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Optimal Energy Decay Rate for Rayleigh Beam Equation with Only One Dynamic Boundary Control

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ABSTRACT: In [21], Wehbe considered the Rayleigh beam equation with two dynamical boundary controls and established the optimal polynomial energy decay rate of type $\frac{1}{t}$. The proof exploits in an explicit way the presence of two boundary controls, hence the case of the Rayleigh beam damped by only one dynamical boundary control remained open. In this paper, we fill this gap by considering a clamped Rayleigh beam equation subject to only one dynamical boundary feedback. First, we consider the Rayleigh beam equation subject to only one dynamical boundary control moment. In that case, we prove a polynomial decay in $\frac{1}{t}$ of the energy by using an observability inequality. For that purpose, we give the asymptotic expansion of eigenvalues and eigenfunctions of the undamped underling system. Moreover, using the real part of the asymptotic expansion of eigenvalues of the damped system, we prove that the obtained energy decay rate is optimal. Next, we consider the Rayleigh beam equation subject to only one dynamical boundary control force. Here we use a Riesz basis approach. As before, we start by giving the asymptotic expansion of the eigenvalues and the eigenfunctions of the damped and undamped systems. We next show that the system of eigenvectors of the damped problem form a Riesz basis. Finally, we deduce the optimal energy decay rate of polynomial type in $\frac{1}{\sqrt{t}}$.

Key Words: Rayleigh beam equation, dynamic boundary control, spectral analysis, observability inequality, Riesz basis.

Contents

1. Introduction

In [21], Wehbe considered a Rayleigh beam clamped at one end and subjected to two dynamical boundary controls at the other end, namely

$$y_{tt} - \gamma y_{xxtt} + y_{xxxx} = 0, \quad 0 < x < 1, \quad t > 0, \tag{1.1}$$
$$y(0,t) = y_x(0,t) = 0, \quad t > 0, \tag{1.2}$$
$$y_{xx}(1,t) + a\eta(t) = 0, \quad t > 0, \tag{1.3}$$
$$y_{xxx}(1,t) - \gamma y_{xtt}(1,t) - b\xi(t) = 0, \quad t > 0, \tag{1.4}$$

where $\gamma > 0$ is the coefficient of moment of inertia, $a > 0$ and $b > 0$ are positive constants, η and ξ denote respectively the dynamical boundary control moment and force. The damping of the system is made via the indirect damping mechanism at the right extremity of the beam that involves the following two first order differential equations:

$$\eta_t(t) - y_{xt}(1,t) + \alpha\eta(t) = 0, \quad t > 0, \tag{1.5}$$
$$\xi_t(t) - y_t(1,t) + \beta\xi(t) = 0, \quad t > 0, \tag{1.6}$$

where $\alpha > 0$ and $\beta > 0$. The notion of indirect damping mechanisms has been introduced by Russell in [18] and since that time, it retains the attention of many authors. In [21], Wehbe considered the Rayleigh beam equation with two dynamical boundary controls moment and force, *i.e.*, under the conditions $a > 0$ and $b > 0$. The lack of uniform stability was proved by a compact perturbation argument of Gibson and a polynomial energy decay rate of type $\frac{1}{t}$ is obtained by a multiplier method usually used for nonlinear problems. Finally, using a spectral method, he proved that the obtained energy decay is optimal in the sense that for any $\varepsilon > 0$, we cannot expect a decay rate of type $\dfrac{1}{t^{1+\varepsilon}}$. But in [21] the effect of each control separately on the stability of the Rayleigh beam equation is not investigated. Indeed, the multiplier method exploits in an explicit way the presence of the two boundary controls. Furthermore, the lack of one of this two controls yield this method ineffective. Then, the important and interesting case when the Rayleigh beam equation is damped by only one dynamical boundary control ($a = 0$ and $b > 0$ or $a > 0$ and $b = 0$) remained open. The aim of this paper is to fill this gap by considering a clamped Rayleigh beam equation subject to only one dynamical boundary feedback.

First, we consider the Rayleigh beam equation (1.1)-(1.4) with only one dynamical boundary control moment η, *i.e.*, when $a = 1$, $b = 0$ and η solution of (1.5).

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Using an explicit approximation of the characteristic equation, we give the asymptotic behavior of eigenvalues and eigenfunctions of the associated undamped system with the help of Rouché's theorem. Then to prove the polynomial energy decay, we apply a methodology introduced in [2]. This requires, on one hand, to establish an observability inequality of solution of the undamped system and on the other hand, to verify the boundedness property of the transfer function. This attend to establish a polynomial energy decay rate of type $\frac{1}{t}$ for smooth initial data. Finally, using a frequency domain approach, we prove that the obtained energy decay rate is optimal in the sense that for any $\varepsilon > 0$, we cannot expect a decay rate of type $\frac{1}{t^{1+\varepsilon}}$.

Next, we consider the Rayleigh beam equation (1.1)-(1.4) with only one dynamical boundary control force, *i.e.*, when $a = 0$, $b = 1$ and ξ solution of (1.6). Here we prefer to use a Riesz basis approach. First, as before we give the asymptotic expansion of the eigenvalues and the eigenfunctions of the damped and undamped systems. Next, we show that the system of eigenvectors of high frequencies of the damped problem is quadratically closed to the system of eigenvectors of high frequencies of the undamped problem. This yields, from [9, Theorem 6.3] and [1, Theorem 1.2.10] that the system of generalized eigenvectors of the damped problem forms a Riesz basis of the energy space. Finally, using [14, Theorem 2.1]) we establish the optimal energy decay rate of polynomial type $\frac{1}{\sqrt{t}}$.

The stabilization of the Rayleigh beam equation retains the attention of many authors. In this regard, different types of dampings have been introduced to the Rayleigh beam equation and several uniform and polynomial stability results have been obtained. Rao [16] studied the stabilization of Rayleigh beam equation subject to a positive internal viscous damping. Using a constructive approximation, he established the optimal exponential energy decay rate. In [12], Lagnese studied the stabilization of system (1.1)-(1.4) with two static boundary controls (the case $a > 0$, $b > 0$, $\eta(t) = y_{xt}(1,t)$ and $\xi(t) = y_t(1,t)$). He proved that the energy decays exponentially to zero for all initial data. Rao in [16] extended the results of [12] to the case of one boundary feedback (the case $a > 0$, $b = 0$ and $\eta(t) = y_{xt}(1,t)$ or $a = 0$, $b > 0$ and $\xi(t) = y_t(1,t)$). In the case of one control moment (the case $a > 0$, $b = 0$ and $\eta(t) = y_{xt}(1,t)$), using a compact perturbation theory due to Gibson [8], he established an exponential stability of system (1.1)-(1.4). Moreover, in the case of one control force ($a = 0$, $b > 0$ and $\xi(t) = y_t(1,t)$), he first proved the lack of exponential stability of the system (1.1)-(1.4). Next, he proved that the Rayleigh beam equation can be strongly stabilized by only one control force if and only if the inertia coefficient γ is large enough but he did not studied the decay rate of the energy of the system. In [3], Bassam *and al.* studied the decay rate of energy of system (1.1)-(1.4) with $a = 0$, $b > 0$ and $\xi(t) = y_t(1,t)$. First, using an explicit approximation, they gave the asymptotic expansion of eigenvalues and eigenfunctions of the undamped system corresponding to (1.1)-(1.4), then they established the optimal polynomial energy decay rate via an observability inequality of solution of the undamped system and the boundedness of the transfer function associated with the undamped problem.

Let us briefly outline the content of this paper. Section 2 considers the Rayleigh beam equation with only one dynamical boundary control moment and is divided into four subsections. In subsection 2.1, we formulate the system into an evolution equation and we recall the well-posedness property of the problem by the semi-group approach (see [15], [16] and [21]). In subsection 2.2, we propose an explicit approximation of the characteristic equation determining the eigenvalues of the corresponding undamped system. Then, we give an asymptotic expansion of eigenvalues and eigenfunctions of the corresponding operator. In subsection 2.3, we establish a polynomial energy decay rate for smooth initial data. In subsection 2.4, we prove that the obtained energy decay rate is optimal. Section 3 considers the Rayleigh beam equation with only one dynamical boundary control force and is divided into 2 subsections. As before our system can be transformed into an evolution equation and we deduce the well-posedness property of the problem by the semi-group approach. We recall the condition to reach the strong stability of our system (see [16]). In subsection 3.1, we proposed also an explicit approximation of the characteristic equation determining the eigenvalues of the damped and undamped system. Then, we give an asymptotic expansion of eigenvalues and eigenfunctions of the corresponding operators. In subsection 3.2, we show that the system of eigenvectors of the damped problem forms a Riesz basis and we establish the optimal polynomial energy decay rate of type $\frac{1}{\sqrt{t}}$.

2. Rayleigh beam equation with only one dynamical control moment

In this section, we consider the Rayleigh beam equation with only one dynamical boundary control moment:

$$\begin{cases} y_{tt} - \gamma y_{xxtt} + y_{xxxx} & = 0, \quad 0 < x < 1, \quad t > 0, \\ y(0,t) = y_x(0,t) & = 0, \quad t > 0, \\ y_{xx}(1,t) + \eta(t) & = 0, \quad t > 0, \\ y_{xxx}(1,t) - \gamma y_{xtt}(1,t) & = 0, \quad t > 0, \\ \eta_t(t) - y_{xt}(1,t) + \alpha\eta(t) & = 0, \quad t > 0. \end{cases} \tag{2.1}$$

Let y and η be smooth solutions of system (2.1), we define their associated energy by:

$$E(t) = \frac{1}{2}\left(\int_0^1 (|y_t|^2 + \gamma|y_{xt}|^2 + |y_{xx}|^2)dx + |\eta(t)|^2\right). \tag{2.2}$$

A direct computation gives

$$\frac{d}{dt}E(t) = -\alpha|\eta(t)|^2 \leq 0. \tag{2.3}$$

Thus the system (2.1) is dissipative in the sense that the energy $E(t)$ is a nonincreasing function of the time variable t.

2.1. Well-posedness and strong stability of the problem

In this subsection, we will study the existence, uniqueness and the asymptotic behavior of the solution of system (2.1). We start our study by formulating the problem in an appropriate Hilbert space. We first introduce the following spaces:

$$V = \left\{y \in H^1(0,1);\ y(0) = 0\right\}, \quad \|y\|_V^2 = \int_0^1 (|y|^2 + \gamma|y_x|^2)dx, \tag{2.4}$$

$$W = \left\{y \in H^2(0,1);\ y(0) = y_x(0) = 0\right\}, \quad \|y\|_W^2 = \int_0^1 |y_{xx}|^2 dx, \tag{2.5}$$

and the energy space

$$\mathcal{H} = W \times V \times \mathbb{C} \tag{2.6}$$

endowed with the usual inner product

$$((y_1,\, z_1,\, \eta_1), (y_2,\, z_2,\, \eta_2))_{\mathcal{H}} = (y_1, y_2)_W + (z_1, z_2)_V + \eta_1\overline{\eta_2},$$

$$\forall (y_1,\, z_1,\, \eta_1), (y_2,\, z_2,\, \eta_2) \in \mathcal{H}.$$

Identify $L^2(0,1)$ with its dual so that we have the following continuous embedding

$$W \subset V \subset L^2(0,1) \subset V' \subset W'. \tag{2.7}$$

Let y and η be smooth solutions of system (2.1). Then, multiplying the first equation of the system (2.1) by $\overline{\varPhi} \in W$ and integrating by parts yields

$$\int_0^1 (y_{tt}\overline{\Phi} + \gamma y_{xtt}\overline{\Phi_x})dx + \int_0^1 y_{xx}\overline{\Phi_{xx}}dx + \eta\overline{\Phi_x(1)} = 0. \tag{2.8}$$

Now we define the following linear operators $A \in \mathcal{L}(W, W')$, $B \in \mathcal{L}(\mathbb{C}, W')$ and $C \in \mathcal{L}(V, V')$ by:

$$< Ay, \Phi >_{W' \times W} = (y, \Phi)_W, \quad \forall y, \Phi \in W, \tag{2.9}$$

$$< B\eta, \Phi >_{W' \times W} = \eta\overline{\Phi_x(1)}, \quad \forall \eta \in \mathbb{C}, \forall \Phi \in W \tag{2.10}$$

and

$$< Cy, \Phi >_{V' \times V} = (y, \Phi)_V, \quad \forall y, \Phi \in V. \tag{2.11}$$

Then, by means of Lax-Milgram theorem (see [6]), we see that A (resp C) is the canonical isomorphism from W into W' (resp from V into V'). On the other hand, using the usual trace theorems and Poincaré inequality, we easily check that the operator B is continuous for the corresponding topology. Therefore, using the operators A, B and C and the continuous embedding (2.7), we formulate the variational equation (2.8) as:

$$Cy_{tt} + Ay + B\eta = 0 \quad \text{in } W'.$$

Assume that $Ay + B\eta \in V'$, then we obtain:

$$y_{tt} + C^{-1}(Ay + B\eta) = 0 \qquad \text{in } V. \tag{2.12}$$

Next we introduce the linear unbounded operator $\mathcal{A}_0$ by

$$D(\mathcal{A}_0) = \{(y, z, \eta) \in \mathcal{H};\ z \in W \ \text{ and } Ay + B\eta \in V'\}, \tag{2.13}$$

$$\mathcal{A}_0 u = \begin{pmatrix} -z \\ C^{-1}(Ay + B\eta) \\ -z_x(1) \end{pmatrix}, \quad \forall u = (y, z, \eta) \in D(\mathcal{A}_0) \tag{2.14}$$

and the linear bounded operator $\mathcal{B}$ as follows

$$\mathcal{B}u = \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix}, \quad \forall u = (y, z, \eta) \in \mathcal{H}. \tag{2.15}$$

Then, denoting $u = (y, y_t, \eta)$ the state of system (2.1) and define $\mathcal{A}_\alpha = \mathcal{A}_0 + \alpha\mathcal{B}$ with $D(\mathcal{A}_\alpha) = D(\mathcal{A}_0)$, we can formulate the system (2.1) into a first-order evolution equation

$$\begin{cases} u_t(t) + \mathcal{A}_\alpha u(t) &= 0, \\ u(0) &= u_0 \in \mathcal{H}. \end{cases} \tag{2.16}$$

It is easy to show that $-\mathcal{A}_0$ is m-dissipative and $-\mathcal{B}$ is dissipative in the energy space $\mathcal{H}$. Therefore the operator $-\mathcal{A}_\alpha$ generates a C_0-semigroup $(e^{-t\mathcal{A}_\alpha})_{t\geq 0}$ of contractions in the energy space $\mathcal{H}$ following Hille-Yosida's theorem (see [15]). Hence, we have the following results concerning the existence and uniqueness of the solution of the problem (2.16):

Theorem 2.1. *For any initial data $u_0 \in \mathcal{H}$, the problem (2.16) has a unique weak solution $u(t) = e^{-t\mathcal{A}_\alpha}u_0$ such that $u \in C^0([0, \infty[, \mathcal{H})$. Moreover, if $u_0 \in D(\mathcal{A}_0)$, then the problem (2.16) has a strong solution $u(t) = e^{-t\mathcal{A}_\alpha}u_0$ such that $u \in C^1([0, \infty[, \mathcal{H}) \cap C^0([0, \infty[, D(\mathcal{A}_0))$.* □

Moreover, we characterize the space $D(\mathcal{A}_0)$ by the following proposition.

Proposition 2.2. *Let $u = (y, z, \eta) \in \mathcal{H}$. Then $u \in D(\mathcal{A}_0)$ if and only if the following conditions hold:*

$$\begin{cases} y \in W \cap H^3(0, 1), \\ z \in W, \\ y_{xx}(1) + \eta = 0. \end{cases} \tag{2.17}$$

In particular, the resolvent $(I + \mathcal{A}_0)^{-1}$ of $-\mathcal{A}_0$ is compact on the energy space $\mathcal{H}$ and the solution of the system (2.1) satisfies

$$y(t) \in C^0([0, \infty[, H^3(0, 1) \cap W) \cap C^1([0, \infty[, W) \cap C^2([0, \infty[, V). \tag{2.18}$$

□

The proof is same as in Rao [16, Proposition 2.3] (see also Wehbe [21]) so we omit the details here.
Now we investigate the strong stability of the problem (2.16) by the following theorem:

Theorem 2.3. *For any $\gamma > 0$, the semigroup of contractions $e^{-t\mathcal{A}_\alpha}$ is strongly asymptotically stable on the energy space $\mathcal{H}$, i.e. for any $u_0 \in \mathcal{H}$, we have*

$$\lim_{t\to+\infty} \|e^{-t\mathcal{A}_\alpha} u_0\|^2_{\mathcal{H}} = 0. \tag{2.19}$$

Proof: The proof is same as in Rao [16, Theorem 3.1], it is based on the spectral decomposition theory of Sz-Nagy-Foias [19], Foguel [7] and Benchimol [4]. In order to prove (2.19), it is sufficient to show that there is no spectrum in imaginary axis. We omit the details here. □

Further, since $\mathcal{A}_0$ is skew adjoint operator and $\mathcal{B}$ is compact, then using a compact perturbation method of Russel [17], we deduce that the system (2.16) is not uniformly stable (see Rao [16], and Wehbe [21]).

2.2. Polynomial Stability for smooth initial data

Our main result in this subsection is the following polynomial-type decay estimate:

Theorem 2.4. *(Polynomial energy decay rate)*
Let $\gamma > 0$. For all initial $U_0 \in D(\mathcal{A}_0)$, there exists a constant $c > 0$ independent of U_0, such that the solution of the problem (2.16) satisfies the following estimate:

$$E(t) \leq \frac{c}{1+t}\|U_0\|^2_{D(\mathcal{A}_0)}, \quad \forall t > 0. \tag{2.20}$$

□

For this aim, we need first to analyze the spectrum of the operator $\mathcal{A}_0$. Next, We will apply a method introduced by Ammari and Tucsnak in [2], where the polynomial stability for the damped problem is reduced to an observability inequality of the corresponding undamped problem (via the spectral analysis), combined with the boundedness property of the transfer function of the associated undamped system.

2.2.1. **Spectral analysis of the operator $\mathcal{A}_0$.**
Since $\mathcal{A}_0$ is closed with a compact resolvent, its spectrum $\sigma(\mathcal{A}_0)$ consists entirely of isolated eigenvalues with finite multiplicities (see [11]). Moreover, as the coefficients of $\mathcal{A}_0$ are real then the eigenvalues appear by conjugate pairs. Further, the eigenvalues of $\mathcal{A}_0$ are on the imaginary axis.

Proposition 2.5. *Let λ be an eigenvalue of $\mathcal{A}_0$ and let $U = (y, z, \eta) \in D(\mathcal{A}_0), U \neq 0$, an associated eigenvector. Then λ is simple and we have $\eta \neq 0$.*

Proof: First, a straightforward computation shows that $0 \in \sigma(\mathcal{A}_0)$ and is simple. An associated eigenvector being $(-\frac{x^2}{2}, 0, 1)$, thus its last component $\eta = 1$ does not vanish.
Next, let $\lambda = i\mu \in \sigma(\mathcal{A}_0)$, $\mu \in \mathbb{R}^*$ and $U = (y, z, \eta)$ an associated eigenvector.
Assume that $\eta = 0$. Using equation (2.15), we get that $\mathcal{B}U = 0$. Thus, we obtain

$$\mathcal{A}_\alpha U = (\mathcal{A}_0 + \alpha\mathcal{B})U = \mathcal{A}_0 U = i\mu U. \tag{2.21}$$

Therefore $\lambda = i\mu$ is also an eigenvalue of $\mathcal{A}_\alpha$ and it is a contradiction with Theorem 2.3 since $\gamma > 0$.
Later, assume that there exists $\lambda \in \sigma(\mathcal{A}_0)$ such that λ is not simple. As $\mathcal{A}_0$ is a skew-adjoint operator we deduce that there correspond at least two independent eigenvectors $U_1 = (y_1, z_1, \eta_1)$ and $U_2 = (y_2, z_2, \eta_2)$. Then, $U_3 = \eta_2 U_1 - \eta_1 U_2 = (y_3, z_3, \eta_3)$ is also an eigenvector associated with λ with $\eta_3 = 0$, hence the contradiction with the first part of the proof. $\square$

Now, in order to get a better knowledge of the spectrum we compute the characteristic equation. Thus let $\lambda = i\mu$, $\mu \in \mathbb{R}^*$, be an eigenvalue of $\mathcal{A}_0$ and $U = (y,\ z,\ \eta) \in D(\mathcal{A}_0)$ be an associated eigenfunction. Then we have

$$\begin{cases} z &= -i\mu y, \\ Ay + B\eta &= i\mu C z, \\ z_x(1) &= -i\mu\eta. \end{cases} \tag{2.22}$$

Then, using (2.9)-(2.11) we interpret (2.22) as the following variational equation

$$\int_0^1 y_{xx}\overline{\Phi_{xx}}dx - \mu^2 \int_0^1 \left(y\overline{\Phi} + \gamma y_x \overline{\Phi_x}\right) dx + y_x(1)\overline{\Phi_x(1)} = 0, \qquad \forall \Phi \in W.$$

Equivalently, the function y is determined by the following system:

$$\begin{cases} y_{xxxx} + \gamma\mu^2 y_{xx} - \mu^2 y &= 0, \\ y(0) = y_x(0) &= 0, \\ y_{xx}(1) + y_x(1) &= 0, \\ y_{xxx}(1) + \gamma\mu^2 y_x(1) &= 0. \end{cases} \tag{2.23}$$

We have found that $\lambda = i\mu \neq 0$ is an eigenvalue of $\mathcal{A}_0$ if and only if there is a non trivial solution of (2.23). The general solution of the first equation of (2.23) is given by

$$y(x) = \sum_{i=1}^{4} c_i e^{r_i(\mu)x}, \tag{2.24}$$

where

$$r_1(\mu) = \sqrt{\frac{-\gamma\mu^2 + \mu\sqrt{\gamma^2\mu^2 + 4}}{2}}, r_2(\mu) = -r_1(\mu), \tag{2.25}$$

$$r_3(\mu) = \sqrt{\frac{-\gamma\mu^2 - \mu\sqrt{\gamma^2\mu^2+4}}{2}}, r_4(\mu) = -r_3(\mu).$$

Here and below, for simplicity we denote $r_i(\mu)$ by r_i. Thus the boundary conditions in (2.23) may be written as the following system:

$$M(\mu)C(\mu) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ r_1 & r_2 & r_3 & r_4 \\ g_1(\mu) & g_2(\mu) & g_3(\mu) & g_4(\mu) \\ h_1(\mu) & h_2(\mu) & h_3(\mu) & h_4(\mu) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0, \tag{2.26}$$

where $g_i(\mu) = r_i\left(r_i^2 + \gamma\mu^2\right)e^{r_i}$ and $h_i(\mu) = r_i\left(r_i+1\right)e^{r_i}$. Consequently (2.23) admits a non-trivial solution if and only if $f(\mu) := \det M(\mu) = 0$. Finally, we have found that $\lambda = i\mu$ is an eigenvalue of $\mathcal{A}_0$ if and only if μ satisfies the characteristic equation $f(\mu) = 0$.

Proposition 2.6. *(Spectrum of $\mathcal{A}_0$)*
There exists $k_0 \in \mathbb{N}^*$, sufficiently large, such that the spectrum $\sigma(\mathcal{A}_0)$ of $\mathcal{A}_0$ is given by:

$$\sigma(\mathcal{A}_0) = \sigma_0 \cup \sigma_1, \tag{2.27}$$

where

$$\sigma_0 = \left\{i\kappa_j^0\right\}_{j\in J_0}, \quad \sigma_1 = \left\{\lambda_k^0 = i\mu_k\right\}_{\substack{k\in\mathbb{Z} \\ |k|\geq k_0}}, \quad \sigma_0 \cap \sigma_1 = \emptyset, \tag{2.28}$$

J_0 is a finite set and κ_j^0, $\mu_k \in \mathbb{R}$. Moreover, μ_k satisfies the following asymptotic behavior:

$$\mu_k = \alpha_k - \frac{F_1}{F_0}\frac{1}{k\pi} + O(\frac{1}{k^2}), \quad |k| \to \infty \tag{2.29}$$

where

$$\alpha_k = \frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}, \tag{2.30}$$

$$F_0 = 2\gamma^{\frac{3}{2}}\cosh(\frac{1}{\sqrt{\gamma}}) \quad \text{and} \quad F_1 = (1-2\gamma)\cosh(\frac{1}{\sqrt{\gamma}}) + 2\sqrt{\gamma}\sinh(\frac{1}{\sqrt{\gamma}}). \tag{2.31}$$

Proof: The proof is decomposed into two steps.
Step 1. First, we start by the expansion of r_1 and r_3 when $|\mu| \to \infty$. After some computations we find

$$r_1 = \frac{1}{\sqrt{\gamma}} + O(\frac{1}{\mu^2}) \tag{2.32}$$

and

$$r_3 = i\sqrt{\gamma}\mu + i\frac{1}{2\gamma^{\frac{3}{2}}\mu} + O(\frac{1}{\mu^3}). \tag{2.33}$$

This gives

$$r_1^2 e^{r_1} = \frac{e^{\frac{1}{\sqrt{\gamma}}}}{\sqrt{\gamma}} + O(\frac{1}{\mu^2}), \tag{2.34}$$

$$r_2^2 e^{r_2} = \frac{e^{-\frac{1}{\sqrt{\gamma}}}}{\sqrt{\gamma}} + O(\frac{1}{\mu^2}), \tag{2.35}$$

$$r_3^2 e^{r_3} = -\gamma e^{i\sqrt{\gamma}\mu}\mu^2 + O(1) \tag{2.36}$$

and

$$r_4^2 e^{r_4} = -\gamma e^{-i\sqrt{\gamma}\mu}\mu^2 + O(1). \tag{2.37}$$

Next, using (2.32)-(2.37), we find the asymptotic behavior of

$$g_1(\mu) = \sqrt{\gamma} e^{\frac{1}{\sqrt{\gamma}}}\mu^2 + O(1), \tag{2.38}$$

$$g_2(\mu) = -\sqrt{\gamma} e^{-\frac{1}{\sqrt{\gamma}}}\mu^2 + O(1), \tag{2.39}$$

$$g_3(\mu) = -\frac{ie^{i\sqrt{\gamma}\mu}\mu}{\sqrt{\gamma}} + O(\frac{1}{\mu}) \tag{2.40}$$

and

$$g_4(\mu) = \frac{ie^{-i\sqrt{\gamma}\mu}\mu}{\sqrt{\gamma}} + O(\frac{1}{\mu}). \tag{2.41}$$

Similarly, we get

$$h_1(\mu) = \left(\frac{1}{\sqrt{\gamma}} + 1\right) \frac{e^{\frac{1}{\sqrt{\gamma}}}}{\sqrt{\gamma}} + O(\frac{1}{\mu}), \tag{2.42}$$

$$h_2(\mu) = \left(\frac{1}{\sqrt{\gamma}} - 1\right) \frac{e^{-\frac{1}{\sqrt{\gamma}}}}{\sqrt{\gamma}} + O(\frac{1}{\mu}), \tag{2.43}$$

$$h_3(\mu) = \left(-\gamma\mu^2 + i(\sqrt{\gamma} - \frac{1}{2\sqrt{\gamma}})\mu\right) e^{i\sqrt{\gamma}\mu} + O(1) \tag{2.44}$$

and

$$h_4(\mu) = \left(-\gamma\mu^2 + i(\frac{1}{2\sqrt{\gamma}} - \sqrt{\gamma})\mu\right) e^{-i\sqrt{\gamma}\mu} + O(1). \tag{2.45}$$

Now, using (2.26) and (2.32)-(2.45), we can write $M(\mu)$ as follows

$$M(\mu)=\begin{pmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{\sqrt{\gamma}}+O(\frac{1}{\mu^2}) & -\frac{1}{\sqrt{\gamma}}+O(\frac{1}{\mu^2}) & P_1+O(\frac{1}{\mu^2}) & P_2+O(\frac{1}{\mu^3}) \\ \sqrt{\gamma}e^{\frac{1}{\sqrt{\gamma}}}\mu^2+O(1) & -\sqrt{\gamma}e^{-\frac{1}{\sqrt{\gamma}}}\mu^2+O(1) & -\frac{ie^{i\sqrt{\gamma}\mu}\mu}{\sqrt{\gamma}}+O(\frac{1}{\mu}) & \frac{ie^{-i\sqrt{\gamma}\mu}\mu}{\sqrt{\gamma}}+O(\frac{1}{\mu}) \\ P_3e^{\frac{1}{\sqrt{\gamma}}}+O(\frac{1}{\mu}) & P_4e^{-\frac{1}{\sqrt{\gamma}}}+O(\frac{1}{\mu}) & P_5e^{i\sqrt{\gamma}\mu}+O(1) & P_6e^{-i\sqrt{\gamma}\mu}+O(\mu) \end{pmatrix}, \tag{2.46}$$

where

$$P_1=i\sqrt{\gamma}\mu+i\frac{1}{2\gamma^{\frac{3}{2}}\mu},\quad P_2=-i\sqrt{\gamma}\mu-i\frac{1}{2\gamma^{\frac{3}{2}}\mu},\quad P_3=\frac{1}{\gamma}+\frac{1}{\sqrt{\gamma}},\quad P_4=\frac{1}{\gamma}-\frac{1}{\sqrt{\gamma}},$$

$$P_5=-\gamma\mu^2+i(\sqrt{\gamma}-\frac{1}{2\sqrt{\gamma}})\mu \quad \text{and} \quad P_6=-\gamma\mu^2+i(\frac{1}{2\sqrt{\gamma}}-\sqrt{\gamma})\mu.$$

Again after some computations, we find the following asymptotic development of $f(\mu)=det(M(\mu))$

$$f(\mu)=\mu^5 f_0(\mu)+\mu^4 f_1(\mu)+O(\mu^3),$$

where

$$f_0(\mu)=-2iF_0\sqrt{\gamma}\cos(\sqrt{\gamma}\mu),\quad \text{and } f_1(\mu)=2i\sqrt{\gamma}F_1\sin(\sqrt{\gamma}\mu). \tag{2.47}$$

For convenience we set

$$S(\mu)=\frac{f(\mu)}{\mu^5}=f_0(\mu)+\frac{f_1(\mu)}{\mu}+O(\frac{1}{\mu^2}), \tag{2.48}$$

that has the same root as f, except 0. **Step 2.** We look at the roots of S. Is is easy to see that the roots of f_0 are given by:

$$\alpha_k=\frac{k\pi}{\sqrt{\gamma}}+\frac{\pi}{2\sqrt{\gamma}},\quad k\in\mathbb{Z}.$$

Then, with the help of Rouché's theorem, there exists $k_0\in\mathbb{N}^*$ large enough, such that for all $|k|\geq k_0$ the large roots of S (denoted by μ_k) are close to α_k. More precisely, there exists $k_0\in\mathbb{N}^*$ large enough, such that the splitting of $\sigma(\mathcal{A}_0)$ given in (2.27)-(2.28) holds and we have

$$\mu_k=\alpha_k+o(1)=\frac{k\pi}{\sqrt{\gamma}}+\frac{\pi}{2\sqrt{\gamma}}+o(1),\quad |k|\to\infty. \tag{2.49}$$

Equivalently we can write

$$\mu_k=\frac{k\pi}{\sqrt{\gamma}}+\frac{\pi}{2\sqrt{\gamma}}+l_k,\quad \lim_{|k|\to\infty} l_k=0. \tag{2.50}$$

It follows that

$$\cos(\sqrt{\gamma}\mu_k) = -(-1)^k \sin(\sqrt{\gamma}l_k) = -(-1)^k\sqrt{\gamma}l_k + o(l_k^2) \tag{2.51}$$

and

$$\sin(\sqrt{\gamma}\mu_k) = (-1)^k \cos(\sqrt{\gamma}l_k) = (-1)^k(1 - \frac{\sqrt{\gamma}l_k^2}{2}) + o(l_k^2). \tag{2.52}$$

Using (2.50), (2.51) and (2.52) then from (2.48) we have

$$0 = S(\mu_k) = 2i\sqrt{\gamma}(-1)^k(F_0\sqrt{\gamma}l_k + \frac{F_1}{k\pi}) + o(l_k^2) + O(\frac{1}{k^2}),$$

which implies

$$l_k = -\frac{F_1}{F_0}\frac{1}{k\pi} + O(\frac{1}{k^2}). \tag{2.53}$$

Inserting the previous identity in (2.50) we directly get (2.29). □

Eigenvectors of $\mathcal{A}_0$. According the decomposition of the spectrum $\sigma(\mathcal{A}_0)$ of $\mathcal{A}_0$, a set of eigenvectors associated with $\sigma(\mathcal{A}_0)$ is given as follows:

$$\left\{\Phi_j = (y_j, z_j, \eta_j) \in D(\mathcal{A}_0)\right\}_{j\in J_0} \bigcup \left\{U_k = (y_k, z_k, \eta_k) \in D(\mathcal{A}_0)\right\}_{\substack{k\in\mathbb{Z}\\ |k|\geq k_0}}, \tag{2.54}$$

where

$$\Phi_j = \begin{pmatrix} y_j \\ -i\kappa_j^0 y_j \\ y_{j,x}(1) \end{pmatrix} \quad \text{and} \quad U_k = \begin{pmatrix} y_k \\ -i\mu_k y_k \\ y_{k,x}(1) \end{pmatrix}. \tag{2.55}$$

Now, for $|k| \geq k_0$ and $\mu = \mu_k$, we give a solution up to a factor of problem (2.23) and some appropriated asymptotic behavior.

Proposition 2.7. *Let $|k| \geq k_0$. Then, a solution y_k of the undamped initial value problem (2.23) with $\mu = \mu_k$ satisfies the following estimations:*

$$y_{k,x}(1) = -(-1)^k k\pi + O(1) \neq 0, \ \|y_k\|_W \sim |k|^2 \text{ and } \|y_k\|_V \sim |k|, \quad |k| \to \infty. \tag{2.56}$$

Moreover we deduce

$$\|U_k\|_{\mathcal{H}} \sim |k|^2, \quad |k| \to \infty. \tag{2.57}$$

Proof: For $\mu = \mu_k, |k| \geq k_0$, solving (2.23) amounts to find a solution $C(\mu_k) \neq 0$ of the system (2.26) of rank three. For clarity, we divide the proof into two steps.
Step 1. Estimate of $y_{k,x}(1)$. For simplicity of notation we write $C(\mu_k) = (c_1, c_2, c_3, c_4)$. Since we search $C(\mu_k)$ up to a factor we choose $c_3 = 1$, the possibility of this choice will be justify later. Therefore (2.26) becomes

$$\begin{cases} c_1 + c_2 + c_4 & = & -1 \\ r_1c_1 + r_2c_2 + r_4c_4 & = & -r_3 \\ r_1(r_1+1)e^{r_1}c_1 + r_2(r_2+1)e^{r_2}c_2 + r_4(r_4+1)e^{r_4}c_4 & = & -r_3(r_3+1)e^{r_3}. \end{cases}$$

Next, using Cramer's rule, we obtain

$$c_1 = \frac{\alpha_1}{\alpha_3}, \quad c_2 = \frac{\alpha_2}{\alpha_3}, \quad c_4 = \frac{\alpha_4}{\alpha_3}, \tag{2.58}$$

where

$$\alpha_1 = 2r_1r_3(1-r_1)e^{-r_1} + r_3(r_3^2 - r_1)(e^{r_3} + e^{-r_3}) + r_3^2(1-r_1)(e^{r_3} - e^{-r_3}), \tag{2.59}$$

$$\alpha_2 = 2r_1r_3(1+r_1)e^{r_1} - r_3(r_3^2 + r_1)(e^{r_3} + e^{-r_3}) - r_3^2(1+r_1)(e^{r_3} - e^{-r_3}), \tag{2.60}$$

$$\alpha_3 = 2r_1r_3(1-r_3)e^{-r_3} + r_1(r_1^2 - r_3)(e^{r_1} + e^{-r_1}) + r_1^2(1-r_3)(e^{r_1} - e^{-r_1}), \tag{2.61}$$

and

$$\alpha_4 = 2r_1r_3(1+r_3)e^{r_3} - r_1(r_1^2 + r_3)(e^{r_1} + e^{-r_1}) - r_1^2(1+r_3)(e^{r_1} - e^{-r_1}). \tag{2.62}$$

First we study the behavior of α_1. Inserting (2.32) and (2.33) (with $\mu = \mu_k$) in (2.59) we find after some computations

$$\alpha_1 = -2i\gamma^{3/2}\cos(\sqrt{\gamma}\mu_k)\mu_k^3 + i(1 + 2\sqrt{\gamma} + 2\gamma)\sin(\sqrt{\gamma}\mu_k)\mu_k^2 + O(\mu_k). \tag{2.63}$$

Now inserting (2.53) in (2.51) and (2.52) we have

$$\cos(\sqrt{\gamma}\mu_k) = (-1)^k \frac{F_1\sqrt{\gamma}}{F_0 k\pi} + O(\frac{1}{k^2}) \quad \text{and} \quad \sin(\sqrt{\gamma}\mu_k) = (-1)^k + O(\frac{1}{k^2}). \tag{2.64}$$

Inserting (2.29) and (2.64) in (2.63) we find again after some computations

$$\begin{aligned} \alpha_1 &= -i(-1)^k \frac{\left(2F_1\gamma^{3/2} + F_0\left(-1 - 2\sqrt{\gamma} + 2\gamma\right)\right)\pi^2}{F_0\gamma} k^2 + O(k) \\ &= -2i(-1)^k \frac{\pi^2\left(\tanh\left(\frac{1}{\sqrt{\gamma}}\right) - 1\right)}{\sqrt{\gamma}} k^2 + O(k). \end{aligned} \tag{2.65}$$

Similarly long computations left to the reader yields

$$\alpha_2 = 2i(-1)^k \frac{\pi^2\left(\tanh\left(\frac{1}{\sqrt{\gamma}}\right) + 1\right)}{\sqrt{\gamma}} k^2 + O(k), \tag{2.66}$$

$$\alpha_3 = -2i(-1)^k \frac{\pi^2}{\sqrt{\gamma}} k^2 + O(k) \tag{2.67}$$

and

$$\alpha_4 = -2i(-1)^k \frac{\pi^2}{\sqrt{\gamma}} k^2 + O(k). \tag{2.68}$$

Remark that $\alpha_3 \neq 0$ provided we have chosen k_0 large enough; for this reason our choice $c_3 = 1$ is valid. Substituting (2.65)-(2.68) into (2.58), we obtain

$$c_1 = \tanh(\frac{1}{\sqrt{\gamma}}) - 1 + O(\frac{1}{k}),\ c_2 = -\tanh(\frac{1}{\sqrt{\gamma}}) - 1 + O(\frac{1}{k}),\ c_3 = 1,\ c_4 = 1 + O(\frac{1}{k}). \tag{2.69}$$

Finally we have found that a solution (2.26) has the form:

$$C(\mu_k) = C_0 + O(\frac{1}{|\mu_k|}), \tag{2.70}$$

where

$$C_0 = (-1 + \tanh(\frac{1}{\sqrt{\gamma}}),\ -1 - \tanh(\frac{1}{\sqrt{\gamma}}),\ 1,\ 1).$$

Note that the corresponding solution y_k of (2.23) is given by (2.24). From equation (2.24), we have

$$y_{k,x}(1) = r_1 c_1 e^{r_1} + r_2 c_2 e^{r_2} + r_3 c_3 e^{r_3} + r_4 c_4 e^{r_4}, \tag{2.71}$$

where we recall that for $i = 1, ..., 4, r_i = r_i(\mu_k)$ are given by (2.25) and $c_i, i = 1, ..., 4$, satisfy (2.69). Therefore using the series expansion (2.29), (2.32), (2.33) and (2.69) we easily find

$$y_{k,x}(1) = -(-1)^k 2k\pi + O(1) \neq 0. \tag{2.72}$$

Step 2. Estimates of $\|y_k\|_W$ and $\|y_k\|_V$. We start with

$$\|y_k\|_W^2 = \int_0^1 |y_{k,xx}|^2 dx = \sum_{i=1}^4 \sum_{j=1}^4 c_i r_i^2 \left(\int_0^1 e^{r_i x} \overline{e^{r_j x}} dx \right) \overline{c_j r_j^2} = C_k G_k \overline{C_k}^T \tag{2.73}$$

where

$$G_k = \left(\int_0^1 e^{(r_i + \overline{r_j})x} dx \right)_{1 \leq i,j \leq 4} \quad \text{and} \quad C_k = (c_i r_i^2)_{i=1,\ldots 4}.$$

First, since $r_2 = -r_1 \in \mathbb{R}$ (for $|k|$ large enough) and $r_3 = -r_4 \in i\mathbb{R}$, we directly find

$$\int_0^1 e^{(r_1 + \overline{r_2})x} dx = \int_0^1 e^{(r_2 + \overline{r_1})x} dx = \int_0^1 e^{(r_3 + \overline{r_3})x} dx = \int_0^1 e^{(r_4 + \overline{r_4})x} = 1. \tag{2.74}$$

In addition using the identity $\int_0^1 e^{rx} dx = \frac{e^r}{r} - \frac{1}{r}$ for $r \neq 0$ and the asymptotic behavior (2.32)-(2.33) we find that

$$G_k = G_0 + O(\frac{1}{k}), \tag{2.75}$$

where

$$G_0 = \begin{pmatrix} \frac{\sqrt{\gamma}}{2}(e^{\frac{2}{\sqrt{\gamma}}} - 1) & 1 & 0 & 0 \\ 1 & \frac{\sqrt{\gamma}}{2}(1 - e^{-\frac{2}{\sqrt{\gamma}}}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.76}$$

and $O(\frac{1}{k})$ is a matrix where all the entries are of order $\frac{1}{k}$. Next, using (2.32), (2.33) and (2.69), we obtain

$$C_k = (0, 0, -\gamma\mu_k^2, -\gamma\mu_k^2) + O(1). \tag{2.77}$$

Finally inserting (2.75) and (2.77) in (2.73) we deduce that

$$\|y_k\|_W^2 = \gamma^2 |\mu_k|^4 + O(|\mu_k|^3) \sim |k|^4, \quad |k| \to \infty. \tag{2.78}$$

Similarly, we easily prove that

$$\int_0^1 |y_k|^2 dx \sim 1, \qquad \int_0^1 |y_{k,x}|^2 dx \sim |\mu_k|^2 \sim |k|^2, \quad |k| \to \infty.$$

Therefore, we deduce that

$$\|y_k\|_V \sim |k|, \quad |k| \to \infty. \tag{2.79}$$

Moreover, using the estimations (2.72), (2.78) and (2.79) then from (2.55) we deduce

$$\|U_k\|_{\mathcal{H}} \sim |k|^2, \quad |k| \to \infty.$$

This completes the proof. □

2.2.2. **Observability inequality and boundedness of the transfer function.**

First, since $\mathcal{B}$ is a self-ajdoint operator and $\mathcal{B}\mathcal{B}^* = \mathcal{B}$, we rewrite the problem (2.16) as follows

$$\begin{cases} U_t(t) + (\mathcal{A}_0 + \alpha\mathcal{B}\mathcal{B}^*)\, U(t) &= 0, \\ U(0) &= U_0 \in \mathcal{H}. \end{cases} \tag{2.80}$$

We will establish an observability inequality for the undamped problem corresponding to (2.80) in following Lemma:

Lemma 2.8. *Let $\gamma > 0$. There exist $T > 0$ and $C_T > 0$ such that the solution U of the problem*

$$\begin{cases} U_t(t) + \mathcal{A}_0 U(t) &= 0, \\ U(0) &= U_0 \end{cases} \tag{2.81}$$

satisfies the following observability inequality

$$\int_0^T \|\mathcal{B}^* U(t)\|_{\mathcal{H}}^2 dt \geq C_T \|U_0\|_{(D(\mathcal{A}_0))'}^2 \tag{2.82}$$

where $(D(\mathcal{A}_0))'$ is the dual of $D(\mathcal{A}_0)$ with respect to the scalar product in $\mathcal{H}$.

Proof: Let $U_0 \in D(\mathcal{A}_0)$, then we can write

$$U_0 = \sum_{j \in J_0} U_0^j \widetilde{\Phi}_j + \sum_{|k| \geq k_0} U_0^k \widetilde{U}_k \tag{2.83}$$

where $\left\{\widetilde{\Phi}_j\right\}_{j \in J_0} \bigcup \left\{\widetilde{U}_k\right\}_{\substack{k \in \mathbb{Z} \\ |k| \geq K_0}}$ denotes the set of normalized eigenvectors of $\mathcal{A}_0$ such that

$$\widetilde{\Phi}_j = (\widetilde{y}_j, \widetilde{z}_j, \widetilde{\eta}_j) = \frac{1}{\|\Phi_j\|_{\mathcal{H}}} \Phi_j, \quad \forall j \in J_0 \tag{2.84}$$

and

$$\widetilde{U}_k = (\widetilde{y}_k,\, \widetilde{z}_k,\, \widetilde{\eta}_k) = \frac{1}{\|U_k\|_{\mathcal{H}}} U_k, \quad \forall |k| \geq k_0. \tag{2.85}$$

From (2.83) we obtain

$$U(t) = \sum_{j \in J_0} U_0^j e^{i\kappa_j t} \widetilde{\Phi}_j + \sum_{|k| \geq k_0} U_0^k e^{i\mu_k t} \widetilde{U}_k. \tag{2.86}$$

Consequently, we have

$$\eta(t) = y_x(1,t) = -\sum_{j \in J_0} U_0^j e^{i\kappa_j t} \widetilde{y}_{j,x}(1) - \sum_{|k| \geq k_0} U_0^k e^{i\mu_k t} \widetilde{y}_{k,x}(1), \quad \forall t > 0.$$

The spectral gap is satisfied by the eigenvalues of $\mathcal{A}_0$ because they are simple and for k large enough, we have $\mu_{k+1} - \mu_k \geq \frac{\pi}{4\sqrt{\gamma}}$, in other words, there exists $d > 0$, such that

$$\min_{\substack{\lambda, \lambda^{'} \in \sigma(\mathcal{A}_0) \\ \lambda \neq \lambda^{'}}} |\lambda - \lambda^{'}| \geq d > 0.$$

Thus, using Ingham's inequality (see [10]), we deduce that there exist $T > 0$ and $c_T > 0$ such that

$$\begin{aligned} \int_0^T \|B^* U(t)\|_{\mathcal{H}}^2 dt &= \int_0^T |\eta(t)|^2 dt \\ &= \int_0^T |y_x(1,t)|^2 dt \\ &\geq c_T \left(\sum_{j \in J_0} |U_0^j|^2 |\widetilde{y}_{j,x}(1)|^2 + \sum_{|k| \geq k_0} |U_0^k|^2 |\widetilde{y}_{k,x}(1)|^2 \right). \end{aligned} \tag{2.87}$$

On the other hand, using (2.56)-(2.57) and (2.85) we get

$$\sum_{|k| \geq k_0} |U_0^k|^2 |\widetilde{y}_{k,x}(1)|^2 \sim \sum_{|k| \geq k_0} \frac{|U_0^k|^2}{|k|^2}. \tag{2.88}$$

Therefore, we deduce from (2.87), Proposition 2.5 and (2.29) that:

$$\int_0^T \|\mathcal{B}^* U(t)\|_{\mathcal{H}}^2 dt \geq c_T \left(\sum_{j \in J_0} |U_0^j|^2 |\widetilde{y}_{j,x}(1)|^2 + \sum_{|k| \geq k_0} |U_0^k|^2 |\frac{1}{|k|^2} \right) \sim c_T \|U_0\|^2_{D(\mathcal{A}_0)'}.$$

The proof of lemma is completed. □

Next, we introduce the transfer function H:

$$H : \mathbb{C}_+ = \{\lambda \in \mathbb{C}; \Re(\lambda) > 0\} \longrightarrow \mathcal{L}(\mathbb{C}) : \lambda \longrightarrow H(\lambda) = -\alpha \mathcal{B}^* (\lambda + \mathcal{A}_0)^{-1} \mathcal{B}. \tag{2.89}$$

Let $\omega > 0$, we define the set $C_\omega = \{\lambda \in \mathbb{C}; \Re(\lambda) = \omega\}$.

Lemma 2.9. *(Boundedness of H on C_ω)*
The transfer function H defined in (2.89) is bounded on C_ω.

Proof: First, since $\mathcal{A}_0$ generate a C_0-semigroup of contractions, we deduce (see Corollary I.3.6 of [15]):

$$\exists c_\omega > 0, \text{ such that } \quad \|(\lambda + \mathcal{A}_0)^{-1}\|_{\mathcal{H}} \leq c_\omega, \quad \forall \lambda \in C_\omega.$$

Next, combining this estimate with the boundedness of the operators $\mathcal{B}$ and $\mathcal{B}^*$, we deduce the boundedness of the function $\mathcal{H}$ on C_ω. □

Proof of the Theorem 2.4. The polynomial energy estimate (2.20) is obtained by application of Theorem 2.4 in [2] on the first order problem with $Y_1 = D(\mathcal{A}_0)$, $X_1 = (D(\mathcal{A}_0))'$ and $\theta = \dfrac{1}{2}$.

2.3. Optimal polynomial decay rate

The aim of this subsection is to prove the following optimality result.

Theorem 2.10. *(Optimal decay rate)*
The energy decay rate (2.20) is optimal in the sense that for any $\epsilon > 0$, we can not expect the decay rate $\dfrac{1}{t^{1+\epsilon}}$ for all initial data $U_0 \in D(\mathcal{A}_0)$.

□

To prove this theorem, we need the asymptotic behavior of the eigenvalues of the operator $\mathcal{A}_\alpha$. Let $\lambda \neq \alpha$ be an eigenvalue of $\mathcal{A}_\alpha$ and $U = (y, z, \eta)$ be an associated eigenfunction, then we obtain $\mathcal{A}_\alpha U = \lambda U$. Equivalently, we have the following system:

$$\begin{cases} y_{xxxx} - \gamma\lambda^2 y_{xx} + \lambda^2 y &= 0, \\ y(0) = y_x(0) &= 0, \\ y_{xxx}(1) - \gamma\lambda^2 y_x(1) &= 0, \\ y_{xx}(1) + \dfrac{\lambda}{\lambda - \alpha} y_x(1) &= 0. \end{cases} \tag{2.90}$$

The general solution of the first equation of (2.90) is given by

$$y(x) = \sum_{i=1}^{4} \widetilde{c}_i e^{R_i(\lambda)x}, \tag{2.91}$$

where

$$R_1(\lambda) = \sqrt{\frac{\gamma\lambda^2 - \lambda\sqrt{\gamma^2\lambda^2 - 4}}{2}},\ R_2(\lambda) = -R_1(\lambda), \tag{2.92}$$
$$R_3(\lambda) = \sqrt{\frac{\gamma\lambda^2 + \lambda\sqrt{\gamma^2\lambda^2 - 4}}{2}},\ R_4(\lambda) = -R_3(\lambda).$$

Here and below, for simplicity we denote $R_i(\lambda)$ by R_i. Thus the boundary conditions in (2.90) may be written as the following system:

$$N(\lambda)\widetilde{C}(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ R_1 & R_2 & R_3 & R_4 \\ \widetilde{g}_1(\lambda) & \widetilde{g}_2(\lambda) & \widetilde{g}_3(\lambda) & \widetilde{g}_4(\lambda) \\ \widetilde{h}_1(\lambda) & \widetilde{h}_2(\lambda) & \widetilde{h}_3(\lambda) & \widetilde{h}_4(\lambda) \end{pmatrix} \begin{pmatrix} \widetilde{c}_1 \\ \widetilde{c}_2 \\ \widetilde{c}_3 \\ \widetilde{c}_4 \end{pmatrix} = 0, \tag{2.93}$$

where we have set $\widetilde{g}_i(\lambda) = R_i\left(R_i^2 - \gamma\lambda^2\right)e^{R_i}$ and $\widetilde{h}_i(\lambda) = R_i\left(R_i + \dfrac{\lambda}{\lambda - \alpha}\right)e^{R_i}$, $i = 1, .., 4$. Since $\mathcal{A}_\alpha$ is closed with a compact resolvent, its spectrum consists entirely of isolated eigenvalues with finite multiplicities. Further as the coefficients of $\mathcal{A}_\alpha$ are real, the eigenvalues appear by conjugate pairs.

Proposition 2.11. *There exists a positive constant c such that any eigenvalue λ of $\mathcal{A}_\alpha$ satisfies*

$$0 < \Re(\lambda) \leq c.$$

Proof: Obviously, we already know that the real part of any eigenvalue of $\mathcal{A}_\alpha$ is positive, so we only have to prove that it is upper bounded.
Let $\lambda \neq \alpha$ be an eigenvalue of $\mathcal{A}_\alpha$ and $U = (y, -\lambda y, y_x(1))$ an associated eigenvector such that $\|U\|_{\mathcal{H}} = 1$. Multiplying the first equation of the system (2.90) by $\overline{y}$ and integrating by parts yields

$$\|y\|_W^2 + \lambda^2\|y\|_V^2 + \frac{\lambda}{\lambda - \alpha}|y_x(1)|^2 = 0. \tag{2.94}$$

Next, set $\lambda = u + iv$, $u \in \mathbb{R}_+^*$ and $v \in \mathbb{R}$. A straightforward computation gives

$$\frac{\lambda}{\lambda - \alpha} = \frac{u(u-\alpha) + v^2}{(u-\alpha)^2 + v^2} + i\frac{\alpha v}{(u-\alpha)^2 + v^2}, \tag{2.95}$$

then the imaginary part of the equation (2.94) gives

$$\left(2u\|y\|_V^2 - \frac{\alpha}{(u-\alpha)^2 + v^2}|y_x(1)|^2\right)v = 0. \tag{2.96}$$

Assume that $v \neq 0$ then

$$|\lambda|^2\|y\|_V^2 = (u^2+v^2)\|y\|_V^2 = \frac{\alpha}{2u}\frac{u^2+v^2}{(u-\alpha)^2+v^2}|y_x(1)|^2.$$

If $u = \Re(\lambda)$ is not bounded and since $|y_x(1)|^2 \leq \|U\|_{\mathcal{H}}^2 = 1$, it follows from the previous identity that for u large

$$|\lambda|^2\|y\|_V^2 = O(\frac{1}{u}).$$

Consequently (2.94) implies

$$\|y\|_W^2 + |y_x(1)|^2 = O(\frac{1}{u}),$$

then

$$\|U\|_{\mathcal{H}}^2 = \|y\|_W^2 + |\lambda|^2\|y\|_V^2 + |y_x(1)|^2 = O(\frac{1}{u}),$$

which is not possible. Therefore, for u large enough, we deduce from (2.96) that $\Im(\lambda) = v = 0$. Finally, taking the real part of the equation (2.92) with $v = 0$, we obtain

$$\|y\|_W^2 + u^2\|y\|_V^2 + \frac{u}{u-\alpha}|y_x(1)|^2 = 0.$$

Hence the contradiction with $\|U\|_{\mathcal{H}}^2 = 1$ if u is large enough. □

In the following proposition we study the spectrum of $\mathcal{A}_\alpha$:

Proposition 2.12. *(Spectrum of $\mathcal{A}_\alpha$)*
There exists $k_1 \in \mathbb{N}^*$ sufficiently large such that the spectrum $\sigma(\mathcal{A}_\alpha)$ of $\mathcal{A}_\alpha$ is given by:

$$\sigma(\mathcal{A}_\alpha) = \widetilde{\sigma}_0 \cup \widetilde{\sigma}_1, \tag{2.97}$$

where

$$\widetilde{\sigma_0} = \{\kappa_j\}_{j\in J}, \quad \widetilde{\sigma}_1 = \{\lambda_k\}_{\substack{k\in\mathbb{Z}\\ |k|\geq k_0}}, \quad \widetilde{\sigma_0}\cap\widetilde{\sigma_1} = \emptyset \tag{2.98}$$

and J is a finite set. Moreover, λ_k is simple and satisfies the following asymptotic behavior

$$\lambda_k = i\left(\frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} + \frac{D}{k} + \frac{E}{k^2}\right) + \frac{\alpha}{\pi^2k^2} + o(\frac{1}{k^2}), \tag{2.99}$$

where

$$D = \frac{2\gamma - 1 - 2\sqrt{\gamma}\tanh(\gamma^{-\frac{1}{2}})}{2\gamma^{\frac{3}{2}}\pi} \tag{2.100}$$

and

$$E = \frac{2(-1)^k}{\gamma^{\frac{3}{2}}\cosh(\gamma^{-\frac{1}{2}})\pi^2} + \frac{4\gamma - 2 - \sqrt{\gamma}\tanh(\gamma^{-\frac{1}{2}})}{2\gamma^{\frac{3}{2}}\pi}. \tag{2.101}$$

Proof: The proof is divided into three steps. Step 1 furnishes an asymptotic development of the characteristic equation for large λ. Step 2 uses Rouché's theorem to localize high frequency eigenvalues. In step 3, we perform a limited development stopped when a non zero real part appear.
Step 1. First, We start by the expansion of R_1 and R_3 when $|\lambda| \to \infty$

$$R_1 = \frac{1}{\sqrt{\gamma}} + \frac{1}{2\gamma^{\frac{5}{2}}\lambda^2} + O(\frac{1}{\lambda^4}) \tag{2.102}$$

and

$$R_3 = \lambda\sqrt{\gamma} - \frac{1}{2\lambda\gamma^{\frac{3}{2}}} + O(\frac{1}{\lambda^3}). \tag{2.103}$$

Next, using (2.102) and (2.103), we find the asymptotic behavior of

$$\widetilde{g}_1(\lambda) = \left(-\sqrt{\gamma}\lambda^2 - \frac{1}{2\gamma^2} + \frac{1}{2\gamma^{\frac{3}{2}}}\right) e^{\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda}), \tag{2.104}$$

$$\widetilde{g}_2(\lambda) = \left(\sqrt{\gamma}\lambda^2 - \frac{1}{2\gamma^2} - \frac{1}{2\gamma^{\frac{3}{2}}}\right) e^{-\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda}), \tag{2.105}$$

$$\widetilde{g}_3(\lambda) = \left(-\frac{\lambda}{\sqrt{\gamma}} + \frac{1}{2\gamma^2}\right) e^{\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda}) \tag{2.106}$$

and

$$\widetilde{g}_4(\lambda) = \left(\frac{\lambda}{\sqrt{\gamma}} + \frac{1}{2\gamma^2}\right) e^{-\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda}). \tag{2.107}$$

Similarly, we get

$$\widetilde{h}_1(\lambda) = \frac{1}{\sqrt{\gamma}}\left(1 + \frac{1}{\sqrt{\gamma}} + \frac{\alpha}{\lambda}\right) e^{\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda^2}), \tag{2.108}$$

$$\widetilde{h}_2(\lambda) = \frac{1}{\sqrt{\gamma}}\left(-1 + \frac{1}{\sqrt{\gamma}} - \frac{\alpha}{\lambda}\right) e^{-\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda^2}), \tag{2.109}$$

$$\widetilde{h}_3(\lambda) = \left(\gamma\lambda^2 + (-\frac{1}{2\sqrt{\gamma}} + \sqrt{\gamma})\lambda + \frac{1 - 12\gamma + 8\gamma^2\sqrt{\gamma}\alpha}{8\gamma^2}\right) e^{\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda}) \tag{2.110}$$

and

$$\widetilde{h}_4(\lambda) = \left(\gamma\lambda^2 + (\frac{1}{2\sqrt{\gamma}} - \sqrt{\gamma})\lambda + \frac{1 - 12\gamma - 8\gamma^2\sqrt{\gamma}\alpha}{8\gamma^2}\right) e^{-\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda}). \tag{2.111}$$

Combining (2.102)-(2.111) and (2.93), we can write the system (2.93) as follow:

$$N(\lambda)\widetilde{C}(\lambda) = 0,$$

where $N(\lambda)$ is given by

$$N(\lambda)=\begin{pmatrix} 1 & 1 & 1 & 1 \\ \widetilde{P}_1+O(\frac{1}{\lambda^2}) & \widetilde{P}_2+O(\frac{1}{\lambda^2}) & \widetilde{P}_3+O(\frac{1}{\lambda^4}) & \widetilde{P}_4+O(\frac{1}{\lambda^4}) \\ e^{\frac{1}{\sqrt{\gamma}}}\widetilde{P}_5+O(\frac{1}{\lambda}) & e^{-\frac{1}{\sqrt{\gamma}}}\widetilde{P}_6+O(\frac{1}{\lambda}) & e^{\sqrt{\gamma}\lambda}\widetilde{P}_7+O(\frac{1}{\lambda^2}) & e^{-\sqrt{\gamma}\lambda}\widetilde{P}_8+O(\frac{1}{\lambda^2}) \\ e^{\frac{1}{\sqrt{\gamma}}}\widetilde{P}_9+O(\frac{1}{\lambda}) & e^{-\frac{1}{\sqrt{\gamma}}}\widetilde{P}_{10}+O(\frac{1}{\lambda}) & e^{\sqrt{\gamma}\lambda}\widetilde{P}_{11}+O(\frac{1}{\lambda^2}) & e^{-\sqrt{\gamma}\lambda}\widetilde{P}_{12}+O(\frac{1}{\lambda^2}) \end{pmatrix},$$

where

$$\widetilde{P}_1=\frac{1}{\sqrt{\gamma}}+\frac{1}{2\gamma^{\frac{5}{2}}\lambda^2},\quad \widetilde{P}_2=-\frac{1}{\sqrt{\gamma}}-\frac{1}{2\gamma^{\frac{5}{2}}\lambda^2},\quad \widetilde{P}_3=\lambda\sqrt{\gamma}-\frac{1}{2\lambda\gamma^{\frac{3}{2}}},$$

$$\widetilde{P}_4=-\lambda\sqrt{\gamma}+\frac{1}{2\lambda\gamma^{\frac{3}{2}}},\quad \widetilde{P}_5=-\sqrt{\gamma}\lambda^2-\frac{1}{2\gamma^2}+\frac{1}{2\gamma^{\frac{3}{2}}}\quad \widetilde{P}_6=\sqrt{\gamma}\lambda^2-\frac{1}{2\gamma^2}-\frac{1}{2\gamma^{\frac{3}{2}}},$$

$$\widetilde{P}_7=-\frac{\lambda}{\sqrt{\gamma}}+\frac{1}{2\gamma^2},\quad \widetilde{P}_8=\frac{\lambda}{\sqrt{\gamma}}+\frac{1}{2\gamma^2},\quad \widetilde{P}_9=\frac{1}{\sqrt{\gamma}}\left(1+\frac{1}{\sqrt{\gamma}}+\frac{\alpha}{\lambda}\right),$$

$$\widetilde{P}_{10}=\frac{1}{\sqrt{\gamma}}\left(-1+\frac{1}{\sqrt{\gamma}}-\frac{\alpha}{\lambda}\right)$$

$$\widetilde{P}_{11}=\gamma\lambda^2+\left(-\frac{1}{2\sqrt{\gamma}}+\sqrt{\gamma}\right)\lambda+\frac{1}{8\gamma^2}-\frac{3}{2\gamma}+\sqrt{\gamma}\alpha$$

and

$$\widetilde{P}_{12}=\gamma\lambda^2+\left(\frac{1}{2\sqrt{\gamma}}-\sqrt{\gamma}\right)\lambda+\frac{1}{8\gamma^2}-\frac{3}{2\gamma}-\sqrt{\gamma}\alpha.$$

Then, after some computations, we find the following asymptotic development of $\widetilde{f}(\lambda)=\det N(\lambda)$

$$\widetilde{f}(\lambda)=\lambda^5\widetilde{f}_0(\lambda)+\lambda^4\widetilde{f}_1(\lambda)+\lambda^3\widetilde{f}_2(\lambda)+O(\lambda^2), \tag{2.112}$$

where

$$\widetilde{f}_0(\lambda)=-4\gamma^2\cosh(\frac{1}{\sqrt{\gamma}})\cosh(\sqrt{\gamma}\lambda), \tag{2.113}$$

$$\widetilde{f}_1(\lambda)=l_1(\gamma)\sinh(\sqrt{\gamma}\lambda), \tag{2.114}$$

where

$$l_1(\gamma)=2\sqrt{\gamma}\left((1-2\gamma)\cosh(\frac{1}{\sqrt{\gamma}})+2\sqrt{\gamma}\sinh(\frac{1}{\sqrt{\gamma}})\right) \tag{2.115}$$

and

$$\widetilde{f}_2(\lambda) = 8 - 4\alpha\gamma^{\frac{3}{2}} \cosh(\frac{1}{\sqrt{\gamma}}) \cosh(\sqrt{\gamma}\lambda) + l_2(\gamma) \sinh(\sqrt{\gamma}\lambda), \tag{2.116}$$

where

$$l_2(\gamma) = (10 - \frac{1}{2\gamma}) \cosh(\frac{1}{\sqrt{\gamma}}) + (4\sqrt{\gamma} - \frac{4}{\sqrt{\gamma}}) \sinh(\frac{1}{\sqrt{\gamma}}). \tag{2.117}$$

As the real part of λ is bounded, then the functions $\widetilde{f}_i$ are bounded for $i \in \{0, 1, 2\}$. For convenience we set

$$\widetilde{S}(\lambda) = \frac{\widetilde{f}(\lambda)}{\lambda^5} = \widetilde{f}_0(\lambda) + \frac{\widetilde{f}_1(\lambda)}{\lambda} + \frac{\widetilde{f}_2(\lambda)}{\lambda^2} + O(\frac{1}{\lambda^3}). \tag{2.118}$$

Step 2. We look at the roots of $\widetilde{S}$. It is easy to see that the roots of $\widetilde{f}_0$ are simple and given by:

$$z_k = i\alpha_k, \quad k \in \mathbb{Z} \tag{2.119}$$

where α_k is defined in (2.30). Then, with the help of Rouché's theorem there exists k_1 large enough such that for all $|k| \geq k_1$ the large eigenvalues of $\sigma(\mathcal{A}_\alpha)$ (denoted by λ_k) are simple and close to z_k. More precisely, there exists $k_1 \in \mathbb{N}^*$ large enough, such that the splitting of $\sigma(\mathcal{A}_\alpha)$ given in (2.97)-(2.98) holds and we have

$$\lambda_k = i\alpha_k + o(1), \quad |k| \to \infty. \tag{2.120}$$

Equivalently, we can write

$$\lambda_k = i\alpha_k + \epsilon_k, \quad \lim_{|k|\to\infty} \epsilon_k = 0. \tag{2.121}$$

Step 3. Determination of ϵ_k. First, using (2.118) and the identities (2.113)-(2.116) we have

$$\begin{aligned} 0 = \widetilde{S}(\lambda_k) = &-4\gamma^2 \cosh(\frac{1}{\sqrt{\gamma}}) \cosh(\sqrt{\gamma}\lambda_k) + \frac{l_1(\gamma) \sinh(\sqrt{\gamma}\lambda_k)}{\lambda_k} + \frac{8}{\lambda_k^2} \\ &- \frac{4\alpha\gamma^{\frac{3}{2}} \cosh(\frac{1}{\sqrt{\gamma}}) \cosh(\sqrt{\gamma}\lambda_k)}{\lambda_k^2} + \frac{l_2(\gamma) \sinh(\sqrt{\gamma}\lambda_k)}{\lambda_k^2} + O(\frac{1}{\lambda_k^3}). \end{aligned} \tag{2.122}$$

On the other hand, using (2.121) we find

$$\cosh(\sqrt{\gamma}\lambda_k) = i(-1)^k \sqrt{\gamma}\epsilon_k + O(\epsilon_k^3) \tag{2.123}$$

and

$$\sinh(\sqrt{\gamma}\lambda_k) = i(-1)^k + O(\epsilon_k^2). \tag{2.124}$$

Then, substituting (2.123) into (2.113) and (2.124) into (2.114) yields

$$\widetilde{f_0}(\lambda_k) = -4i\gamma^{\frac{5}{2}}(-1)^k \cosh(\frac{1}{\sqrt{\gamma}})\epsilon_k + O(\epsilon_k^3) \tag{2.125}$$

and

$$\widetilde{f_1}(\lambda_k) = i(-1)^k l_1(\gamma) + O(\epsilon_k^2). \tag{2.126}$$

Similarly, we get

$$\widetilde{f_2}(\lambda_k) = 8 - 4i\alpha\gamma^{\frac{3}{2}} \cosh(\frac{1}{\sqrt{\gamma}}) + \sqrt{\gamma}i(-1)^k l_2(\gamma)\epsilon_k + O(\epsilon_k^3). \tag{2.127}$$

Now, using (2.121), (2.126) and (2.127) we get

$$\frac{\widetilde{f_1}(\lambda_k)}{\lambda_k} = \frac{(-1)^k l_1(\gamma)}{\alpha_k} + O(\frac{\epsilon_k}{k}) \tag{2.128}$$

and

$$\frac{\widetilde{f_2}(\lambda_k)}{\lambda_k^2} = -\frac{8}{\alpha_k^2} + \frac{4\alpha i\gamma^{\frac{3}{2}} \cosh(\frac{1}{\sqrt{\gamma}})(-1)^k}{\alpha_k^2} + O(\frac{\epsilon_k}{k}). \tag{2.129}$$

Next, substituting (2.125), (2.128) and (2.129) into (2.122) yields

$$\begin{aligned} 0 \quad = \quad & -4i\gamma^{\frac{5}{2}}(-1)^k \cosh(\frac{1}{\sqrt{\gamma}})\epsilon_k + \frac{(-1)^k l_1(\gamma)}{\alpha_k} - \frac{8}{\alpha_k^2} \\ & + \frac{4\alpha i\gamma^{\frac{3}{2}} \cosh(\frac{1}{\sqrt{\gamma}})(-1)^k}{\alpha_k^2} + O(\frac{\epsilon_k}{k}). \end{aligned} \tag{2.130}$$

Therefore

$$\epsilon_k = -\frac{\frac{8}{\alpha_k^2} - \frac{(-1)^k l_1(\gamma)}{\alpha_k}}{4i\gamma^{\frac{5}{2}}(-1)^k \cosh(\frac{1}{\sqrt{\gamma}})} + \frac{\alpha}{\gamma\alpha_k^2} + O(\frac{\epsilon_k}{k}). \tag{2.131}$$

Moreover, substituting (2.30) and (2.115) into (2.131) then a long computation gives

$$\epsilon_k = i\left(\frac{D}{k} + \frac{E}{k^2}\right) + \frac{\alpha}{\pi^2 k^2} + o(\frac{1}{k^2}) \tag{2.132}$$

where D and E are given in (2.100)-(2.101). Finally, substituting (2.132) into (2.121), we directly get (2.99). □

Numerical validation. The asymptotic behavior of λ_k in (2.99) can be numerically validated. For instance, with $\alpha = 1$ and $\gamma = 2$ then from (2.99) we have

$$\lim_{k\to+\infty} k^2\Re(\lambda_k) = \frac{1}{\pi^2}(\approx 0.101321).$$

The table below confirms this behavior.

k	100	150	200	250	300	350	400	450	500
$k^2\Re(\lambda_k)$	0.100312	0.100647	0.100816	0.100917	0.100984	0.101032	0.101068	0.101096	0.101119

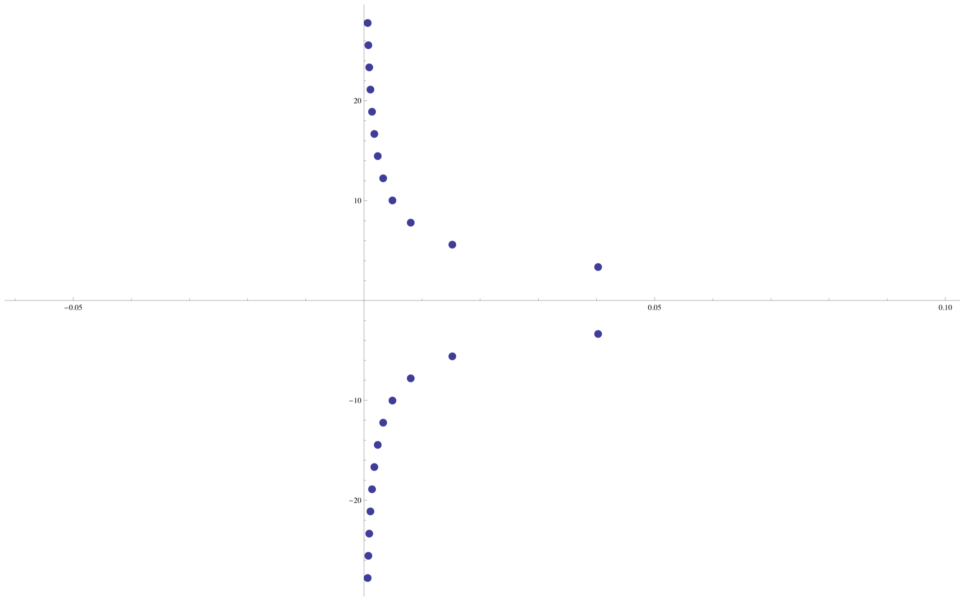

Figure 1: Eigenvalues of $\mathcal{A}_1$ with $\gamma = 2$

In addition, figure 1 represents some eigenvalues in this case. Note that for a scale reason three eigenvalues (with a small imaginary part) do not appear in the previous figure. Their approximated value are

$$0.13825 \pm i1.30223, \qquad \text{and} \quad 0.54640.$$

Proof of Theorem 2.10. Let $\epsilon > 0$ and set $l = \dfrac{\epsilon}{1+\epsilon}$. First, for $|k| \geq k_1$, let λ_k be an eigenvalue of the operator $\mathcal{A}_\alpha$ and $U_k \in D(\mathcal{A}_0)$ the associated normalized eigenfunction. Moreover, we introduce the following sequence

$$\beta_k = -\Im(\lambda_k), \quad |k| \geq k_1.$$

Next, using (2.99), we have

$$(iI\beta_k + \mathcal{A}_\alpha)U_k = (iI\beta_k + \lambda_k)U_k \left(\frac{\alpha}{\pi^2 k^2} + o(\frac{1}{k^2})\right) U_k, \quad \forall |k| \geq k_0.$$

Therefore

$$\beta_k^{2-2l} \|(i\beta_k I + \mathcal{A}_\alpha)U_k\|_{\mathcal{H}} \sim \frac{\alpha}{\pi^2} \times \frac{1}{k^{\frac{2\epsilon}{1+\epsilon}}}, \quad \forall |k| \geq k_0.$$

Thus, we deduce

$$\lim_{k\to+\infty} \beta_k^{2-2l} \|(i\beta_k I + \mathcal{A}_\alpha)U_k\|_{\mathcal{H}} = 0.$$

Finally, thanks to Theorem 2.4 in [5], we deduce that the trajectory $e^{t\mathcal{A}_\alpha}U_0$ decays slower that $\dfrac{1}{t^{\frac{1}{2-2l}}}$ on the time $t\to+\infty$. Then we cannot expect the energy decay rate $\dfrac{1}{t^{1+\epsilon}}$. □

3. Rayleigh beam equation with only one dynamical boundary control force

In this section, we consider the Rayleigh beam equation with only one dynamical boundary control force:

$$\begin{cases} y_{tt} - \gamma y_{xxtt} + y_{xxxx} & = 0, \quad 0 < x < 1, \quad t > 0, \\ y(0,t) = y_x(0,t) & = 0, \quad t > 0, \\ y_{xx}(1,t) & = 0, \quad t > 0, \\ y_{xxx}(1,t) - \gamma y_{xtt}(1,t) - \xi(t) & = 0, \quad t > 0, \\ \xi_t(t) - y_t(1,t) + \beta\xi(t) & = 0, \quad t > 0. \end{cases} \tag{3.1}$$

First, let y and ξ be smooth solutions of system (3.1). We define its associated energy by:

$$E(t) = \frac{1}{2}\left(\int_0^1 (|y_t|^2 + \gamma|y_{xt}|^2 + |y_{xx}|^2)dx + |\xi(t)|^2\right). \tag{3.2}$$

A direct computation gives

$$\frac{d}{dt}E(t) = -\beta|\xi(t)|^2 \leq 0,$$

Then the system (3.1) is dissipative in the sense that its energy $E(t)$ is a nonincreasing function of the time variable t. Let $\overline{\Phi} \in W$. Integrating by parts, we transform (3.1) into a variational equation:

$$\int_0^1 (y_{tt}\overline{\Phi} + \gamma y_{xtt}\overline{\Phi_x})dx + \int_0^1 y_{xx}\overline{\Phi_{xx}}dx + \xi\overline{\Phi(1)} = 0. \tag{3.3}$$

According, we define the continuous operator $\widetilde{B}$ as follows:

$$\widetilde{B} \in \mathcal{L}(\mathbb{C}, V'), \quad <\widetilde{B}\xi, \Phi>_{V'\times V} = \xi\overline{\Phi(1)}, \quad \forall\xi\in\mathbb{C}, \forall\Phi\in V. \tag{3.4}$$

Assume that $Ay \in V^{'}$, then we can formulate the variational equation (3.4) as:

$$y_{tt} + C^{-1}Ay + C^{-1}\widetilde{B}\xi = 0, \tag{3.5}$$

where the operators A and C are defined in (2.9) and (2.11). Now define the energy space $\mathcal{H} = W \times V \times \mathbb{C}$ endowed with the usual inner product and where W and V are given in (2.4) and (2.5). Next, we introduce the linear unbounded operator $\widetilde{\mathcal{A}}_0$ and the linear bounded operator $\widetilde{\mathcal{B}}$ as follows:

$$D(\widetilde{\mathcal{A}}_0) = \left\{(y,z,\xi)\in\mathcal{H};\ z\in W \text{ and } Ay\in V'\right\}, \tag{3.6}$$

$$\widetilde{\mathcal{A}}_0 U = \begin{pmatrix} -z \\ C^{-1}Ay + C^{-1}\widetilde{B}\xi \\ -z(1) \end{pmatrix}, \qquad U = (y,z,\xi)\in D(\widetilde{\mathcal{A}}_0), \tag{3.7}$$

and

$$\widetilde{\mathcal{B}}U = \begin{pmatrix} 0 \\ 0 \\ \xi \end{pmatrix}, \qquad U = (y, z, \xi) \in \mathcal{H}. \tag{3.8}$$

Then, denoting by $U = (y,\, y_t,\, \xi)$ the state of the system (3.1) and define $\widetilde{\mathcal{A}}_\beta = \widetilde{\mathcal{A}}_0 + \beta \widetilde{\mathcal{B}}$ with $D(\widetilde{\mathcal{A}}_\beta) = D(\widetilde{\mathcal{A}}_0)$, we can formulate the system into an evolution equation

$$\begin{cases} U_t(t) + \widetilde{\mathcal{A}}_\beta U(t) & = & 0, \\ \quad U(0) & = & U_0 \in \mathcal{H}. \end{cases} \tag{3.9}$$

It is easy to prove that $-\widetilde{\mathcal{A}}_\beta$ is a maximal dissipative operator in the energy space $\mathcal{H}$, therefore it generates a C_0-semigroup $(e^{-t\widetilde{\mathcal{A}}_\beta})_{t\geq 0}$ of contractions in the energy space $\mathcal{H}$ using Hille-Yosida's theorem (see Pazy [15]). In addition, it is easy to show that an element $U = (y, z, \xi) \in D(\mathcal{A}_\beta)$ if and only if $y \in H^3(0,1) \cap W$, $z \in W$ and $y_{xx}(1) = 0$. In particular, the resolvent $(I + \mathcal{A}_\beta)^{-1}$ of $-\mathcal{A}_\beta$ is compact in the energy space $\mathcal{H}$. Consequently, the spectrum of $\mathcal{A}_\beta$ (respectively $\mathcal{A}_0$) consists entirely of isolated eigenvalues with finite multiplicities. Moreover, since the coefficients of $\mathcal{A}_\beta$ (respectively $\mathcal{A}_0$) are real, their eigenvalues appear by conjugate pairs.

Theorem 4.2 of [16] shows that the semi-group of contractions $(e^{-t\mathcal{A}})_{t\geq 0}$ is strongly asymptotically stable in the energy space $\mathcal{H}$, *i.e.* for any $u_0 \in \mathcal{H}$, we have $\lim\limits_{t\to+\infty} \|e^{-t\mathcal{A}} u_0\|^2_{\mathcal{H}} = 0$ if $\gamma \geq \gamma_0$ where $\sqrt{\gamma_0}\sinh^{-1}(\sqrt{\gamma_0}\pi)$. Using a numerical program we find

$$\gamma_0 \simeq 0.45001246517627713.$$

Moreover, from Theorem 4.3 of [16] there exists a infinite numbers of $0 < \gamma < \gamma_0$ such that the operator $\mathcal{A}_\beta$ has eigenvalues on the imaginary axis and therefore for which problem (3.1) is not stable. Further, we know that the Rayleigh beam is not uniformly exponentially stable neither with one boundary direct control force (see [16]) nor with two dynamical boundary control (see [21]). Then, we look for a optimal polynomial energy decay rate for smooth initial data.

3.1. Analysis of eigenvalues and eigenvectors of the operator $\widetilde{\mathcal{A}}_\beta$ for $\beta \geq 0$

In this subsection, we study the eigenvalues and the eigenvectors of the operator $\widetilde{\mathcal{A}}_\beta$ for $\beta \geq 0$. First, let $\lambda \neq \beta$ be an eigenvalue of the operator $\widetilde{\mathcal{A}}_\beta$ and $U = (y,\; z,\; \xi)$ be an associated eigenfunction, then we have $\widetilde{\mathcal{A}}_\beta U = \lambda U$. Equivalently, λ and y verify the following system:

$$\begin{cases} y_{xxxx} - \gamma\lambda^2 y_{xx} + \lambda^2 y & = & 0, \\ y_{xxx}(1) - \gamma\lambda^2 y_x(1) - \dfrac{\lambda}{\lambda - \beta} y(1) & = & 0, \\ y(0) = y_x(0) = y_{xx}(1) & = & 0. \end{cases} \tag{3.10}$$

The general solution of the system (3.10) is

$$y = \sum_{i=1}^{4} c_i(\lambda) e^{R_i(\lambda)x}, \tag{3.11}$$

where $R_i(\lambda)$, $i = 1, .., 4$ are given in (2.92). Next, using the boundary conditions, we may write the system (3.10) as follows:

$$M_\beta(\lambda) \cdot C(\lambda) = 0, \tag{3.12}$$

where

$$M_\beta(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ R_1(\lambda) & R_2(\lambda) & R_3(\lambda) & R_4(\lambda) \\ R_1^2(\lambda)e^{R_1(\lambda)} & R_2^2(\lambda)e^{R_2(\lambda)} & R_3^2(\lambda)e^{R_3(\lambda)} & R_4^2(\lambda)e^{R_4(\lambda)} \\ T_{1,\beta}(\lambda) & T_{2,\beta}(\lambda) & T_{3,\beta}(\lambda) & T_{4,\beta}(\lambda) \end{pmatrix}, \tag{3.13}$$

$$C(\lambda) = \begin{pmatrix} c_1(\lambda) \\ c_2(\lambda) \\ c_3(\lambda) \\ c_4(\lambda) \end{pmatrix}$$

and where

$$T_{i,\beta}(\lambda) = \left(R_i(\lambda)^3 - \gamma\lambda^2 R_i(\lambda) - \frac{\lambda}{\lambda-\beta}\right) e^{R_i(\lambda)}.$$

Remark 3.1. *First, like we did in Proposition 2.11, we find that the real part of any eigenvalue λ of $\widetilde{\mathcal{A}}_\beta$ is bounded,* i.e.

$$\exists c > 0, \quad \forall \lambda \in \sigma(\widetilde{\mathcal{A}}_\beta), \quad 0 < \Re(\lambda) \leq c.$$

Next, let λ^0 be an eigenvalue of $\widetilde{\mathcal{A}}_0$ and $U^0 = (y^0, z^0, \xi^0) \in D(\widetilde{\mathcal{A}}_0)$ an associated eigenvector. Then, like we did in Proposition 2.5, we can easily prove that λ^0 is simple and $\xi^0 \neq 0$.

$\square$

Next, we study the asymptotic behavior of the eigenvalues of the operators $\widetilde{\mathcal{A}}_\beta$ in the following proposition:

Proposition 3.2. *(Spectrum of $\widetilde{\mathcal{A}}_\beta$)*
Let $\beta \geq 0$. Then there exists $k_\beta \in \mathbb{N}^*$ sufficiently large such that the spectrum $\sigma(\mathcal{A}_\beta)$ of $\mathcal{A}_\beta$ is given by

$$\sigma(\mathcal{A}_\beta) = \sigma_{\beta,0} \cup \sigma_{\beta,1}, \tag{3.14}$$

where

$$\sigma_{\beta,0} = \{\kappa_{\beta,j}\}_{j\in J_\beta}, \quad \sigma_{\beta,1} = \{\lambda_{\beta,k}\}_{\substack{k\in\mathbb{Z} \\ |k|\geq k_\beta}}, \quad \sigma_{\beta,0} \cap \sigma_{\beta,1} = \emptyset, \tag{3.15}$$

where J_β is a finite set and $\lambda_{\beta,k}$ is simple and satisfies the following asymptotic behavior:

$$\begin{aligned} \lambda_{\beta,k} \quad = \quad & i\left(\alpha_k - \frac{\left(\frac{1}{2\sqrt{\gamma}} + \tanh\left(\frac{1}{\sqrt{\gamma}}\right)\right)}{\gamma^{\frac{3}{2}}\alpha_k} + \frac{2(-1)^k}{\gamma^{\frac{5}{2}}\cosh\left(\frac{1}{\sqrt{\gamma}}\right)\alpha_k^2} + \frac{E}{\alpha_k^3} + \frac{F}{\alpha_k^4}\right) \\ & + \frac{\beta}{\pi^4\cosh\left(\frac{1}{\sqrt{\gamma}}\right)} \times \frac{1}{k^4} + o\left(\frac{1}{k^4}\right), \end{aligned} \tag{3.16}$$

with

$$\alpha_k = \frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}, \tag{3.17}$$

$$\begin{aligned} E &= \frac{1}{3\gamma^{\frac{7}{2}}}\tanh(\frac{1}{\sqrt{\gamma}})^3 - \frac{1}{\gamma^2}\left(1+\frac{1}{\gamma}+\frac{1}{2\gamma^2}\right)\tanh(\frac{1}{\sqrt{\gamma}})^2 \\ &+\frac{1}{\gamma^{\frac{7}{2}}}\tanh(\frac{1}{\sqrt{\gamma}}) + \frac{1}{\gamma^2} + \frac{1}{\gamma^4}, \end{aligned} \tag{3.18}$$

$$\begin{aligned} F &= \frac{(-1)^k}{\gamma^3\cosh(\frac{1}{\sqrt{\gamma}})}\left[(2+\frac{6}{\gamma}+\frac{1}{\gamma^2})\tanh(\frac{1}{\sqrt{\gamma}}) - \right. \\ &\left. \frac{1}{\gamma^{\frac{3}{2}}}(1+\tanh(\frac{1}{\sqrt{\gamma}})^2)\right]. \end{aligned} \tag{3.19}$$

Proof:

The proof uses the same strategy than the one from Proposition 2.12. For the sake of completeness, we give the details. For simplicity, we denote $R_i(\lambda)$ by R_i.

Step 1. First, using the expansions (2.102) and (2.103), we find the following asymptotic behavior:

$$R_1^2 e^{R_1} = \left(\frac{1}{\gamma} + (\frac{1}{2\gamma^{\frac{7}{2}}} + \frac{1}{\gamma^3})\frac{1}{\lambda^2}\right) e^{\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda^4}), \tag{3.20}$$

$$R_1^2 e^{R_1} = \left(\frac{1}{\gamma} + (\frac{1}{2\gamma^{\frac{7}{2}}} + \frac{1}{\gamma^3})\frac{1}{\lambda^2}\right) e^{\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda^4}), \tag{3.21}$$

$$R_3^2 e^{R_3} = \left(\gamma\lambda^2 - \frac{\lambda}{2\sqrt{\gamma}} + \frac{1}{8\gamma^2} - \frac{1}{\gamma} - (\frac{1}{8\gamma^{\frac{5}{2}}} + \frac{1}{48\gamma^{\frac{7}{2}}})\frac{1}{\lambda}\right) e^{\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda^2}) \tag{3.22}$$

and

$$R_4^2 e^{R_4} = \left(\gamma\lambda^2 + \frac{\lambda}{2\sqrt{\gamma}} + \frac{1}{8\gamma^2} - \frac{1}{\gamma} + (\frac{1}{8\gamma^{\frac{5}{2}}} + \frac{1}{48\gamma^{\frac{7}{2}}})\frac{1}{\lambda}\right) e^{-\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda^2}). \tag{3.23}$$

Similarly, we get

$$T_{\beta,1}(\lambda) = \left(-\sqrt{\gamma}\lambda^2 - \frac{1}{2\gamma^2} + \frac{1}{2\gamma^{\frac{3}{2}}} - 1 - \frac{\beta}{\lambda}\right) e^{\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda^2}), \tag{3.24}$$

$$T_{\beta,2}(\lambda) = \left(\sqrt{\gamma}\lambda^2 - \frac{1}{2\gamma^2} - \frac{1}{2\gamma^{\frac{3}{2}}} - 1 - \frac{\beta}{\lambda}\right) e^{-\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda^2}), \tag{3.25}$$

$$\begin{aligned} T_{\beta,3}(\lambda) = &\left[-\frac{\lambda}{\sqrt{\gamma}} + \frac{1}{2\gamma^2} - 1 + \left(\frac{1}{2\gamma^{\frac{3}{2}}} - \frac{1}{2\gamma^{\frac{5}{2}}} - \frac{1}{8\gamma^{\frac{7}{2}}} - \beta\right)\frac{1}{\lambda}\right. \\ &\left. + \left(-\beta^2 + \frac{\beta}{2\gamma^{\frac{3}{2}}} - \frac{1}{8\gamma^3} + \frac{7}{8\gamma^4} + \frac{1}{48\gamma^5}\right)\frac{1}{\lambda^2}\right] e^{\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda^3}) \end{aligned} \tag{3.26}$$

and

$$T_{\beta,4}(\lambda)=\left[\frac{\lambda}{\sqrt{\gamma}}+\frac{1}{2\gamma^2}-1+\left(-\frac{1}{2\gamma^{\frac{3}{2}}}+\frac{1}{2\gamma^{\frac{5}{2}}}+\frac{1}{8\gamma^{\frac{7}{2}}}-\beta\right)\frac{1}{\lambda}\right.$$
$$\left.+\left(-\beta^2-\frac{\beta}{2\gamma^{\frac{3}{2}}}-\frac{1}{8\gamma^3}+\frac{7}{8\gamma^4}+\frac{1}{48\gamma^5}\right)\frac{1}{\lambda^2}\right]e^{-\sqrt{\gamma}\lambda}+O(\frac{1}{\lambda^3}). \tag{3.27}$$

Combining (2.102)-(2.103), (3.20)-(3.27) and (3.13), we can write

$$M_\beta(\lambda)=\begin{pmatrix} 1 & 1 & 1 & 1 \\ q_1+O(\frac{1}{\lambda^4}) & q_2+O(\frac{1}{\lambda^4}) & q_3+O(\frac{1}{\lambda^2}) & q_4+O(\frac{1}{\lambda^2}) \\ q_5e^{\frac{1}{\sqrt{\gamma}}}+O(\frac{1}{\lambda^4}) & q_6e^{-\frac{1}{\sqrt{\gamma}}}+O(\frac{1}{\lambda^4}) & q_7e^{\sqrt{\gamma}\lambda}+O(\frac{1}{\lambda^2}) & q_8e^{-\sqrt{\gamma}\lambda}+O(\frac{1}{\lambda^2}) \\ q_9e^{\frac{1}{\sqrt{\gamma}}}+O(\frac{1}{\lambda^2}) & q_{10}e^{-\frac{1}{\sqrt{\gamma}}}+O(\frac{1}{\lambda^2}) & q_{11}e^{\sqrt{\gamma}\lambda}+O(\frac{1}{\lambda^3}) & q_{12}e^{-\sqrt{\gamma}\lambda}+O(\frac{1}{\lambda^3}) \end{pmatrix},$$

where

$$q_1=\frac{1}{\sqrt{\gamma}}+\frac{1}{2\gamma^{\frac{5}{2}}\lambda},\quad q_2=-\frac{1}{\sqrt{\gamma}}-\frac{1}{2\gamma^{\frac{5}{2}}\lambda^2},\quad q_3=\lambda\sqrt{\gamma}-\frac{1}{2\lambda\gamma^{\frac{3}{2}}},\quad q_4=-\lambda\sqrt{\gamma}+\frac{1}{2\lambda\gamma^{\frac{3}{2}}},$$

$$q_5=\frac{1}{\gamma}+(\frac{1}{2\gamma^{\frac{7}{2}}}+\frac{1}{\gamma^3})\frac{1}{\lambda^2},\ q_6=\frac{1}{\gamma}+(-\frac{1}{2\gamma^{\frac{7}{2}}}+\frac{1}{\gamma^3})\frac{1}{\lambda^2},$$

$q_7=\gamma\lambda^2-\dfrac{\lambda}{2\sqrt{\gamma}}+\dfrac{1}{8\gamma^2}-\dfrac{1}{\gamma}-(\dfrac{1}{8\gamma^{\frac{5}{2}}}+\dfrac{1}{48\gamma^{\frac{7}{2}}})\dfrac{1}{\lambda},\ q_8=\gamma\lambda^2+\dfrac{\lambda}{2\sqrt{\gamma}}+\dfrac{1}{8\gamma^2}-\dfrac{1}{\gamma}+(\dfrac{1}{8\gamma^{\frac{5}{2}}}+\dfrac{1}{48\gamma^{\frac{7}{2}}})\dfrac{1}{\lambda},$

$$q_9=-\sqrt{\gamma}\lambda^2-\frac{1}{2\gamma^2}+\frac{1}{2\gamma^{\frac{3}{2}}}-1-\frac{\beta}{\lambda},\quad q_{10}=\sqrt{\gamma}\lambda^2-\frac{1}{2\gamma^2}-\frac{1}{2\gamma^{\frac{3}{2}}}-1-\frac{\beta}{\lambda},$$

$$q_{11}=-\frac{\lambda}{\sqrt{\gamma}}+\frac{1}{2\gamma^2}-1+\left(\frac{1}{2\gamma^{\frac{3}{2}}}-\frac{1}{2\gamma^{\frac{5}{2}}}-\frac{1}{8\gamma^{\frac{7}{2}}}-\beta\right)\frac{1}{\lambda}$$
$$+\left(-\beta^2+\frac{\beta}{2\gamma^{\frac{3}{2}}}-\frac{1}{8\gamma^3}+\frac{7}{8\gamma^4}+\frac{1}{48\gamma^5}\right)\frac{1}{\lambda^2}$$

and

$$q_{12}=\frac{\lambda}{\sqrt{\gamma}}+\frac{1}{2\gamma^2}-1+\left(-\frac{1}{2\gamma^{\frac{3}{2}}}+\frac{1}{2\gamma^{\frac{5}{2}}}+\frac{1}{8\gamma^{\frac{7}{2}}}-\beta\right)\frac{1}{\lambda}$$
$$+\left(-\beta^2+\frac{\beta}{2\gamma^{\frac{3}{2}}}-\frac{1}{8\gamma^3}+\frac{7}{8\gamma^4}+\frac{1}{48\gamma^5}\right)\frac{1}{\lambda^2}.$$

Then, after long computations, we find the following asymptotic development of $f_\beta(\lambda) = \det(M_\beta(\lambda))$:

$$f_\beta(\lambda) = \lambda^5 f_0(\lambda) + \lambda^4 f_1(\lambda) + \lambda^3 f_2(\lambda) + \lambda^2 f_{\beta,3}(\lambda) + \lambda f_{\beta,4}(\lambda) + O(1), \qquad (3.28)$$

where

$$f_0(\lambda) = L_0(\gamma)\cosh(\sqrt{\gamma}\lambda),\ L_0(\gamma) = 4\gamma^2\cosh(\frac{1}{\sqrt{\gamma}}), \qquad (3.29)$$

$$f_1(\lambda) = L_1(\gamma)\sinh(\sqrt{\gamma}\lambda),\ L_1(\gamma) = -2\sqrt{\gamma}\left(\cosh(\frac{1}{\sqrt{\gamma}}) + 2\sqrt{\gamma}\sinh(\frac{1}{\sqrt{\gamma}})\right), \qquad (3.30)$$

$$f_2(\gamma) = -8 + L_2(\gamma)\cosh(\sqrt{\gamma}\lambda), \qquad (3.31)$$

$$L_2(\gamma) = \left(\frac{1}{2\gamma} - 8\right)\cosh(\frac{1}{\sqrt{\gamma}}) + \left(\frac{4}{\sqrt{\gamma}} + 4\gamma\sqrt{\gamma}\right)\sinh(\frac{1}{\sqrt{\gamma}}),$$

$$f_{\beta,3}(\lambda) = L_{\beta,3c}(\gamma)\cosh(\sqrt{\gamma}\lambda) + L_{3s}(\gamma)\sinh(\sqrt{\gamma}\lambda), \qquad (3.32)$$

$$L_{\beta,3c}(\gamma) = 4\beta\gamma^{\frac{3}{2}}\sinh(\frac{1}{\sqrt{\gamma}}),$$

$$L_{3s}(\gamma) = -\left(2 + \frac{3}{2\gamma^2}\right)\sinh(\frac{1}{\sqrt{\gamma}}) - \left(\frac{1}{12\gamma^{\frac{5}{2}}} + \frac{1}{2\gamma\sqrt{\gamma}} + 4\sqrt{\gamma}\right)\cosh(\frac{1}{\sqrt{\gamma}}), \qquad (3.33)$$

and

$$f_{\beta,4}(\lambda) = L_{4c}(\gamma)\cosh(\sqrt{\gamma}\lambda) + L_{\beta,4s}(\gamma)\sinh(\sqrt{\gamma}\lambda), \qquad (3.34)$$

$$\begin{aligned} L_{4c}(\gamma) \quad = \quad & \left(\frac{1}{3\gamma^{\frac{7}{2}}} - \frac{1}{2\gamma^{\frac{5}{2}}} + \frac{1}{2\gamma^{\frac{3}{2}}} - \frac{10}{\sqrt{\gamma}}\right)\sinh(\frac{1}{\sqrt{\gamma}}) \qquad (3.35) \\ & + \left(\frac{2}{\gamma} + \frac{13}{2\gamma^2} + \frac{1}{2\gamma^3}\right)\cosh(\frac{1}{\sqrt{\gamma}}), \end{aligned}$$

$$L_{\beta,4s}(\gamma) = -4\sqrt{\gamma}\beta\left(\cosh(\frac{1}{\sqrt{\gamma}}) + \frac{1}{\sqrt{\gamma}}\sinh(\frac{1}{\sqrt{\gamma}})\right). \qquad (3.36)$$

Since the real part of λ is bounded, the functions f_i, $i \in \{0, 1, 2, 3, 4\}$ are bounded. For convenience we set

$$S_\beta(\lambda) = \frac{f_\beta(\lambda)}{\lambda^5} = f_0(\lambda) + \frac{f_1(\lambda)}{\lambda} + \frac{f_2(\lambda)}{\lambda^2} + \frac{f_{\beta,3}(\lambda)}{\lambda^3} + \frac{f_{\beta,4}(\lambda)}{\lambda^4} + O(\frac{1}{\lambda^5}). \qquad (3.37)$$

Step 2. We look at the roots of S_β. It is easy to see that the roots of f_0 are simple and given by:

$$z_k = i\alpha_k = i\left(\frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}\right).$$

Then, with the help of Rouché's theorem, there exists $k_\beta \in \mathbb{N}^*$ large enough, such that $\forall |k| \geq k_\beta$ the large eigenvalues of $\mathcal{A}_\beta$ (denoted by $\lambda_{\beta,k}$) are simple and close to z_k, *i.e.*

$$\lambda_{\beta,k} = i\alpha_k + o_\beta(1), \quad |k| \to \infty. \qquad (3.38)$$

Equivalently we can write

$$\lambda_{\beta,k} = i\alpha_k + \zeta_{\beta,k}, \quad \lim_{|k|\to\infty} \zeta_{\beta,k} = 0. \tag{3.39}$$

Step 3. Determination of $\zeta_{\beta,k}$. First, using (3.37) and the identities (3.29)-(3.36) we have

$$\begin{aligned} 0 = S_\beta(\lambda_{\beta,k}) =& L_0(\gamma)\cosh(\sqrt{\gamma}\lambda_{\beta,k}) + \frac{L_1(\gamma)\sinh(\sqrt{\gamma}\lambda_{\beta,k})}{\lambda_{\beta,k}} \\ &+ \frac{-8 + L_2(\gamma)\cosh(\sqrt{\gamma}\lambda_{\beta,k})}{\lambda_{\beta,k}^2} + \frac{L_{\beta,3c}(\gamma)\cosh(\sqrt{\gamma}\lambda_{\beta,k})}{\lambda_{\beta,k}^3} \\ &+ \frac{L_{3s}(\gamma)\sinh(\sqrt{\gamma}\lambda_{\beta,k})}{\lambda_{\beta,k}^3} + \frac{L_{4c}(\gamma)\cosh(\sqrt{\gamma}\lambda_{\beta,k})}{\lambda_{\beta,k}^4} \\ &+ \frac{L_{\beta,4s}(\gamma)\sinh(\sqrt{\gamma}\lambda_{\beta,k})}{\lambda_{\beta,k}^4} + O(\frac{1}{\lambda_{\beta,k}^5}). \end{aligned} \tag{3.40}$$

On the other hand, using (3.39) we obtain

$$\begin{aligned} \cosh(\sqrt{\gamma}\lambda_{\beta,k}) &= i(-1)^k \sinh(\sqrt{\gamma}\zeta_{\beta,k}) \\ &= i(-1)^k \left(\sqrt{\gamma}\zeta_{\beta,k} + \frac{\gamma\sqrt{\gamma\zeta_{\beta,k}^3}}{9} + o(\zeta_{\beta,k}^4) \right), \end{aligned} \tag{3.41}$$

$$\begin{aligned} \sinh(\sqrt{\gamma}\zeta_{\beta,k}) &= i(-1)^k \cosh(\sqrt{\gamma}\zeta_{\beta,k}) \\ &= i(-1)^k \left(1 + \frac{\gamma\zeta_{\beta,k}^2}{2} + \frac{\gamma^2\zeta_{\beta,k}^4}{6} + o(\zeta_{\beta,k}^4) \right) \end{aligned} \tag{3.42}$$

and

$$\frac{1}{\lambda_{\beta,k}} = \frac{1}{i\alpha_k}\left(1 - \frac{\zeta_{\beta,k}}{i\alpha_k} + o(\frac{\zeta_{\beta,k}^2}{\alpha_k^2})\right) = -\frac{i}{\alpha_k} + \frac{\zeta_{\beta,k}}{\alpha_k^2} + o(\frac{\zeta_{\beta,k}^2}{\alpha_k^2}). \tag{3.43}$$

Similarly we get

$$\frac{1}{\lambda_{\beta,k}^2} = -\frac{1}{\alpha_k^2} - 2i\frac{\zeta_{\beta,k}^2}{\alpha_k^3} + o(\frac{\zeta_{\beta,k}^2}{\alpha_k^2}), \tag{3.44}$$

$$\frac{1}{\lambda_{\beta,k}^3} = \frac{i}{\alpha_k^3} - 3\frac{\zeta_{\beta,k}}{\alpha_k^4} + o(\frac{\zeta_{\beta,k}^2}{\alpha_k^2}) \tag{3.45}$$

and

$$\frac{1}{\lambda_{\beta,k}^4} = \frac{1}{\alpha_k^4} + 4i\frac{\zeta_{\beta,k}}{\alpha_k^5} + o(\frac{\zeta_{\beta,k}^2}{\alpha_k^2}). \tag{3.46}$$

Then, substituting (3.41)-(3.46) into (3.40) and after some computation yields

$$iL_0(\gamma)\sqrt{\gamma}\zeta_{\beta,k}+\frac{i\gamma\sqrt{\gamma}L_0(\gamma)}{6}\zeta_{\beta,k}^3+\frac{L_1(\gamma)}{\alpha_k}+\frac{iL_1(\gamma)}{\alpha_k^2}\zeta_{\beta,k}+\frac{\gamma L_1(\gamma)}{2\alpha_k}\zeta_{\beta,k}^2+ \tag{3.47}$$
$$\frac{8(-1)^k}{\alpha_k^2}+\frac{16i(-1)^k}{\alpha_k^3}\zeta_{\beta,k}-\frac{iL_2(\gamma)}{\alpha_k^2}\zeta_{\beta,k}-\frac{\sqrt{\gamma}L_{\beta,3c}(\gamma)}{\alpha_k^3}\zeta_{\beta,k}-\frac{L_{3s}(\gamma)}{\alpha_k^3}+$$
$$\frac{iL_{\beta,4s}(\gamma)}{\alpha_k^4}+o(\zeta_{\beta,k}^4)+o(\frac{\zeta_{\beta,k}^2}{\alpha_k^2})+o(\frac{1}{\alpha_k^4})=0.$$

Next, using (3.47) we find the first development of $\zeta_{k,\beta}$ given by

$$\zeta_{\beta,k}=\frac{iL_1(\gamma)}{\sqrt{\gamma}L_0(\gamma)\alpha_k}+e_{\beta.1} \tag{3.48}$$

where $e_{\beta,1}=O_\beta(\frac{1}{\alpha_k^2})$. Then, inserting (3.48) in (3.47) we obtain

$$e_{\beta,1}=\frac{8i(-1)^k}{\sqrt{\gamma}L_0(\gamma)\alpha_k^2}+e_{\beta,2} \tag{3.49}$$

where $e_{\beta,2}=O_\beta(\frac{1}{\alpha_k^3})$. Substituting (3.49) into (3.48) yields

$$\zeta_{\beta,k}=\frac{iL_1(\gamma)}{\sqrt{\gamma}L_0(\gamma)\alpha_k}+\frac{8i(-1)^k}{\sqrt{\gamma}L_0(\gamma)\alpha_k^2}+e_{\beta,2}. \tag{3.50}$$

Next, inserting (3.50) in (3.47) we obtain

$$e_{\beta,2}=\frac{iQ_1}{\alpha_k^3}+e_{\beta,3} \tag{3.51}$$

where

$$\begin{aligned} Q_1 &= \frac{1}{3\gamma L_0^3(\gamma)}\left[-\sqrt{\gamma}L_1^3(\gamma)-3L_0(\gamma)L_1^2(\gamma)+3\sqrt{\gamma}L_0(\gamma)L_1(\gamma)L_2(\gamma)\right. \\ &\qquad \left.-3\sqrt{\gamma}L_0^2(\gamma)L_{3s}(\gamma)\right] \end{aligned} \tag{3.52}$$

and where $e_{\beta,3}=O_\beta(\frac{1}{\alpha_k^4})$. Then, substituting (3.51) into (3.50) yields

$$\zeta_{\beta,k}=\frac{iL_1(\gamma)}{\sqrt{\gamma}L_0(\gamma)\alpha_k}+\frac{8i(-1)^k}{\sqrt{\gamma}L_0(\gamma)\alpha_k^2}+\frac{iQ_1}{\alpha_k^3}+e_{3,\beta}. \tag{3.53}$$

Later, inserting (3.53) in (3.47) we obtain

$$e_{\beta,3}=\frac{iQ_2}{\alpha_k^4}+\frac{Q_{\beta,3}}{\alpha_k^4}+o_1(\frac{1}{\alpha_k^4}), \tag{3.54}$$

where

$$Q_2 = \frac{4(-1)^k}{\gamma L_0^3(\gamma)} \left[2\sqrt{\gamma} L_0(\gamma) L_2(\gamma) - \sqrt{\gamma} L_1^2(\gamma) - 6 L_0(\gamma) L_1(\gamma)\right] \tag{3.55}$$

and

$$Q_{\beta,3} = \frac{L_{\beta,3c}(\gamma) L_1(\gamma) - L_0(\gamma) L_{\beta,4s}(\gamma)}{\sqrt{\gamma} L_0^2}. \tag{3.56}$$

Then, substituting (3.54) into (3.53) yields

$$\zeta_{\beta,k} = \frac{i L_1(\gamma)}{\sqrt{\gamma} L_0(\gamma) \alpha_k} + \frac{8i(-1)^k}{\sqrt{\gamma} L_0(\gamma) \alpha_k^2} + \frac{i Q_1}{\alpha_k^3} + i \frac{Q_2}{\alpha_k^4} + \frac{Q_{\beta,3}}{\alpha_k^4} + o_1(\frac{1}{k^4}). \tag{3.57}$$

Moreover, using (3.29)-(3.33) and (3.36), then from (3.57) and after long computations we obtain

$$\begin{aligned} \zeta_{\beta,k} \quad &= \quad i\left(-\frac{(\frac{1}{2\sqrt{\gamma}} + \tanh(\frac{1}{\sqrt{\gamma}}))}{\gamma^{\frac{3}{2}} \alpha_k} + \frac{2(-1)^k}{\gamma^{\frac{5}{2}} \cosh(\frac{1}{\sqrt{\gamma}}) \alpha_k^2} + \frac{E}{\alpha_k^3} + \frac{F}{\alpha_k^4}\right) \\ &\quad + \frac{\beta}{\pi^4 \cosh(\frac{1}{\sqrt{\gamma}})} \times \frac{1}{k^4} + o_1(\frac{1}{k^4}), \end{aligned}$$

where E and F are given in (3.18) and (3.19) respectively. Finally inserting the previous identity in (3.39) we directly get (3.16). □

Graphical Interpretation. Figure 2 represents the eigenvalues of $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_0$ for $\beta = 1$ and $\gamma = 10$.

Note that for a scale reason seven eigenvalues do no appear in the previous figure. Their approximates values are

$$0.0152039 \pm 5.58917i, \quad 0.0402791 \pm 3.3494i, \quad 0.138254 \pm 1.30223i \quad \text{and} \quad 0.546406.$$

From Proposition 3.2 we denote that

$$\Phi_{\beta,k} = (y_{\beta,k}, -\lambda_{\beta,k} y_{\beta,k}, y_{\beta,k}(1)) \tag{3.58}$$

is the eigenvector associated with the eigenvalue $\lambda_{\beta,k}$ of high frequency, and by $\{\Phi_{\beta,j,l}\}_{l=1}^{m_{\beta,j}}$ the Jordan chain of root vectors associated with the eigenvalue $\lambda_{\beta,j}$ of low frequency ($\Phi_{0,j,l}$ are in fact eigenvectors of $\mathcal{A}_0$) . Thus we obtain a system of root vectors of $_\beta$:

$$\{\Phi_{\beta,k},\ |k| \geq k_\beta\} \cup \{\Phi_{\beta,j,l},\ 1 \leq l \leq m_{\beta,j},\ j \in J_\beta\}. \tag{3.59}$$

Now, we solve the problem (3.10) for $\lambda = \lambda_{\beta,k}$ (for $\beta \geq 0$) and we give a solution up to factor in the following proposition:

Proposition 3.3. *For $\beta \geq 0$ and $|k| \geq k_\beta$, a solution $y_{\beta,k}$ of the problem (3.10) with $\lambda = \lambda_{\beta,k}$ satisfies the following estimations:*

$$y_{\beta,k}(1) = -\frac{2}{\cosh(\frac{1}{\sqrt{\gamma}})} + o(1) \neq 0,\ \|y_{\beta,k}\|_W \sim |k|^2,\ \|y_{\beta,k}\|_V \sim |k|,\ |k| \to \infty \tag{3.60}$$

and we deduce that

$$\|\Phi_{\beta,k}\|_{\mathcal{H}} \sim |k|^2, \quad |k| \to \infty. \tag{3.61}$$

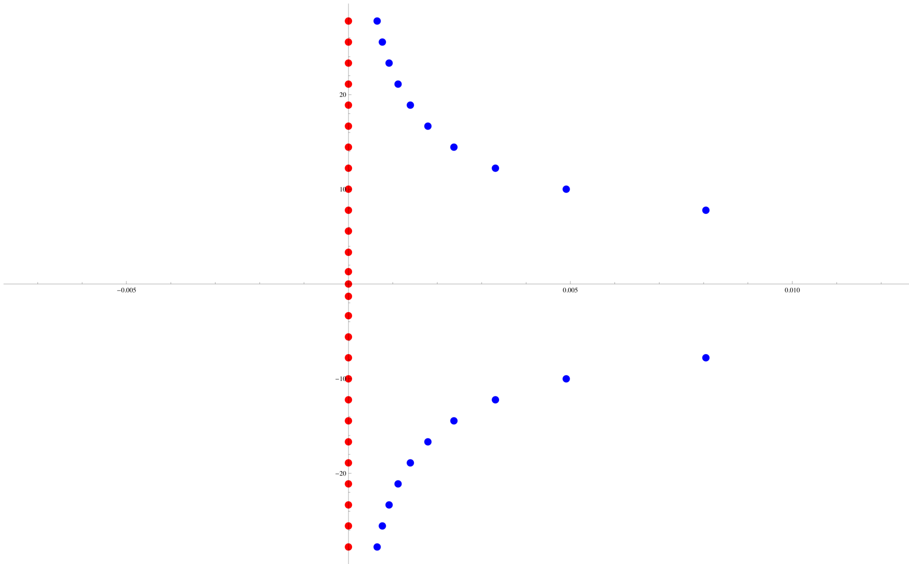

Figure 2: Eigenvalues of $\widetilde{\mathcal{A}}_1$ (in blue) and $\widetilde{\mathcal{A}}_0$ (in red) with $\beta = 1$ and $\gamma = 10$

Proof: For simplicity, in this proof we denote $\lambda_{\beta,k}$ by λ_k and $y_{\beta,k}$ by y_k. For $\beta \geq 0$, $\lambda = \lambda_k$ and $|k| \geq k_\beta$, solving (3.10) amounts to find a solution $C(\lambda_k) \neq 0$ of system (3.12) of rank three. For clarity, we divide the proof to several steps.

Step 1. Determination of y_k. Since we search $C(\lambda_k)$ up to factor we choose $c_4(\lambda_k) = 1$, the possibility of this choice will be justify later. Therefore (3.12) becomes

$$\left\{\begin{array}{ccc} c_1(\lambda_k) + c_2(\lambda_k) + c_3(\lambda_k) & = & -1, \\ R_1(\lambda_k)c_1(\lambda_k) + R_2(\lambda_k)c_2(\lambda_k) + R_3(\lambda_k)c_3(\lambda_k) & = & -R_4(\lambda_k), \\ R_1^2(\lambda_k)e^{R_1(\lambda_k)}c_1(\lambda_k) + R_2^2(\lambda_k)e^{R_2(\lambda_k)}c_2(\lambda_k) + R_3^2(\lambda_k)e^{R_3(\lambda_k)} & = & -R_4^2(\lambda_k)e^{R_4(\lambda_k)}. \end{array}\right.$$

Next, using Cramer's rule, we obtain

$$c_1(\lambda_k) = \frac{b_1}{b_4}, \quad c_2(\lambda_k) = \frac{b_2}{b_4}, \quad c_3(\lambda_k) = \frac{b_3}{b_4}, \tag{3.62}$$

where

$$\begin{aligned} b_1 = & 2R_1(\lambda_k)R_3(\lambda_k)^2\sinh(R_3(\lambda_k)) - 2R_3(\lambda_k)^3\cosh(R_3(\lambda_k)) \\ & + 2R_1(\lambda_k)^2R_3(\lambda_k)e^{-R_1(\lambda_k)}, \end{aligned} \tag{3.63}$$

$$\begin{aligned} b_2 = & 2R_1(\lambda_k)R_3(\lambda_k)^2\sinh(R_3(\lambda_k)) + 2R_3(\lambda_k)^3)\cosh(R_3(\lambda_k)) \\ & - 2R_1(\lambda_k)^2R_3(\lambda_k)e^{R_1(\lambda_k)}, \end{aligned} \tag{3.64}$$

$$\begin{aligned} b_3 = & 2R_1(\lambda_k)^2R_3(\lambda_k)\sinh(R_1(\lambda_k)) - 2R_1(\lambda_k)^3\cosh(R_1(\lambda_k)) \\ & + R_1(\lambda_k)R_3(\lambda_k)^2e^{-R_3(\lambda_k)} \end{aligned} \tag{3.65}$$

and

$$b_4 = 2R_1(\lambda_k)^2 R_3(\lambda_k)\sinh(R_1(\lambda_k)) + 2R_1(\lambda_k)^3\cosh(R_1(\lambda_k)) \\ - R_1(\lambda_k)R_3(\lambda_k)^2 e^{R_3(\lambda_k)}. \tag{3.66}$$

First, we study the behavior of b_1. Inserting (2.102) and (2.103) (with $\lambda = \lambda_k$) in (3.63) we find after some computations

$$b_1 = -2\gamma^{\frac{3}{2}}\lambda_k^3\cosh(\sqrt{\gamma}\lambda_k) + (1+2\sqrt{\gamma})\lambda_k^2\sinh(\sqrt{\gamma}\lambda_k). \tag{3.67}$$

Now, using the asymptotic behavior (3.16) we find

$$\begin{cases} \cosh(\sqrt{\gamma}\lambda_k) &= \dfrac{i(-1)^k(1+2\sqrt{\gamma}\tanh(\frac{1}{\sqrt{\gamma}}))}{2\gamma^{\frac{3}{2}}\lambda_k} + O(\dfrac{1}{\lambda_k^2}), \\ \sinh(\sqrt{\gamma}\lambda_k) &= i(-1)^k + O(\dfrac{1}{\lambda_k^2}). \end{cases} \tag{3.68}$$

Then, inserting (3.68) in (3.67) we find again after some computations

$$b_1 = 2\sqrt{\gamma}i(-1)^k\left(1-\tanh(\frac{1}{\sqrt{\gamma}})\right)\lambda_k^2 + O(\lambda_k). \tag{3.69}$$

Similarly long computations left to the reader yield

$$b_2 = 2i(-1)^k\sqrt{\gamma}\left(1+\tanh(\frac{1}{\sqrt{\gamma}})\right)\lambda_k^2 + O(\lambda_k), \tag{3.70}$$

$$b_3 = -2\sqrt{\gamma}i(-1)^k\lambda_k^2 + O(\lambda_k) \tag{3.71}$$

and

$$b_4 = -2\sqrt{\gamma}i(-1)^k\lambda_k^2 + O(\lambda_k). \tag{3.72}$$

Remark that $b_4 \neq 0$ provided we have chosen k_β large enough, for this reason our choice $c_4(\lambda_k) = 1$ is valid. Substituting (3.69)-(3.72) into (3.62), we deduce

$$c_1(\lambda_k) = -1 + \tanh(\frac{1}{\sqrt{\gamma}}) + O(\frac{1}{|\lambda_k|}), \quad c_2(\lambda_k) = -1 - \tanh(\frac{1}{\sqrt{\gamma}}) + O(\frac{1}{|\lambda_k|}), \tag{3.73}$$

$$c_3(\lambda_k) = 1 + O(\frac{1}{|\lambda_k|}) \quad \text{and} \quad c_4(\lambda_k) = 1.$$

Finally we have found that a solution of (3.12) has the form:

$$C(\lambda_k) = C_0 + O(\frac{1}{|\lambda_k|}), \tag{3.74}$$

where

$$C_0 = (-1 + \tanh(\frac{1}{\sqrt{\gamma}}),\ -1 - \tanh(\frac{1}{\sqrt{\gamma}}),\ 1,\ 1). \tag{3.75}$$

Note that the corresponding solution y_k of (3.10) is given by:

$$y_k = \sum_i^4 c_i(\lambda_k)e^{R_i(\lambda_k)}. \tag{3.76}$$

Step 2. Estimate of $y_k(1)$. From equation (3.76), we have

$$y_k(1) = c_1(\lambda_k)e^{R_1(\lambda_k)} + c_2(\lambda_k)e^{R_2(\lambda_k)} + c_3(\lambda_k)e^{R_3(\lambda_k)} + c_4(\lambda_k)e^{R_4(\lambda_k)},$$

where we recall that for $i \in \{1,2,3,4\}$ $R_i(\lambda_k)$ are given in (2.92) and c_i satisfy (3.73). Therefore using the series expansions (3.16) and (2.102)-(2.103) for $\lambda = \lambda_k$ we easily find

$$y_k(1) = -\frac{2}{\cosh(\frac{1}{\sqrt{\gamma}})} + o(1) \neq 0. \tag{3.77}$$

Step 3. Estimates of $\|y_k\|_W$ and $\|y_k\|_V$. We start with

$$\begin{aligned} \|y_k\|_W^2 &= \int_0^1 |y_{k,xx}|^2 dx \\ &= \sum_{i=1}^4 \sum_{j=1}^4 c_i(\lambda_k)R_i(\lambda_k)^2 \left(\int_0^1 e^{R_i(\lambda_k)x} \overline{e^{R_j(\lambda_k)x}} dx \right) \overline{c_j(\lambda_k)R_j(\lambda_k)^2} \\ &= C_k G_k \overline{C_k}^T \end{aligned} \tag{3.78}$$

where

$$G_k = \left(\int_0^1 e^{(R_i(\lambda_k) + \overline{R_j(\lambda_k)})x} dx \right)_{1 \le i,j \le 4} \quad \text{and where} \quad C_k = (c_i(\lambda_k)R_i(\lambda_k)^2)_{i=1,..,4}.$$

First, using (3.16), then from (2.102) and (2.103) we can write $R_1(\lambda_k)$ and $R_3(\lambda_k)$ as follows

$$R_1(\lambda_k) = q_1 + ir_1, \tag{3.79}$$

where

$$q_1 = \frac{1}{\sqrt{\gamma}} + O(\frac{1}{\lambda_k^2}), \quad r_1 = -\frac{\gamma^{\frac{7}{2}}\beta}{\cosh(\frac{1}{\sqrt{\gamma}})\lambda_k^7} + O(\frac{1}{\lambda_k^8})$$

and

$$R_3(\lambda_k) = q_3 + ir_3, \tag{3.80}$$

where

$$q_3 = \frac{\gamma^{\frac{5}{2}}\beta}{\lambda_k^4} + O(\frac{1}{\lambda_k^6}), \quad r_3 = \sqrt{\gamma}\lambda_k + O(1).$$

Then, the fact that $R_2(\lambda_k) = -R_1(\lambda_k)$ and $R_4(\lambda_k) = -R_3(\lambda_k)$ and using the asymptotic behavior (3.79)-(3.80) we directly find

$$\begin{aligned} \int_0^1 e^{(R_1(\lambda_k) + \overline{R_2(\lambda_k)})x} dx &= \int_0^1 e^{(R_2(\lambda_k) + \overline{R_1(\lambda_k)})x} dx \\ &= \int_0^1 e^{(R_3(\lambda_k) + \overline{R_3(\lambda_k)})x} dx \\ &= \int_0^1 e^{(R_4(\lambda_k) + \overline{R_4(\lambda_k)})x} dx = 1 + O(\frac{1}{\lambda_k^4}). \end{aligned}$$

Moreover, using the asymptotic behavior (3.79)- (3.80) we find that G_k is given as follows:

$$G_k = G_0 + O(\frac{1}{\lambda_k}), \tag{3.81}$$

where G_0 was defined by (2.76) and $O(\frac{1}{\lambda_k})$ is a matrix where all entries are if order $\frac{1}{\lambda_k}$. Next, using (3.73) and (3.79)-(3.80) we obtain

$$C_k = (0, 0, \gamma\lambda_k^2, \gamma\lambda_k^2) + O(1). \tag{3.82}$$

Finally, using (3.81) and (3.82) then from (3.78) we deduce

$$\|y_k\|_W^2 = \gamma^2|\lambda_k|^4 + O(|\lambda_k|^3) \sim |k|^4, \quad |k| \to \infty. \tag{3.83}$$

Similarly, we easily prove that

$$\|y_k\|_{L^2(0,1)}^2 \sim 1, \quad \|y_{k,x}\|_{L^2(0,1)}^2 \sim |k|^2, \quad |k| \to \infty.$$

Therefore, we deduce that

$$\|y_k\|_V \sim |k|, \quad |k| \to \infty. \tag{3.84}$$

Finally, using the estimations (3.77), (3.83) and (3.84) then from (3.58) we deduce (3.61). This completes the proof. $\square$

3.2. Riesz basis and polynomial stability with optimal decay rate

Our main result is the following optimal polynomial-type decay estimation.

Theorem 3.4. *(Optimal energy decay rate)*
Assume that $\beta > 0$ and that $\gamma \geq \gamma_0$. Then, for all initial data $U_0 \in D(\widetilde{\mathcal{A}}_\beta)$, there exists a constant $c > 0$ independent of U_0, such that the energy of the problem (3.9) satisfies the following estimation

$$E(t) \leq \frac{c}{\sqrt{t}}\|U_0\|_{D(\widetilde{\mathcal{A}}_\beta)}^2. \tag{3.85}$$

Moreover, the energy decay rate (3.85) is optimal.

$\square$

First, we prove that the set of the generalized eigenvectors associated with $\widetilde{\mathcal{A}}_\beta$ forms a Riesz basis in $\mathcal{H}$ in the following proposition:

Theorem 3.5. *The set of generalized eigenvectors associated with $\sigma(\widetilde{\mathcal{A}}_\beta)$ forms a Riesz basis of $\mathcal{H}$.*

Proof: First, since $\widetilde{\mathcal{A}}_0$ is a skew-adjoint operator, its set of normalized eigenvectors form an orthonormal basis in $\mathcal{H}$. Next, we prove the following property:

$$\sum_{k=\max\{k_0,k_\beta\}}^{+\infty} \|\widetilde{\Phi}_{\beta,k} - \widetilde{\Phi}_{0,k}\|_{\mathcal{H}} < +\infty \tag{3.86}$$

where

$$\widetilde{\Phi}_{\beta,k} = (\widetilde{y}_{\beta,k}, \widetilde{z}_{\beta,k}, \widetilde{\xi}_{\beta,k}) = \frac{1}{\|\Phi_{0,k}\|_{\mathcal{H}}}\Phi_{\beta,k}, \quad \forall |k| \geq k_\beta, \tag{3.87}$$

and

$$\widetilde{\Phi}_{0,k} = (\widetilde{y}_{0,k}, \widetilde{z}_{0,k}, \widetilde{\xi}_{0,k}) = \frac{1}{\|\Phi_{0,k}\|_{\mathcal{H}}}\Phi_{0,k}, \quad \forall |k| \geq k_0. \tag{3.88}$$

We first estimate

$$\|\widetilde{\Phi}_{\beta,k} - \widetilde{\Phi}_{0,k}\|^2_{\mathcal{H}} = \|\widetilde{y}_{\beta,k} - \widetilde{y}_{0,k}\|^2_W + \|\widetilde{z}_{\beta,k} - \widetilde{z}_{0,k}\|^2_V + |\widetilde{\xi}_{\beta,k} - \widetilde{\xi}_{0,k}|^2. \tag{3.89}$$

For clarity, we divide the proof into several steps.
Step 1. Estimate of $\|\widetilde{y}_{\beta,k} - \widetilde{y}_{0,k}\|^2_W$. First, since $\|\Phi_{0,k}\|_{\mathcal{H}} \sim |k|^2$ then from (3.87) and (3.88) we obtain

$$\|\widetilde{y}_{\beta,k} - \widetilde{y}_{0,k}\|_W \sim \frac{1}{|k|^2}\|y_{\beta,k} - y_{0,k}\|_W. \tag{3.90}$$

Next, using (3.76) we obtain

$$\begin{aligned}\widetilde{y}_{\beta,k,xx} - \widetilde{y}^0_{0,k,xx} \sim{}& \frac{1}{|k|^2}\left(R_1^2(\lambda_{\beta,k})c_1(\lambda_{\beta,k})e^{R_1(\lambda_{\beta,k})x} - R_1^2(\lambda_{0,k})c_1(\lambda_{0,k})e^{R_1(\lambda_{0,k})x}\right)\\
&+\frac{1}{|k|^2}\left(R_1^2(\lambda_{\beta,k})c_2(\lambda_{\beta,k})e^{-R_1(\lambda_{\beta,k})x} - R_1^2(\lambda_{0,k})c_2(\lambda_{0,k})e^{-R_1(\lambda_{0,k})x}\right)\\
&+\frac{1}{|k|^2}\left(R_3^2(\lambda_{\beta,k})c_3(\lambda_{\beta,k})e^{R_3(\lambda_{\beta,k})x} - R_3^2(\lambda_{0,k})c_3(\lambda_{0,k})e^{R_3(\lambda_{0,k})x}\right)\\
&+\frac{1}{|k|^2}\left(R_3^2(\lambda_{\beta,k})c_4(\lambda_{\beta,k})e^{-R_3(\lambda_{\beta,k})x} - R_3^2(\lambda_{0,k})c_4(\lambda_{0,k})e^{-R_3(\lambda_{0,k})x}\right).\end{aligned}$$

For simplicity we denote $c_i(\lambda_{\beta,k})$ by $c_i^{\beta,k}$ and $c_i(\lambda_{0,k})$ by $c_i^{0,k}$ for $i \in \{1,2,3,4\}$. Then, a direct computation gives

$$\|\widetilde{y}_{\beta,k} - \widetilde{y}_{0,k}\|^2_W \lesssim J_1 + J_2 + J_3 + J_4 \tag{3.91}$$

where

$$\begin{aligned}J_1 ={}& \frac{1}{|k|^4}\int_0^1 |R_1^2(\lambda_{\beta,k}) - R_1^2(\lambda_{0,k})|^2 |c_1^{\beta,k}|^2 |e^{R_1(\lambda_{\beta,k})x}|^2 dx\\
&+\frac{1}{|k|^4}\int_0^1 |R_1^2(\lambda_{0,k})|^2 |c_1^{\beta,k} - c_1^{0,k}|^2 |e^{R_1(\lambda_{\beta,k})x}|^2 dx\\
&+\frac{1}{|k|^4}\int_0^1 |R_1^2(\lambda_{0,k})|^2 |c_1^{0,k}|^2 |e^{R_1(\lambda_{\beta,k})x} - e^{R_1(\lambda_{0,k})x}|^2 dx,\end{aligned} \tag{3.92}$$

$$\begin{aligned}J_2 ={}& \frac{1}{|k|^4}\int_0^1 |R_1^2(\lambda_{\beta,k}) - R_1^2(\lambda_{0,k})|^2 |c_2^{\beta,k}|^2 |e^{-R_1(\lambda_{\beta,k})x}|^2 dx\\
&+\frac{1}{|k|^4}\int_0^1 |R_1^2(\lambda_{0,k})|^2 |c_2^{\beta,k} - c_2^{0,k}|^2 |e^{-R_1(\lambda_{\beta,k})x}|^2 dx\\
&+\frac{1}{|k|^4}\int_0^1 |R_1^2(\lambda_{0,k})|^2 |c_2^{0,k}|^2 |e^{-R_1(\lambda_{\beta,k})x} - e^{-R_1(\lambda_{0,k})x}|^2 dx,\end{aligned} \tag{3.93}$$

$$\begin{aligned}J_3 ={}& \frac{1}{|k|^4}\int_0^1 |R_3^2(\lambda_{\beta,k}) - R_3^2(\lambda_{0,k})|^2 |c_3^{\beta,k}|^2 |e^{R_3(\lambda_{\beta,k})x}|^2 dx\\
&+\frac{1}{|k|^4}\int_0^1 |R_3^2(\lambda_{0,k})|^2 |c_3^{\beta,k} - c_3^{0,k}|^2 |e^{R_3(\lambda_{\beta,k})x}|^2 dx\\
&+\frac{1}{|k|^4}\int_0^1 |R_3^2(\lambda_{0,k})|^2 |c_3^{0,k}|^2 |e^{R_3(\lambda_{\beta,k})x} - e^{R_3(\lambda_{0,k})x})|^2 dx,\end{aligned} \tag{3.94}$$

and

$$\begin{aligned} J_4 &= \frac{1}{|k|^4}\int_0^1 |R_3^2(\lambda_{\beta,k}) - R_3^2(\lambda_{0,k})|^2 |e^{-R_3(\lambda_{\beta,k})x}|^2 dx \\ &+ \frac{1}{|k|^4}\int_0^1 |R_3(\lambda_{0,k})|^2 |c_4^{\beta,k} - c_4^{0,k}|^2 |e^{-R_3(\lambda_{\beta,k})x}|^2 dx \\ &+ \frac{1}{|k|^4}\int_0^1 |R_3^2(\lambda_{0,k})|^2 |e^{-R_3(\lambda_{\beta,k})x} - e^{-R_3(\lambda_{0,k})x}|^2 dx. \end{aligned} \tag{3.95}$$

Now, using (3.16) ,(3.73) and the asymptotic behavior (2.102)-(2.103) for $\lambda = \lambda_{\beta,k}$ and for $\lambda = \lambda_{0,k}$ we find the following equivalences:

$$\begin{cases} R_1(\lambda_{\beta,k})^2 - R_1(\lambda_{0,k})^2 & \sim \dfrac{1}{|k|^4}, \\ e^{R_1(\lambda_{\beta,k})} - e^{R_1(\lambda_{0,k})} & \sim \dfrac{1}{|k|^2}, \\ c_1^{\beta,k} - c_1^{0,k} & \sim \dfrac{1}{|k|} \end{cases} \tag{3.96}$$

and

$$\begin{cases} R_3(\lambda_{\beta,k})^2 - R_3(\lambda_{0,k})^2 & \sim \dfrac{1}{|k|^3}, \\ e^{R_3(\lambda_{\beta,k})x} - e^{R_3(\lambda_{0,k})x} & \sim \dfrac{1}{|k|}, \\ c_3^{\beta,k} - c_3^{0,k} & \sim \dfrac{1}{|k|}. \end{cases} \tag{3.97}$$

Since $c_1^{\beta,k} \sim c_1^{0,k} \sim e^{R_1(\lambda_{\beta,k})x} \sim R_1^2(\lambda_{0,k}) \sim 1$ and using (3.96) then from (3.92) we obtain

$$J_1 \sim \frac{1}{|k|^4}\int_0^1 \frac{1}{|k|^8}dx + \frac{1}{|k|^4}\int_0^1 \frac{1}{|k|^2}dx + \frac{1}{|k|^4}\int_0^1 \frac{1}{|k|^4}dx \sim \frac{1}{|k|^6}. \tag{3.98}$$

Similarly, we get

$$J_2 \sim \frac{1}{|k|^8}. \tag{3.99}$$

In the same way, since $c_3^{\beta,k} \sim c_3^{0,k} \sim e^{R_3(\lambda_{\beta,k})} \sim 1$, $R_1(\lambda_{\beta,k})^2 \sim |k|^2$ and using (3.97) then from (3.94) we obtain

$$J_3 \sim \frac{1}{|k|^4}\int_0^1 \frac{1}{|k|^6}dx + \frac{1}{|k|^4}\int_0^1 |k|^2 dx + \frac{1}{|k|^4}\int_0^1 |k|^2 dx \sim \frac{1}{|k|^2}. \tag{3.100}$$

Similarly, we get

$$J_4 \sim \frac{1}{|k|^2}. \tag{3.101}$$

Finally, using (3.98)-(3.101) then from (3.91) we deduce

$$\|\widetilde{y}_{\beta,k} - \widetilde{y}_{0,k}\|_W^2 \lesssim \frac{1}{|k|^2}. \tag{3.102}$$

Step 2. Estimates $\|\widetilde{z}_{\beta,k} - \widetilde{z}_{0,k}\|_V^2$ and $|\widetilde{\xi}_{\beta,k} - \widetilde{\xi}_{0,k}|^2$. First, since $\|\Phi_{0,k}\|_{\mathcal{H}} \sim |k|^2$ and using (3.87)-(3.88) we obtain

$$\|\widetilde{z}_k - \widetilde{z}_k^0\|_V^2 \sim \frac{1}{|k|^4}\|z_k - z_k^0\|_V^2. \tag{3.103}$$

Then, using (3.58) we obtain

$$\begin{aligned} \|\widetilde{z}_{\beta,k} - \widetilde{z}_{0,k}\|_V^2 &\sim \frac{1}{|k|^4}\|\lambda_{\beta,k}y_{\beta,k} - \lambda_{0,k}y_{0,k}\|_V^2 \\ &\leq \frac{1}{|k|^4}|\lambda_{\beta,k} - \lambda_{0,k}|^2\|y_{\beta,k}\|_V^2 + \frac{|\lambda_{0,k}|^2}{|k|^4}\|y_{\beta,k} - y_{0,k}\|_V^2. \end{aligned} \tag{3.104}$$

Now, since $|\lambda_{\beta,k} - \lambda_{0,k}| \sim \dfrac{1}{|k|^4}$ and $\|y_{\beta,k}\|_V \sim |k|^2$ we get

$$\frac{1}{|k|^4}|\lambda_{\beta,k} - \lambda_{0,k}|^2\|y_{\beta,k}\|_V^2 \sim \frac{1}{|k|^8}. \tag{3.105}$$

Next, using the same strategy as in Step 1, we find after long computations that

$$\|y_{\beta,k} - y_{0,k}\|_V^2 \sim 1. \tag{3.106}$$

Then inserting (3.105)-(3.106) in (3.104) and the fact that $|\lambda_{0,k}|^2 \sim |k|^2$ we deduce

$$\|\widetilde{z}_{\beta,k} - \widetilde{z}_{0,k}\|_V^2 \lesssim \frac{1}{|k|^2}. \tag{3.107}$$

Similarly, we can easily find that

$$|\widetilde{\xi}_{\beta,k} - \widetilde{\xi}_{0,k}|^2 \lesssim \frac{1}{|k|^{10}}. \tag{3.108}$$

Step 3. Finally, inserting the estimations (3.102), (3.107) and (3.108) into (3.89) we obtain

$$\|\widetilde{\Phi}_{\beta,k} - \widetilde{\Phi}_{0,k}\|_{\mathcal{H}}^2 \lesssim \frac{1}{|k|^2},$$

and consequently

$$\sum_{k=\max\{k_0,k_\beta\}}^{\infty} \|\widetilde{\Phi}_{\beta,k} - \widetilde{\Phi}_{0,k}\|_{\mathcal{H}}^2 < +\infty.$$

Therefore, using a clarified form of Guo's Theorem (see [9, Theorem 6.3] and [1, Theorem 1.2.10]) we deduce that the set of generalized eigenvectors associated with $\sigma(\widetilde{\mathcal{A}}_\beta)$ forms a Riesz basis in $\mathcal{H}$. □

Proof of Theorem 3.4: First, using (3.16) we have $\Re(\lambda_k) \sim \dfrac{1}{k^4}$. Next, from Theorem 3.5 we know that the set of generalized eigenvectors associated with $\sigma(\widetilde{\mathcal{A}}_\beta)$ form a Riesz basis in $\mathcal{H}$. Then, applying [14, Theorem 2.1]) (see also [13] and [20]) we deduce the optimal polynomial energy decay rate (3.85) for smooth initial data.

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