrightarrow{*} 1$$

and $|x_j^n/x_j|^* \xrightarrow{*} 1$ as $n \to \infty$ for each $j \in \mathbb{N}$. Therefore $X(\Delta_*^m)$ is a Banach space with *continuous coordinates. Hence $X(\Delta_*^m)$ is *BK-space. □

4. Dual properties

In this section, following [19] we give the α-, β- and γ-duals of a set $\lambda \subset \omega^*$ which are respectively denoted by λ^α, λ^β and λ^γ, as follows:

$$\lambda^\alpha := \left\{x = (x_k) \in \omega^* : (x_k \odot y_k) \in \ell_1^* \text{ for all } y = (y_k) \in \lambda\right\},$$
$$\lambda^\beta := \left\{x = (x_k) \in \omega^* : (x_k \odot y_k) \in cs^* \text{ for all } y = (y_k) \in \lambda\right\},$$
$$\lambda^\gamma := \left\{x = (x_k) \in \omega^* : (x_k \odot y_k) \in bs^* \text{ for all } y = (y_k) \in \lambda\right\}.$$

A multiplicative infinite matrix $A = (a_{ij})$ of positive real numbers is defined by a function A from the set $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}^+$. The addition $(\oplus)$ and scalar multiplication $(\odot)$ of the infinite matrices $A = (a_{ij})$ and $B = (b_{ij})$ are defined by

$$A \oplus B = (a_{ij} \oplus b_{ij}) = (a_{ij} \cdot b_{ij}) \text{ and } \lambda \odot A = (\lambda \odot a_{ij}) = ({a_{ij}}^{\lambda})$$

for all $i, j \in \mathbb{N}$. Also, the product $(A \odot B)_{ij}$ of $A = (a_{ij})$ and $B = (b_{ij})$ can be interpreted as

$$(A \odot B)_{ij} = \prod_{k=0}^{\infty} (a_{ik} \odot b_{kj}), \quad \left(= \prod_{k=0}^{\infty} b_{kj}^{a_{ik}} \right) \tag{4.1}$$

and provided the infinite product on the right hand side *converge when the *limit exists. Further the product (4.1) may *diverge for some, or all, values of i, j; the product $A \odot B$ may not exist.

Let μ_1, $\mu_2 \subset w^*$ and $A = (a_{nk})$ be a multiplicative infinite matrix. Then, we say that A defines a matrix mapping from μ_1 into μ_2, and denote it by writing $A : \mu_1 \to \mu_2$, if for every sequence $x = (x_k) \in \mu_1$ the sequence $A \odot x = \{(Ax)_n\}$, the multiplicative A-transform of x, exists and is in μ_2. In this way, we transform the sequence $x = (x_k)$ into the sequence $\{(Ax)_n\}$ defined by

$$(Ax)_n = \prod_{k=0}^{\infty} (a_{nk} \odot x_k) \tag{4.2}$$

for all $k, n \in \mathbb{N}$. Thus, $A \in (\mu_1 : \mu_2)$ if and only if the infinite product on the right side of (4.2) *converges for each $n \in \mathbb{N}$. A sequence z is said to be A-*summable to γ if $A \odot x$ *converges to $\gamma \in \mathbb{R}^+$ which is called as the A-*lim of z.

Lemma 4.1. *Let $\mathcal{F}$ be the collection of all finite subsets of $\mathbb{N}$ and $K \in \mathcal{F}$. Then, we have*

(i) (cf. [16]) $A = (a_{nk}) \in (\ell_\infty^* : \ell_\infty^*)$ if and only if

$$\sup_{n \in \mathbb{N}} \prod_k |a_{nk}|^* < \infty. \tag{4.3}$$

(ii) $A = (a_{nk}) \in (\ell_\infty^* : \ell_1^*)$ if and only if

$$\sup_{K \in \mathcal{F}} \prod_n \left| \prod_{k \in K} a_{nk} \right|^* < \infty. \tag{4.4}$$

(iii) $A = (a_{nk}) \in (\ell_\infty^* : c^*)$ if and only if there exists $(\delta_k) \in \omega^*$ such that

$${}^*\lim_{n \to \infty} a_{nk} = \delta_k \text{ for all } k, \text{ and} \tag{4.5}$$

$${}^*\lim_{n \to \infty} \prod_k |a_{nk}|^* = \prod_k |\delta_k|^*. \tag{4.6}$$

Proof:

(ii) Let $A \in (\ell_\infty^* : \ell_1^*)$ and $x = (x_k) \in \ell_\infty^*$. Then, the infinite product $\prod_k (a_{nk} \odot x_k)$ is *convergent for each fixed $n \in \mathbb{N}$, since $A \odot x$ exists. Thus, the sequence $\{a_{nk}\}_{k=0}^{\infty} \in \{\ell_\infty^*\}^\beta = \ell_1^*$ (see. [9]) for all $n \in \mathbb{N}$ and $A \odot x \in \ell_1^*$ which yields that

$$
\begin{aligned}
\prod_{n=0}^{\infty} |(Ax)_n|^* &= \prod_{n=0}^{\infty} \left| \prod_{k \in K} (a_{nk} \odot x_k) \right|^* \\
&= \prod_{n=0}^{\infty} \left(\left| \prod_{k \in K} a_{nk} \right|^* \odot |x_k|^* \right)
\end{aligned}
$$

Thus, the sequence $\left\{ \left| \prod_{k \in K} a_{nk} \right|^* \odot |x_k|^* \right\}$ is *bounded (see Def 2.2) and the condition (4.4) holds.

Conversely suppose that (4.4) holds and $x = (x_k) \in \ell_\infty^*$. Then, since $\{a_{nk}\}_{k=0}^{\infty} \in \{\ell_\infty^*\}^\beta$ for each $n \in \mathbb{N}$, $A \odot x$ exists. Therefore, one can observe by using (4.3) that

$$
\begin{aligned}
\prod_{n=0}^{\infty} |(Ax)_n|^* &= \prod_{n=0}^{\infty} \left(\left| \prod_{k \in K} a_{nk} \right|^* \odot |x_k|^* \right) \\
&\leq \sup_{K \in \mathcal{F}} |x_k|^* \odot \sup_{K \in \mathcal{F}} \left\{ \prod_{n=0}^{\infty} \left| \prod_{k \in K} a_{nk} \right|^* \right\} < \infty.
\end{aligned}
$$

Hence $A \odot x \in \ell_1^*$.

(iii) The necessary and sufficient conditions can be obtained by taking into account [20, Theorem 10, pp. 223-225] and [16, Theorem 23], respectively.

∎

Define the matrices $B = (b_{nk})$ and $C = (c_{nk})$ as follows:

$$
\begin{aligned}
b_{nk} &= \begin{cases} \left(\prod_{i=0}^{m} (-1)^i \binom{-m}{i}\right) \odot x_k &, \quad (k \geq n), \\ 1 &, \quad (k < n). \end{cases} \\
c_{nk} &= \begin{cases} \left(\prod_{i=0}^{m} (-1)^i \binom{-m}{i}\right) \odot \left(\prod_{j=n}^{k} x_j\right) &, \quad (k \geq n), \\ 1 &, \quad (k < n). \end{cases}
\end{aligned}
$$

where $\prod_{i=0}^{m} (-1)^i \binom{-m}{i} = m \left(\frac{m(m+1)}{2}\right) \left(\frac{m(m+1)(m+2)}{3!}\right) \cdots (-1)^m \binom{-m}{m}$ for each $m \in \mathbb{N}$.

Theorem 4.2. *Consider the sets h_1 and h_2 defined by*

$$\begin{aligned} h_1 &:= \left\{ (x_k) \in \omega^* : \sup_{K \in \mathcal{F}} \prod_n \left| \prod_{k \in K} \left[\left(\prod_{i=0}^{m} (-1)^i \binom{-m}{i} \right) \odot x_n \right] \right|^* < \infty \right\}, \\ h_2 &:= \left\{ (x_k) \in \omega^* : \prod_n \left| \prod_k \left[\left(\prod_{i=0}^{m} (-1)^i \binom{-m}{i} \right) \odot x_n \right] \right|^* < \infty \right\}. \end{aligned}$$

Then, $\{\ell_\infty(\Delta_*^m)\}^\alpha = h_1$, $\{c_1(\Delta_*^m)\}^\alpha = h_1$ *and* $\{c(\Delta_*^m)\}^\alpha = h_1 \cap h_2$.

Proof: Since the proof can also be obtained in the similar way for other cases, we prove only case $\{\ell_\infty(\Delta_*^m)\}^\alpha = h_1$.

For a sequence $v = (v_k) \in \ell_\infty^*$ and $u = (u_k) = \left(\prod_{i=0}^{m} (-1)^i \binom{-m}{i}\right) \odot (v_k/v_{k-1}) \in \ell_\infty(\Delta_*^m)$. Then we obtain

$$\begin{aligned} x_k \odot u_k &= \prod_{j=0}^{k} \left\{ \left(\prod_{i=0}^{m} (-1)^i \binom{-m}{i} \right) \odot (v_k/v_{k-1}) \right\} \odot x_k \\ &= (B \odot v)_k \end{aligned}$$

for each $k \in \mathbb{N}$ and $v_{-1} = 1$. Therefore, we observe that $(x_k \odot u_k) \in \ell_1^*$ whenever $u \in \ell_\infty(\Delta_*^m)$ if and only if $B \odot v \in \ell_1^*$ whenever $v \in \ell_\infty^*$. This yields that $x = (x_k) \in \{\ell_\infty(\Delta_*^m)\}^\alpha$ if and only if $B \in (\ell_\infty^* : \ell_1^*)$. By using Lemma 4.1(ii), we have $\{\ell_\infty(\Delta_*^m)\}^\alpha = h_1$. □

Theorem 4.3. *Consider the sets h_3, h_4, h_5 and h_6 defined by*

$$\begin{aligned} h_3 &:= \left\{ (x_k) \in \omega^* : \ {}^*\lim_{n\to\infty} \prod_k |c_{nk}|^* = \prod_k | \ {}^*\lim_{n\to\infty} c_{nk}|^* \right\}, \\ h_4 &:= \left\{ (x_k) \in \omega^* : \ {}^*\lim_{n\to\infty} \left(c_{nk}/\ell_k\right) = 1 \ \textit{for all } k \right\}, \ (\ell = (\ell_k) \in \omega^*) \\ h_5 &:= \left\{ (x_k) \in \omega^* : \ {}^*\lim_{n\to\infty} \left(\prod_k c_{nk}/\ell' \right) = 1 \right\}, \ (\ell' \in \mathbb{R}^+) \\ h_6 &:= \left\{ (x_k) \in \omega^* : \sup_n \prod_k |c_{nk}|^* < \infty \right\}. \end{aligned}$$

Then

(i) $\{\ell_\infty(\Delta_*^m)\}^\beta = h_3 \cap h_4$, $\{c_0(\Delta_*^m)\}^\beta = h_4 \cap h_6$, $\{c(\Delta_*^m)\}^\beta = h_4 \cap h_5 \cap h_6$.

(ii) $\{\ell_\infty(\Delta_*^m)\}^\gamma = h_6$, $\{c_0(\Delta_*^m)\}^\gamma = h_4 \cap h_6$, $\{c(\Delta_*^m)\}^\gamma = h_6$.

Proof: We give the proof only for the space $\ell_\infty(\Delta_*^m)$, since other part can be obtained similarly.

(i) To prove the β-dual, we must show that $(x_k \odot u_k) \in cs^* \subset c^*$ for all $x \in \{\ell_\infty(\Delta_*^m)\}^\beta$. For this, we can write

$$\begin{aligned}\prod_{p=0}^{k}(x_p \odot u_p) &= \prod_{p=0}^{k}\left(\prod_{j=0}^{p}\left\{\left(\prod_{i=0}^{m}(-1)^i\binom{-m}{i}\right)\odot(v_k/v_{k-1})\right\}\odot x_p\right)\\ &= (C\odot v)_k\end{aligned}$$

Thus, we observe that $(x_k \odot u_k) \in cs^*$ whenever $u \in \ell_\infty(\Delta_*^m)$ if and only if $C \odot v \in c^*$ whenever $v \in \ell_\infty^*$. This implies that $x \in \{\ell_\infty(\Delta_*^m)\}^\beta$ if and only if $C \in (\ell_\infty^* : c^*)$. By using Lemma 4.1(iii) we get $\{\ell_\infty(\Delta_*^m)\}^\beta = h_3 \cap h_4$.

(ii) Again for the proof of γ-dual may be obtained by using similar techniques in case (i). Hence, omitted.

□

Concluding Remarks

The main results given in the present paper will base on examining the concept of multiplicative difference sequence spaces using a multiplicative linear operator Δ_*^m. This is a new development of the difference sequence spaces over real numbers. In particular using the inverse operator Δ_*^{-m} some new classes of matrix transformations can be extended and the table on the characterizations of the matrix transformations between certain spaces can be investigated. Finally, we should note from now on that our next papers will be devoted to the corresponding table for multiplicative difference sequence spaces of order m.

Computing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

References

1. H. Kızmaz, On certain sequence spaces, Canad. Math. Bull. 24 (2) (1981) 169-176.
2. M. Et, R. Çolak, On some generalized difference sequence spaces, Soochow J. Math. 21 (4) (1995) 377-386.
3. M. Mursaleen, Generalized spaces of difference sequences, J. Math. Anal. Appl., 203(1996) 738-745.
4. E. Malkowsky, Mursaleen and S. Suantai, The dual spaces of sets of difference sequences of order m and matrix transformations, Acta Math. Sinica (Engl. Ser.), 23(3)(2007) 521-532.
5. M. Mursaleen and A. K. Noman, On some new difference sequence spaces of non-absolute type, Math. Comput. Modelling, 52 (2010) 603-617.
6. M. Mursaleen, A. Alotaibi and Sunil K. Sharma, New classes of generalized seminormed difference sequence spaces, Abstract Appl. Anal., Volume 2014, Article ID 461081, 8 pages.
7. M. Mursaleen, N.A. Sheikh and Ab H. Ghanie, New type of generalized difference sequence space of non-absolute type and some matrix transformations, Filomat, 28(7) (2014) 1381–1392.

8. A. F. Çakmak, F. Başar, Some new results on sequence spaces with respect to non-Newtonian calculus, J. Inequal. Appl., vol. 2012, no. 1, pp. 228, 2012.

9. U. Kadak, Determination of the Kothe-Toeplitz duals over the non-Newtonian complex field, Sci. World J., vol. 2014, Article ID 438924, 10 pages, 2014.

10. M. Grossman, R. Katz,Non-Newtonian Calculus, Lee Press, Pigeon Cove, MA, 1972.

11. M. Grossman, Bigeometric Calculus, Archimedes Foundation, Rockport, Mass, USA, 1983.

12. M. Grossman, The First Nonlinear System of Differential and Integral Calculus, Mathco, 1979.

13. A. E. Bashirov, E. Kurpınar, A. Özyapıcı, Multiplicative calculus and its applications, Math. Anal. Appl., vol. 337, no. 1, pp. 36-48, 2008.

14. A. F. Çakmak, F. Başar, Certain spaces of functions over the field of non-Newtonian complex numbers, Abstr. Appl. Anal., vol. 2014, Article ID 236124, 12 pages, 2014.

15. C. Türkmen, F. Başar, Some basic results on the sets of sequences with geometric calculus, Commun. Fac. Sci. Univ. Ank. Series A1, vol.61, no. 2, pp. 17ÂŨ-34, 2012.

16. U. Kadak, H. Efe, Matrix transformations between certain sequence spaces over the non-Newtonian complex field, Sci. World J., vol. 2014, Article ID 705818, 12 pages, 2014.

17. U. Kadak, H. Efe, The construction of Hilbert spaces over the Non-Newtonian field,Int. J. Anal., vol. 2014, Article ID 746059.

18. U. Kadak, M. Özlük, Generalized Runge-Kutta method with respect to the non-Newtonian calculus, Abstr. Appl. Anal., vol. 2014, Article ID 594685.

19. F. Başar, *Summability Theory and Its Applications, Normed and Paranormed Sequence Spaces*, Bentham Science Publishers, e-books, Monographs, İstanbul, 2012.

20. I.J. Maddox, Elements of Functional Analysis, The University Press, Cambridge, second edition, 1988.

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