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Fractional Tarig Transform and Mittag-Leffler Function

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ABSTRACT: In the present paper the Tarig transform of fractional order is studied by employing Mittag - Leffler function. Properties of Tarig transform are proved using the same fractional Tarig transform.

Key Words: Tarig transform, Laplace transform, Mittag - Leffler function, convolution.

Contents

1. Introduction

One may, among several , can justify efficiencies of the applications of integral transforms, viz. Fourier, Laplace and Hankel (among other) to solve ordinary and partial differential equations besides their applications in other areas. Consider a set A of function $f(t)$ of exponential order, expressed as

$$A = \{f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{|t|/k_j}; t \in (-1)^j \times [0, \infty)\},$$

where M is a constant of finite number and k_1 and k_2 may be finite or infinite.

The integral transform, known as Tarig transform, is introduced and studied by Elzaki, et al. [1,2,3], defined by

$$T[f(t), u] = G(u) = \frac{1}{u}\int_0^\infty e^{-(\frac{t}{u^2})} f(t)dt \quad , \quad u \neq 0 \tag{1.1}$$

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Tarig transform that is denoted by the operator $T[\cdot]$, is defined by

$$T[f(t)] = G(u) = \int_0^\infty f(ut)e^{-(\frac{t}{u})}dt \quad , \quad u \neq 0 \tag{1.2}$$

Properties of the said transform are given as follows :

1. The Tarig transform of *derivative* and n*th order derivative* of $f(t)$, respectively, are defined

$$T[f'(t)] = \frac{G(u)}{u^2} - \frac{f(0)}{u} \tag{1.3}$$

$$T[f^{(n)}(t)] = G^{(n)}(u) = \frac{G(u)}{u^{2n}} - \sum_{i=1}^{n} u^{2(i-n)-1} f^{(i-1)}(0). \tag{1.4}$$

2. When $f(t) = \delta(t)$ (the Dirac delta function), the Tarig transform becomes

$$T[\delta(t)] = \frac{1}{u} \ , \tag{1.5}$$

more such for other values are tabulated in [2,3].

When $f(t) = 1$, the Tarig transform becomes

$$T(1) = u, \tag{1.6}$$

and when $f(t) = t^n$,the same yields

$$T[t^n] = n!u^{2n+1} \tag{1.7}$$

$$= \Gamma(n+1)u^{2n+1} \ . \tag{1.8}$$

3. If $F(u)$ and $G(u)$ are Tarig transforms of the functions $f(t)$ and $g(t)$, then the *convolution* is given by

$$T[(f * g)(t)] = u \ F(u)G(u) \ . \tag{1.9}$$

4. If α, β are any constants and $f(t)$ and $g(t)$ are real functions, then the linear property is defined by

$$T[\alpha f(t) + \beta g(t)] = \alpha F(u) + \beta G(u) \ . \tag{1.10}$$

5.The relation between the Laplace transform $F(s)$ and Tarig transform $G(u)$ is defined by

$$G(u) = \frac{F\left(\frac{1}{u^2}\right)}{u}. \tag{1.11}$$

In [12], the Tarig transform is extended to the distribution spaces and some other properties have been formulated. In [11] the Parseval equation of the Tarig transform for distribution spaces is established and solution of Abel integral equation is obtained related to the distribution spaces.

Tarig transform for fractional integrals and derivatives for distribution spaces are employed in [13]. The Mittag - Leffler function and its applications with the integral transforms viz., Fourier, Laplace, Sumudu and Natural transform are given in [6,7,8,9,10]. Different techniques are employed to solve fractional differential equations [1,9,15,16,17,18]. Using the Mittag – Leffler function and its properties, some functions are defined and studied by the researchers [9,15,16,17,18].

The Mittag – Leffler function [15] is a direct generalization of the exponential function, and has an affinity for fractional calculus.

One parameter representation of the Mittag – Leffler function is given by [14]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{(\alpha k + 1)} \quad , \alpha > 0 \tag{1.12}$$

Whereas two parameter Mittag – Leffler function [4] is represented as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{(\alpha k + \beta)} \quad , \alpha, \beta, z \in \mathbb{C}, \mathrm{Re}(\alpha, \beta) > 0 \tag{1.13}$$

with $\mathbb{C}$ being the set of complex numbers.

Special cases of the Mittag – Leffler function are
(i) $E_\alpha(z) = \frac{1}{1-z} \quad , |z| < 1$
(ii) $E_1(z) = e^z$
(iii) $E_2(z) = \cosh(\sqrt{z}), z \in \mathbb{C}$
(iv) $E_{1,1}(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum\limits_{k=0}^{\infty} \frac{z^k}{(k)!} = e^z$
(v) $E_{1,2}(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \frac{1}{z} \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \frac{e^z - 1}{z}$.

Following relations, related to the Mittag – Leffler function, may be useful.
(i) $\frac{d^m}{dz^m} E_m(z^m) = E_m(z^m)$
(ii) $E_{\alpha,\beta}(z) = z E_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}$
(iii) $\frac{d}{dz} E_{\alpha,\beta}(z) = \frac{E_{\alpha,\beta-1}(z) - (\beta-1) E_{\alpha,\beta}(z)}{\alpha z}$

Having accommodated possible sources and terminologies on Tarig transform and the Mittag - Leffler function, in Section 2 we write concepts of fractional derivative, define modified Riemann - Liouville fractional derivative, and Taylor series of fractional order. In Section 3, we establish main results.

2. Preliminaries on Fractional Derivatives

2.1. Fractional derivative via fractional difference

Definition 2.1. *Let there be a continuous function* $f : R \to R,\ \ t \to f(t)$ *(but not necessarily differentiable). Let* $h > 0$ *be a constant discretization span. The forward operator* $FW(h)$ *is given by* .

$$FW(h)f(t) = f(t+h). \tag{2.11}$$

With regard to (2.11), the fractional difference of order $\alpha, 0 < \alpha < 1$, *of the function* $f(t)$ *is given by*

$$\Delta^{\alpha} f(t) = (FW-1)^{\alpha} = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[t+(\alpha-k)h],$$

and the fractional derivative of order α *is defined by the limit*

$$f^{(\alpha)}(t) = \lim_{h\to 0} \frac{\Delta^{\alpha} f(t)}{h^{\alpha}} \quad . \tag{2.12}$$

2.2. Modified Riemann-Liouville fractional derivative

. To overcome some drawbacks with the Riemann – Liouville fractional derivative, the modified version is devised [6,7].

Definition 2.2. *Let* $f : R \to R, t \to f(t)$ *is a continuous function.*

(i) When $f(t)$ *is constant* K, *its fractional derivative of order* α , *is given by*

$$\begin{aligned} D_t^{\alpha} K &= K\frac{1}{\Gamma(1-\alpha)} \cdot \frac{1}{t^{\alpha}}, \alpha \geq 0 \\ &= 0 \qquad , \qquad \alpha > 0 \end{aligned}$$

(ii) For $f(t)$ *being not a constant, we have*

$$f(t) = f(0) + (f(t) - f(0))$$

and its fractional derivative is defined by

$$f^{(\alpha)}(t) = D_t^{\alpha} f(0) + D_t^{\alpha}(f(t) - f(0)) \quad , \tag{2.21}$$

which, when $\alpha < 0$, *is given by*

$$D_t^{\alpha}(f(t) - f(0)) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\xi)^{-\alpha-1} f(\xi) d\xi, \qquad \alpha < 0 \tag{2.22}$$

whereas for $\alpha > 0$, we have

$$D_t^\alpha(f(t) - f(0)) = D_t^\alpha(f(t)) = D_t(f^{(\alpha-1)}(t)) \tag{2.23}$$

and

$$f^{(\alpha)}(t) = (f^{(\alpha-n)}(t))^{(n)} \quad , n \le \alpha < n+1 \; . \tag{2.24}$$

2.3. Taylor series of fractional order

Definition 2.3. *The continuous function $f : R \to R, t \to f(t)$ has a fractional derivative of order $k\alpha$. For any positive integer k and for any α, $0 < \alpha \le 1$, we have*

$$f(t+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{(\alpha k)!} f^{(\alpha k)}(t) \quad , 0 < \alpha \le 1, \tag{2.31}$$

where $\Gamma(1 + \alpha k) = (\alpha k)!$.

2.4. Integration with respect to $(dt)^\alpha$

. The integral with respect to $(dt)^\alpha$, is defined as the solution of the fractional differential equation

$$dy = f(t)(dt)^\alpha \quad , \quad t \ge 0, y(0) = 0 \tag{2.41}$$

Lemma 2.4. [6,7] *Let $f(t)$be a continuous function. Then the solution $y(t), y(0) = 0$, is given by*

$$\begin{aligned} y &= \int_0^t f(\xi)(d\xi)^\alpha \\ &= \alpha \int_0^t (t-\xi)^{(\alpha-1)} f(\xi) d\xi, \quad 0 < \alpha < 1 \;\; . \end{aligned} \tag{2.42}$$

3. Tarig Transform of Fractional Order and the Mittag-Leffler Function

In this section Tarig transform of fractional order is defined by using the Mittag – Leffler function, which is the generalization of the exponential function. Properties and convolution theorem are proved using the Tarig transform of fractional order.

By virtue of terminologies used in the preceding sections and recalling those described for Fourier and the Lapalce transforms, respectively , through the Mittag – Leffler function [6,7], following definition results.

Definition 3.1. *Let $f(t)$ be a function that vanishes for negative values of t . Then the Tarig transform of order α, for finite $f(t)$, is defined by*

$$
\begin{aligned}
T_\alpha[f(t)] &= G_\alpha(u) = \int_0^\infty f(ut) E_\alpha \left(-\frac{t}{u}\right)^\alpha (dt)^\alpha && (3.1)\\
&= \frac{1}{u} \int_0^\infty E_\alpha \left(-\frac{t}{u^2}\right)^\alpha f(t) (dt)^\alpha && (3.2)\\
&= \lim_{M\uparrow\infty} \frac{1}{u} \int_0^M E_\alpha \left(-\frac{t}{u^2}\right)^\alpha f(t) (dt)^\alpha && (3.3)
\end{aligned}
$$

where E_α is the Mittag – Leffler function, given by (1.12).

Theorem 3.2. (Tarig Laplace Duality of Fractional order) *If the Laplace transform of fractional order of a function $f(t)$ is $L_\alpha\{f(t)\} = F_\alpha(s)$ and the Tarig transform $T_\alpha[f(t)] = G_\alpha(u)$ is of order α, then*

$$
G_\alpha(u) = \frac{F_\alpha\left(\frac{1}{u^2}\right)}{u} \tag{3.4}
$$

Proof: Invoking Eqn. (2.42) in the definition of Tarig transform of fractional order (3.1), we write

$$
\begin{aligned}
T_\alpha[f(t)] &= G_\alpha(u) = \int_0^\infty f(ut) E_\alpha \left(-\frac{t}{u}\right)^\alpha (dt)^\alpha \\
&= \lim_{M\uparrow\infty} \alpha \int_0^M (M-t)^{\alpha-1} f(ut) E_\alpha \left(-\frac{t}{u}\right)^\alpha dt\ . && (3.5)
\end{aligned}
$$

By using the change of variable $ut \to w$, i.e., $dt = \frac{dw}{u}$, we get the right hand side

$$
\begin{aligned}
&= \lim_{M\uparrow\infty} \alpha \int_0^M (M - \frac{w}{u})^{\alpha-1} f(w) E_\alpha \left(-\frac{w}{u^2}\right)^\alpha \frac{dw}{u} \\
&= \lim_{M\uparrow\infty} \alpha \int_0^M (Mu - w)^{\alpha-1} f(w) E_\alpha \left(-\frac{w}{u^2}\right)^\alpha \frac{dw}{u^\alpha}.
\end{aligned}
$$

Using the definition of Laplace transform, we have

$$
T_\alpha[f(t)] = G_\alpha(u) = \frac{F_\alpha\left(\frac{1}{u^2}\right)}{u^\alpha} \tag{3.6}
$$

which, proves the theorem. □

Theorem 3.3. (Change of Scale Property) *Let $f(at)$ be a function in the set A , where a is non – zero constant. Then*

$$
T_\alpha[f(at)] = \frac{1}{a^\alpha} G_\alpha\left(\frac{u}{a}\right)\ . \tag{3.7}
$$

Conditions as mentioned above, are applicable.

Proof: Using (3.3), we have

$$\begin{aligned} T_\alpha[f(at)] &= \lim_{M\uparrow\infty} \frac{1}{u} \int_0^M E_\alpha \left(-\frac{t}{u^2}\right)^\alpha f(at)(dt)^\alpha \\ &= \lim_{M\uparrow\infty} \frac{1}{u} \alpha \int_0^M (M-t)^{\alpha-1} f(at) E_\alpha \left(-\frac{t}{u^2}\right)^\alpha dt \end{aligned} \tag{3.8}$$

By using the change of variable $at \to t'$, $dt = \frac{dt'}{a}$, we get

$$= \lim_{M\uparrow\infty} \frac{1}{u} \alpha \int_0^{Ma} (M - \frac{t'}{a})^{\alpha-1} f(t') E_\alpha \left(-\frac{t'}{au^2}\right)^\alpha \frac{dt'}{a}$$

$$= \int_0^{Ma} \frac{(Ma-t')^{\alpha-1}}{a^\alpha} f(t') E_\alpha \left(-\frac{t'}{au^2}\right)^\alpha dt'$$

i.e.

$$T_\alpha[f(at)] = \frac{1}{a^\alpha} G_\alpha\left(\frac{u}{a}\right). \tag{3.9}$$

Theorem is proved. □

Theorem 3.4. *Let $f(t-b)$ is a function of fractional Tarig transform. Then*

$$T_\alpha[f(t-b)] = E_\alpha\left(-\frac{b}{u^2}\right)^\alpha G_\alpha(u) \quad . \tag{3.10}$$

Proof: By (3.3) of Definition 3.1, we have.

$$\begin{aligned} T_\alpha[f(t-b)] &= \lim_{M\uparrow\infty} \frac{1}{u} \int_0^M E_\alpha \left(-\frac{t}{u^2}\right)^\alpha f(t-b)(dt)^\alpha \\ &= \lim_{M\uparrow\infty} \alpha \int_0^M (M-t)^{\alpha-1} v\, f(t-b) E_\alpha \left(-\frac{t}{u^2}\right)^\alpha dt \quad . \end{aligned} \tag{3.11}$$

Considering $t-b=x$, we have the right hand side .

$$= \lim_{M\uparrow\infty} \alpha \frac{1}{u} \int_0^{M-b} (M-b-x)^{\alpha-1} f(x) E_\alpha \left(-\frac{(b+x)}{u^2}\right)^\alpha dx$$

$$= \lim_{M\uparrow\infty} \alpha \frac{1}{u} \int_0^{M-b} (M-b-x)^{\alpha-1} f(x) E_\alpha \left(-\frac{x}{u^2}\right)^\alpha E_\alpha \left(-\frac{b}{u^2}\right)^\alpha dx$$

i.e.

$$T_\alpha[f(t-b)] = E_\alpha\left(-\frac{b}{u^2}\right)^\alpha G_\alpha(u) \quad , \tag{3.12}$$

which is due to $E_\alpha(\lambda(x+y)^\alpha) = E_\alpha(\lambda x^\alpha) E_\alpha(\lambda y^\alpha)$. □

Theorem 3.5. *If $f(t)$ is $E_\alpha(a^\alpha t^\alpha)f(t)$, then the Tarig transform is given by*

$$T_\alpha[E_\alpha(a^\alpha t^\alpha)f(t)] = \left(\frac{1}{1-au^2}\right)^\alpha G_\alpha\left(\frac{u}{1-au^2}\right). \tag{3.13}$$

Proof: Using (3.3) again of Definition 3.1, we have.

$$\begin{aligned} T_\alpha[E_\alpha(a^\alpha t^\alpha)f(t)] &= \lim_{M\uparrow\infty}\frac{1}{u}\int_0^M E_\alpha\left(-\frac{t}{u^2}\right)^\alpha E_\alpha(a^\alpha t^\alpha)f(t)(dt)^\alpha \\ &= \lim_{M\uparrow\infty}\alpha\frac{1}{u}\int_0^M (M-t)^{\alpha-1}\ f(t)E_\alpha(a^\alpha t^\alpha)E_\alpha\left(-\frac{t}{u^2}\right)^\alpha dt \end{aligned}$$

i.e.

$$= \lim_{M\uparrow\infty}\alpha\frac{1}{u}\int_0^M (M-t)^{\alpha-1}v\ f(t)E_\alpha\left(-\left(\frac{t-avt}{u^2}\right)\right)^\alpha dt$$

Setting $(1-au^2)t = w$, we have the right hand side, reduced to .

$$= \lim_{M\uparrow\infty}\alpha\frac{1}{u}\int_0^{M-au^2}\left(M-\frac{w}{1-au^2}\right)^{\alpha-1} f\left(\frac{w}{1-au^2}\right)E_\alpha\left(-\frac{w}{u^2}\right)^\alpha\frac{dw}{(1-au^2)}$$

$$= \int_0^{M-au^2}\left(\frac{1}{1-au^2}\right)^\alpha (M(1-au^2)-w))^{\alpha-1} f\left(\frac{w}{1-au^2}\right)E_\alpha\left(-\frac{w}{u^2}\right)^\alpha dw \quad ,$$

i.e.

$$T_\alpha[E_\alpha(a^\alpha t^\alpha)f(t)] = \left(\frac{1}{1-au^2}\right)^\alpha G_\alpha\left(\frac{u}{1-au^2}\right) \quad . \tag{3.14}$$

Hence, the theorem is proved. □

Theorem 3.6. *Let the convolution of two functions $f(t)$ and $g(t)$ of order α is given by*

$$(f(t)*g(t))_\alpha = \int_0^\infty f(t-\xi)g(\xi)(d\xi)^\alpha. \tag{3.15}$$

Then the convolution of Tarig transform of order α is

$$T_\alpha[(f(t)*g(t))_\alpha] = u^\alpha M_\alpha(v)N_\alpha(v) \quad . \tag{3.16}$$

Proof: The convolution of Laplace transform of order α is given by

$$L_\alpha[(f(t)*g(t))_\alpha] = L_\alpha\{f(t)\}L_\alpha\{g(t)\} \tag{3.17}$$

Now using Tarig – Laplace duality (Theorem 3.1, (3.4)), we have

$$\begin{aligned} T_\alpha[(f(t)*g(t))_\alpha] &= \frac{1}{u^\alpha} L_\alpha\{f(t)\} L_\alpha\{g(t)\} \\ &= \frac{1}{u^\alpha}\left[F_\alpha\left(\frac{1}{u^2}\right) G_\alpha\left(\frac{1}{u^2}\right)\right] \quad , \quad \text{as } M_\alpha(u) = \frac{F_\alpha\left(\frac{1}{u^2}\right)}{u^\alpha} \\ &= \frac{1}{u^\alpha}\left[u^\alpha M_\alpha(u) \cdot u^\alpha N_\alpha(u)\right] \quad . \end{aligned}$$

i.e.

$$T_\alpha[(f(t)*g(t))_\alpha] = u^\alpha M_\alpha(v) N_\alpha(v) \quad . \tag{3.18}$$

The theorem is proved. □

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References

1. Tarig M. Elzaki and Salih M. Elzaki, . *New integral transform "Tarig Transform" and system of integro differential equations*, Elixir Appl. Math. 57 (2013), 13982 - 13985.
2. Tarig M. Elzaki and Salih M. Elzaki, *Application of Tarig transform to Integral and Integro-Differential Equations*, Elixir Appl. Math. 57 (2013), 13978 - 13981.
3. Tarig M. Elzaki and Salih M. Elzaki, *Application of new transform "Tarig transform" to partial differential equations,* Elixir Appl. Math. 42 (2012), 6369 - 6372.
4. Tarig M. Elzaki and Salih M Elzaki, *On the relationship between Laplace and new integral transform "Tarig transform"*, Elixir Appl. Math. 36 (2011), 3230 - 3233.
5. A.Erdelyi, (ed.) Higher Transcendental Functions, Vol. 1, McGraw Hill, New York. (1955)
6. G. Jumarie, *Laplace's transform of fractional order via the Mittag – Leffler function and modified Riemann – Liouville derivative*, Appl. Math. Letters, 22 (2009) 1659 – 1664.
7. G. Jumarie, *Fourier's transform of fractional order via the Mittag – Leffler function and modified Riemann – Liouville derivative*, J. Appl. Math. & Informatics, 26 (5-6), (2008) 1101 - 1121.
8. Adem Kilicman, and Omer Altun, *Some remarks on the Fractional Sumudu transform and Applications*, Appl. Math. Inf. Sci. 8 (6) (2014), 2881 - 2888.
9. Muhammet Kurulay, and Mustafa Bayram, *Some properties of the Mittag – Leffler functions and their relation with the Wright functions,* Advances in Difference Equation, 2012, (2012) 181 – 188.
10. Deshna Loonker, *Fractional Natural Transform and the Mittag - Leffler function*, J. Indian Acad. Math. 37 (2) (2015) (To Appear)
11. Deshna Loonker and P. K. Banerji, *Distributional Abel integral equation for distributional Tarig transform*, Int. J. Appl. Math. 27 (3) (2014), 245 - 254
12. Deshna Loonker and P. K. Banerji, *Tarig transformation for Distribution and Boehmian Spaces,* Indian Journal of Mathematics Research 2 (1) (2014), 1-8

13. Deshna Loonker and P. K Banerji, *On Tarig Fractional Differintegral Transform on Distribution Spaces*, Pure and Applied Mathematics Letters 2 (2014), 19-25

14. K. S. Miller, and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons Inc., New York (1993)

15. G. Mittag – Leffler, *Sur la nouvelle fonction* , Comptes Rendus Hebdomadaires Des Seances Del Academie Des Sciences, Paris 2 (137), (1903) 554 – 558.

16. K. B. Oldham, and J. Spanier, The Fractional Calculus, Academic Press, New York. (1974)

17. I. Podlubny, Fractional Differential Equations, Academic Press, San Diego. (1999)

18. S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach Science Publishers, London.(1987)

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