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On statistical acceleration convergence of double sequences

Bipan Hazarika

abstract: In this article the notion of statistical acceleration convergence of double sequences in Pringsheim's sense has been introduced. We prove the decomposition theorems for statistical acceleration convergence of double sequences and some theorem related to that concept have been established using the four dimensional matrix transformations. We provided some examples, where the results of acceleration convergence fails to hold for the statistical cases.

Key Words: Converging faster, converging at the same rate, acceleration field, double natural density; statistical convergence; st_2-convergence, st_2-lim-inf, st_2-lim-sup.

Contents

1. Introduction

The idea of statistical convergence was formerly studied under the name almost convergence by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [29]. The concept was formally introduced by Steinhaus [25] and Fast [7] and later was introduced independently by Schoenberg [23] and Buck [4]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Mursaleen and Edely [12] extended the above idea from single to double sequences by using two dimensional analogue of natural density and established relations between statistical convergence and strongly Cesáro summable double sequences. Mursaleen and Mohiuddine [14] defined this notions for double sequences intuitionistic fuzzy normed spaces. Recently, Mohiuddine et al., [13] introduced this notions for double sequences locally solid Riesz spaces.

Let $E \subseteq \mathbb{N}$. Then the natural density of E is denoted by $\delta(E)$ and is defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : k \in E\}| \text{ exists,}$$

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where the vertical bar denotes the cardinality of the respective set.

A sequence $x = (x_k)$ of real numbers is said to be *statistically convergent* to ℓ if for every $\varepsilon > 0$,

$$\delta\left(\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}\right) = 0.$$

In this case, we write $st - \lim x = \ell$ or $(x_k) \xrightarrow{st} \ell$ and st denotes the set of all statistically convergent sequences.

Let $x = (x_k)$ and $y = (y_k)$ be two sequences , then we say that $x_k = y_k$ for *almost of all* k (in short $a.a.k$) if $\delta(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$.

The faster convergence of sequences particularly the acceleration of convergence of sequence of partial sums of series via linear and nonlinear transformations are widely used in finding solutions of mathematical as well as different scientific and engineering problems. The problem of acceleration convergence often occurs in numerical analysis. To accelerate the convergence, the standard interpolation and extrapolation methods of numerical mathematics are quite helpful. It is useful to study about the acceleration of convergence methods, which transform a slowly converging sequence into a new sequence, converging to the same limit faster than the original sequence. The speed of convergence of sequences is of the most importance in the theory of sequence transformation.

A sequence transformation T is a function $T : (x_k) \to (x_k^*)$ which maps a slowly convergent sequence to another sequence with better numerical properties. If $\lim_k x_k = x$ and $\lim_k x_k^* = x^*$ with r_k and r_k^* as the truncated errors. Then we have $x_k = x + r_k, x_k^* = x^* + r_k^*$. We say that the sequence (x_k) converge more rapidly than the sequence (x_k^*) if

$$\lim_k \frac{x_k^* - x^*}{x_k - x} = \lim_k \frac{r_k^*}{r_k} = 0.$$

The convergence rate of a sequence is defined as follows:

Let (x_k) be a real valued sequence with limit x. Then the convergence rate of (x_k) is characterized by

$$\alpha = \lim_k \frac{x_{k+1} - x_k}{x_k - x},$$

which closely resembles the ratio test in the theory of infinite series. If $0 < \alpha < 1$, then (x_k) is said to be linearly convergent. If $\alpha = 1$, then (x_k) is said to be logarithmically convergent, if $\alpha = 0$ then (x_k) is said to converge hyper-linearly and obviously $\alpha > 1$ stands for divergence of the sequence.

In 1968, D.F. Dawson [5] had characterized the summability field of matrix $A = (a_{kn})$ by showing A is convergence preserving over the set of all sequences which converge faster than some fixed sequence x or A only preserves the limit of a set of constant sequences. In 1979, Smith and Ford [24] introduced the concept of acceleration of linear and logarithmic convergence. Subsequently, Keagy and Ford [11] had established the results analogues to the results of Dawson [5] dealing with the acceleration field of subsequence transformation. Later, Brezinski, Delahaye and Gesmain-Bonne [3], Brezinski [2] and many other authors have worked in this areas (see [1,18,21,22,26,27]). The concept of acceleration convergence of real sequence as well as sequence of fuzzy real numbers in the ideal context was introduced by Dutta and Tripathy [6] and Tripathy and Mahanta [28], respectively.

The sequence $x = (x_n)$ converges to σ faster than the sequence $y = (y_n)$ converges to λ, usually written as $x < y$, if

$$\lim_{n\to\infty} \frac{|x_n - \sigma|}{|y_n - \lambda|} = 0, \text{ provided } y_n - \lambda \neq 0 \text{ for all } n \in \mathbb{N}.$$

The matrix $A = (a_{kn})$ is said to accelerate the convergence of the sequence $x = (x_n)$ if $Ax < x$. The acceleration field of A is defined to be the class of sequences $\{x = (x_n) \in w : Ax < x\}$, where w is the space of all sequences.

The sequence $x = (x_n)$ converges to σ at the same rate as the sequence $y = (y_n)$ converges to λ, written as $x \approx y$, if

$$0 < \lim - \inf \left| \frac{x_n - \sigma}{y_n - \lambda} \right| \leq \lim - \sup \left| \frac{x_n - \sigma}{y_n - \lambda} \right| < \infty.$$

Let $A = (a_{kn})$ be an infinite matrix. For a sequence $x = (x_n)$, the A-transform of x is defined as

$$Ax = \sum_{k=1}^{\infty} a_{kn} x_n, \text{ for all } n \in \mathbb{N}.$$

The subsequence (x_{k_n}) of the sequence $x = (x_n)$ can be represented by a matrix transformation Ax, where the matrix $A = (a_{kn})$ is defined by

$$a_{i,k_i} = \begin{cases} 1, & \text{for } i \in \mathbb{N}; \\ 0, & \text{otherwise} \end{cases}$$

For a matrix summabilty transformation A, we define the domain of A, usually denoted by $d(A)$, as

$$d(A) = \left\{ x = (x_n) \in w : \lim_k \sum_{n=1}^{\infty} a_{kn} x_n \text{ exists} \right\}.$$

We denote

$$l^1 = \left\{ x = (x_n) \in w : \sum_{n=1}^{\infty} |x_n| < \infty \right\},$$

$S_\delta = \{x = (x_n) \in w : x_n \geq \delta > 0 \text{ for all } n\}$,
S_0 = the set of all nonnegative sequences which have at most a finite number of zero entries.

2. Definitions and Preliminaries

In 1900, A. Pringsheim [19] introduced the notion of double sequence and presented the definition of convergence of a double sequence. Robinson [20] studied the divergent of double sequences and series. Later on Hamilton ([8], [9]) introduced the transformation of multiple sequences. Subsequently Patterson [16] introduced the concept of rate of convergence of double sequences. Recently, Hazarika [10] introduced the concept of acceleration convergence of double sequences and proved some interesting results.

In this article various notions and definitions on double sequences have been presented. Some interesting results on spaces of statistically convergent double sequences and statistically bounded double sequences have been presented.

Definition 2.1. *[19]: A double sequence $x = (x_{m,n})$ is said to converge to a number L in Pringsheim's sense, symbolically we write $P - \lim_{m,n\to\infty} x_{m,n} = L$, if for every $\varepsilon > 0$, there exists a positive integer n_0 depending upon ε, such that $|x_{m,n} - L| < \varepsilon$, whenever $m, n \geq n_0$. The number L is called the Pringsheim's limit of the sequence x.*

Definition 2.2. *[19]: A double sequence $x = (x_{m,n})$ is said to be bounded if there exists a real number $M > 0$ such that $|x_{m,n}| < M$ for all m and n.*

Definition 2.3. *[17]: The double sequence y is a double subsequence of a sequence $x = (x_{m,n})$, if there exists two increasing index sequences $\{m_j\}$ and $\{n_j\}$ such that $y = (x_{m_j,n_j})$.*

Definition 2.4. *[15]: A number β is said to be a Pringsheim limit point of the double sequence $x = (x_{m,n})$ if there exists a subsequence y of x such that $P - \lim y = \beta$.*

Definition 2.5. *[15]: Let $x = (x_{m,n})$ be a double sequence of real numbers and for each k, let $\alpha_k = \sup_k\{x_{m,n} : m, n \geq k\}$. Then the Pringsheim limit superior of x is defined as follows:*

(i) If $\alpha_k = +\infty$, for each k, then $P - \lim - \sup x = +\infty$;

(ii) If $\alpha_k < +\infty$, for each k, then $P - \lim - \sup x = \inf_k\{\alpha_k\}$.

Similarly, let $\beta_k = \inf_k\{x_{m,n} : m, n \geq k\}$. Then the Pringsheim limit inferior of x is defined as follows:

(iii) If $\beta_k = -\infty$, for each k, then $P - \lim - \sup x = -\infty$;

(iv) If $\beta_k > -\infty$, for each k, then $P - \lim - \sup x = \sup_k\{\beta_k\}$.

Definition 2.6. *A four-dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.*

Definition 2.7. *Let $A \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers. The cardinality of A, usually denoted by $|A(m,n)|$, is defined to be the number of (i,j) in A such that $i \leq m$ and $j \leq n$.*

Definition 2.8. *[12,10] A two-dimensional set of positive integers A is said to have a double natural density, if the sequence $\left(\frac{|A(m,n)|}{mn}\right)$ has a limit in Pringsheim's sense. If this exists, it is denoted by $\delta_2(A)$. Thus we have*

$$\delta_2(A) = P - \lim_{m,n\to\infty} \frac{|A(m,n)|}{mn}.$$

Clearly we have $\delta_2(A^c) = \delta_2(\mathbb{N} \times \mathbb{N} - A) = 1 - \delta_2(A)$. Further, it is also clear that all finite subsets of $\mathbb{N} \times \mathbb{N}$ have zero double natural density. Moreover, some infinite subsets also have zero density. For example, the set

$$A = \left\{(i,j) : i \in [2^k, 2^k + k) \text{ and } j \in [2^l, 2^l + l), k, l = 1, 2, 3...\right\}$$

has double natural density zero.

Let $A = \{(i^2, j^2) : i, j \in \mathbb{N}\}$. Then

$$\delta_2(A) = P - \lim_{m,n\to\infty} \frac{|A(m,n)|}{mn} \leq P - \lim_{m,n\to\infty} \frac{\sqrt{m}\sqrt{n}}{mn} = 0,$$

i.e. the set A has double natural density zero, while the set $\{(i, 3j) : i, j \in \mathbb{N}\}$ has double natural density $\frac{1}{3}$.

Definition 2.9. *[12,10] A real double sequence $x = (x_{m,n})$ is said to be statistically convergent to the number ℓ in Pringsheim's sense if for each $\varepsilon > 0$, the set*

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : |x_{m,n} - \ell| \geq \varepsilon\}$$

has double natural density zero. In this case, we write $st_2 - \lim x = \ell$ or $(x_k) \xrightarrow{st_2} \ell$ and st_2 denotes the set of all statistically convergent double sequences. It is clear that $st_2 - \lim x = \ell$ if and only if $st_2 - \liminf x = st_2 - \limsup x = \ell$.

Definition 2.10. *A double sequence $x = (x_{m,n})$ is said to satisfy a property P for "almost all (m,n)" if it satisfies the property P for all (m,n) except a set of double natural density zero. We abbreviate this by "a.a.(m,n)".*

Definition 2.11. *Let $x = (x_{m,n})$ be a double sequence of real numbers and let $\alpha_x = \{b \in \mathbb{R} : \delta_2(\{(m,n) \in \mathbb{N} \times \mathbb{N} : x_{m,n} > b\}) \neq 0\}$. Then the st_2-limit superior of x is defined as follows:*

(i) If $\alpha_x \neq \phi$, then $st_2 - \lim - \sup x = \sup \alpha_x$;

(ii) If $\alpha_x = \phi$, then $st_2 - \lim - \sup x = -\infty$.

Similarly, let $\beta_x = \{a \in \mathbb{R} : \delta_2(\{(m,n) \in \mathbb{N} \times \mathbb{N} : x_{m,n} < a\}) \neq 0\}$. *Then the* st_2*-limit inferior of* x *is defined as follows:*

(iii) If $\beta_x \neq \phi$*, then* $st_2 - \lim - \inf x = \inf \beta_x$;

(iv) If $\beta_x = \phi$*, then* $st_2 - \lim - \inf x = \infty$.

Throughout this paper we use the following notations:

${}_2w =$ the space of all double sequences of real numbers.
${}_2l_\infty =$ the space of all bounded double sequences of real numbers.
${}_2c =$ the space of all convergent in Pringsheim's sense double sequences of real numbers.
${}_2c_0 =$ the space of all null in Pringsheim's sense double sequences of real numbers.
${}_2\overline{c} =$ the space of all statistically convergent in Pringsheim's sense double sequences of real numbers.
${}_2\overline{c}_0 =$ the space of all statistically null in Pringsheim's sense double sequences of real numbers.
${}_2\overline{c}_0^B = {}_2\overline{c}_0 \cap {}_2l_\infty$.
${}_2\overline{S}_0^B =$ the subset with only finitely many nonzero entries of the space ${}_2\overline{c}_0^B$.
${}_2S_\delta =$ the set of all real double sequences $x = (x_{m,n})$ such that
$x_{m,n} \geq \delta > 0$, for all m and n.
${}_2S_0 =$ the subset with only finitely many nonzero entries of the space ${}_2c_0$.

$$ {}_2l = \left\{ x = (x_{m,n}) : \sum_{m,n=1,1}^{\infty,\infty} |x_{m,n}| < \infty \right\}. $$

$$ {}_2d(A) = \left\{ x = (x_{m,n}) : P - \lim_{k,l} \sum_{m,n=1,1}^{\infty,\infty} a_{k,l,m,n} x_{m,n} \text{ exists} \right\}. $$

μ is a mapping from ${}_2l$ to ${}_2w$ with the coordinate maps defined as

$$ \mu_{k,l}(x) = \sup_{m \geq k \text{ and } n \geq l} |x_{m,n}|, \text{ for } x \in {}_2l. $$

Definition 2.12. *Let* $A = (a_{k,l,m,n})$ *be a four-dimensional matrix. For any* $x = (x_{m,n}) \in {}_2w$*, the* A*-transform of* x *is defined as*

$$ Ax = \sum_{m,n=1,1}^{\infty,\infty} a_{k,l,m,n} x_{m,n} \text{ for all } m,n \in \mathbb{N}. \tag{2.1} $$

The subsequence (x_{m_k,n_l}) *of the sequence* $x = (x_{m,n})$ *can be represented by a matrix transformation represented by (2.1), where*

$$ a_{k,l,m,n} = \begin{cases} 1, & \text{if } (m,n) = (m_k, n_l); \\ 0, & \text{if } (m,n) \neq (m_k, n_l) \end{cases} \tag{2.2} $$

It can be easily verified that the matrix as defined in (2.2) is a RH-regular matrix.

Definition 2.13. *A matrix transformation associated with the four-dimensional matrix A is said to be an ${}_2c_0 - {}_2c_0$ if Ax is in the set ${}_2c_0$, whenever x is in ${}_2c_0$ and is bounded (for details see [15]).*

The following is an important result on the characterization of ${}_2c_0 - {}_2c_0$ matrices:

Lemma 2.14. *[8] A four-dimensional matrix $A = (a_{k,l,m,n})$ is an ${}_2c_0 - {}_2c_0$, if and only if,*

(a) $\sum_{p,q=1,1}^{\infty,\infty} |a_{k,l,p,q}| < \infty$ *for all* k, l;

(b) for $q = q_0$, *there exists* $C_q(k,l)$ *such that* $a_{k,l,p,q} = 0$, *whenever* $q > C_q(k,l)$ *for all* k, l, p;

(c) for $p = p_0$, *there exists* $C_p(k,l)$ *such that* $a_{k,l,p,q} = 0$, *whenever* $p > C_p(k,l)$ *for all* k, l, q;

(d) $P - \lim_{k,l} a_{k,l,p,q} = 0$, *for all* p *and* q.

3. Statistical acceleration convergence of double sequences

In this section statistical acceleration convergence of double sequences have been defined and some interesting theorems regarding statistical acceleration convergence of double sequences have been discussed.

In [10], Hazarika introduced the notion of acceleration convergence of double sequences as follows:

Definition 3.1. *[10] Let $x = (x_{m,n})$ and $y = (y_{m,n})$ be two double sequences of real numbers. Then the sequence x is said to converge P-faster than the sequence y, written as $x <^P y$, if*

$$P - \lim_{m,n} \frac{|x_{m,n}|}{|y_{m,n}|} = 0.$$

Definition 3.2. *[10] The sequence $x = (x_{m,n})$ is said to converge at the same rate in Pringsheim's sense as the sequence $y = (y_{m,n})$, written as $x \approx^P y$, if*

$$0 < P - \lim - \inf \left| \frac{x_{m,n}}{y_{m,n}} \right| \leq P - \lim - \sup \left| \frac{x_{m,n}}{y_{m,n}} \right| < \infty.$$

Definition 3.3. *[10] The four-dimensional matrix $A = (a_{k,l,m,n})$ is said to P-accelerate the convergence of the sequence $x = (x_{m,n})$ if $Ax <^P x$.*

We define the P-acceleration field of A as the set

$$\{x = (x_{m,n}) \in {}_2w : Ax <^P x\}.$$

Now we introduce the following definitions.

Definition 3.4. *Let $x = (x_{m,n})$ and $y = (y_{m,n})$ be two double sequences of real numbers. Then the sequence x is said to statistically converge st_2-faster than the sequence y, written as $x <^{st_2} y$, if*

$$st_2 - \lim_{m,n} \frac{|x_{m,n}|}{|y_{m,n}|} = 0.$$

Definition 3.5. *The sequence $x = (x_{m,n})$ is said to st_2-converge at the same rate as the sequence $y = (y_{m,n})$, written as $x \approx^{st_2} y$, if*

$$0 < st_2 - \lim - \inf \left|\frac{x_{m,n}}{y_{m,n}}\right| \leq st_2 - \lim - \sup \left|\frac{x_{m,n}}{y_{m,n}}\right| < \infty.$$

Definition 3.6. *The four-dimensional matrix A=$(a_{k,l,m,n})$ is said to st_2-accelerate the convergence of the sequence $x = (x_{m,n})$ if $Ax <^{st_2} x$.*

We define the st_2-acceleration field of A as the set

$$\{x = (x_{m,n}) \in {}_2w : Ax <^{st_2} x\}.$$

Theorem 3.7. *Let $x = (x_{m,n})$ and $y = (y_{m,n})$ be two elements of ${}_2\overline{S}_0^B$ such that $x <^{st_2} y$, then there exists an element $z = (z_{m,n})$ in ${}_2\overline{S}_0^B$ such that $x <^{st_2} z <^{st_2} y$.*

Proof: Let $x, y \in {}_2\overline{S}_0^B$ be such that $x <^{st_2} y$. Define the sequence $z = (z_{m,n})$ as follows:

$$z = x_{m,n}^{\frac{4}{5}}\ y_{m,n}^{\frac{1}{5}}.$$

This implies that $x <^{st_2} z <^{st_2} y$. □

Theorem 3.8. *Let $x <^{st_2} y$ and $y \approx^{st_2} z$, then $x <^{st_2} z$.*

The proof is omitted as it is straight forward.

Theorem 3.9. *Let $x = (x_{m,n}), y = (y_{m,n}) \in {}_2\overline{S}_0^B$, then the following are equivalent:*

(a) $x <^{st_2} y$.

(b) there exists $x' = (x'_{m,n}), y' = (y'_{m,n}) \in {}_2S_0$ such that $x_{m,n} = x'_{m,n}$ for a.a.(m.n); $y_{m,n} = y'_{m,n}$ for a.a.(m.n) and $x' <^P y'$.

(c) there exists a subset $K = \{(m_i, n_j) : i, j \in \mathbb{N}\}$ of $\mathbb{N} \times \mathbb{N}$ such that $\delta_2(K) = 1$ and $(x_{m_i,n_j}) <^P (y_{m_i,n_j})$.

Proof: (a)⇒ (b). Let $x = (x_{m,n}) \in {}_2\overline{S}_0^B$, then there exists a subset A of $\mathbb{N} \times \mathbb{N}$ with $\delta_2(A) = 1$ such that $P - \lim_{m,n} x_{m,n} = 0$ over A.
Next $y = (y_{m,n}) \in {}_2\overline{S}_0^B$, implies there exists a subset B of $\mathbb{N} \times \mathbb{N}$ with $\delta_2(B) = 1$

such that $P-\lim_{m,n} x_{m,n}=0$ over B.
Since $x <^{st_2} y$, then there exists a subset C of $\mathbb{N}\times\mathbb{N}$ with $\delta_2(C)=1$ such that

$$P-\lim_{m,n}\frac{|x_{m,n}|}{|y_{m,n}|}=0 \text{ over } C.$$

Let $D=A\cap B\cap C$, then clearly $\delta_2(D)=1$.
Define the sequences $x'=(x'_{m,n})$ and $y'=(y'_{m,n})$ as follows:

$$x'_{m,n}=\begin{cases} x_{m,n} & \text{if } (m,n)\in D;\\ (mn)^{-3} & \text{otherwise}\end{cases}$$

and

$$y'_{m,n}=\begin{cases} x_{m,n} & \text{if } (m,n)\in D;\\ (mn)^{-2} & \text{otherwise}\end{cases}$$

Then clearly $x'=(x'_{m,n}), y'=(y'_{m,n})\in {}_2S_0, x_{m,n}=x'_{m,n}$ for $a.a.(m,n)$ and $y_{m,n}=y'_{m,n}$ for $a.a.(m,n)$. Also $x' <^P y'$.

(b)$\Rightarrow$ (c). Let $x'=(x'_{m,n}), y'=(y'_{m,n})\in {}_2S_0$ be such that $x_{m,n}=x'_{m,n}$ for $a.a.(m.n)$; $y_{m,n}=y'_{m,n}$ for $a.a.(m.n)$ and $x' <^P y'$.

Let $E=\{(m,n)\in\mathbb{N}\times\mathbb{N}: x_{m,n}=x'_{m,n}\}$ and $F=\{(m,n)\in\mathbb{N}\times\mathbb{N}: y_{m,n}=y'_{m,n}\}$. Then we have $\delta_2(E)=1=\delta_2(F)$. Let $K=E\cap F$. Then $\delta_2(K)=1$. Since $K\subset\mathbb{N}\times\mathbb{N}$, we can enumerate K as $K=\{(m_i,n_j): i,j\in\mathbb{N}\}$.

Then we have $(x_{m_i,n_j})=(x'_{m_i,n_j})\in {}_2S_0$ and $(y_{m_i,n_j})=(y'_{m_i,n_j})\in {}_2S_0$. Also

$$P-\lim_{i,j}-\frac{|x_{m_i,n_j}|}{|y_{m_i,n_j}|}=P-\lim_{i,j}\frac{|x'_{m_i,n_j}|}{|y'_{m_i,n_j}|}=0.$$

Hence $(x_{m_i,n_j}) <^P (y_{m_i,n_j})$.

(c)$\Rightarrow$ (a). It is obvious from the definition.

This establishes the result. □

Lemma 3.10. *A four-dimensional matrix $A=(a_{k,l,m,n})$ is an ${}_2\bar{c}_0-{}_2\bar{c}_0$, if and only if,*

(a) $\sum_{p,q=1,1}^{\infty,\infty}|a_{k,l,p,q}|<\infty$ *for all* k,l;

(b) for $q=q_0$, *there exists* $C_q(k,l)$ *such that* $a_{k,l,p,q}=0$, *whenever* $q>C_q(k,l)$ *for all* k,l,p;

(c) for $p=p_0$, *there exists* $C_p(k,l)$ *such that* $a_{k,l,p,q}=0$, *whenever* $p>C_p(k,l)$ *for all* k,l,q;

(d) $st_2 - \lim_{k,l} a_{k,l,p,q} = 0$ *for all* p *and* q.

Theorem 3.11. *Let* A *be a nonnegative* ${}_2\overline{c}_0 - {}_2\overline{c}_0$ *summability matrix and let* x *and* y *be two elements in* ${}_2l$ *such that* $x <^{st_2} y$ *with* $x \in {}_2\overline{S}_0^B$ *and* $y \in {}_2S_\delta$ *for some* $\delta > 0$, *then* $\mu(Ax) <^{st_2} \mu(Ay)$.

Proof: Since $x <^{st_2} y$, then there exists a bounded double sequence $z = (z_{m,n})$ with Pringsheim's limit zero on some subset K of $\mathbb{N} \times \mathbb{N}$ with $\delta_2(K) = 1$, such that $x_{m,n} = y_{m,n}\ z_{m,n}$. For each k and l, we have

$$\begin{aligned}
\frac{\mu_{k,l}(Ax)}{\mu_{k,l}(Ay)} &= \frac{\sup_{r\geq k \text{ and } s\geq l}(Ax)_{r,s}}{\sup_{r,s\geq k,l}(Ay)_{r,s}} \\
&= \frac{\sup_{r\geq k \text{ and } s\geq l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ x_{m,n}}{\sup_{r\geq k \text{ and } s\geq l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ y_{m,n}} \\
&= \frac{\sup_{r\geq k \text{ and } s\geq l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ y_{m,n}\ z_{m,n}}{\sup_{r\geq k \text{ and } s\geq l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ y_{m,n}} \\
&\leq \frac{\sup_{r\geq k \text{ and } s\geq l}\left|\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ y_{m,n}\ z_{m,n}\right|}{\sup_{r\geq k \text{ and } s\geq l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ y_{m,n}} \\
&\leq \frac{\sup_{r\geq k \text{ and } s\geq l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ y_{m,n}\ |z_{m,n}|}{\sup_{r\geq k \text{ and } s\geq l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ y_{m,n}} \\
&\leq \frac{\sup_{r\geq k \text{ and } s\geq l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ y_{m,n}\ |z_{m,n}|}{\delta\sup_{r\geq k \text{ and } s\geq l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}}.
\end{aligned}$$

Since y and z are bounded real double sequences with z is in ${}_2\overline{c}_0$ and A is a nonnegative ${}_2\overline{c}_0 - {}_2\overline{c}_0$ matrix , then

$$st_2 - \lim_{k,l} \sup_{r\geq k \text{ and } s\geq l} \sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ y_{m,n}\ |z_{m,n}| = 0.$$

Hence

$$st_2 - \lim_{k,l}\frac{\mu_{k,l}(Ax)}{\mu_{k,l}(Ay)} \leq 0. \tag{3.1}$$

In a similar manner we can establish

$$st_2 - \lim_{k,l}\frac{\mu_{k,l}(Ax)}{\mu_{k,l}(Ay)} \geq 0. \tag{3.2}$$

Hence from (3.1) and (3.2), we have

$$st_2 - \lim_{k,l}\frac{\mu_{k,l}(Ax)}{\mu_{k,l}(Ay)} = 0$$

which implies $\mu(Ax) <^{st_2} \mu(Ay)$. This establishes the result. $\square$

Theorem 3.12. *Let $x = (x_{m,n}) \in {}_2\overline{S}_0^B$ and A be a subsequence transformation such that $Ax <^{st_2} x$. Then there exists $y = (y_{m,n}) \in {}_2S_0^B$ such that $x_{m,n} = y_{m,n}$ a.a.(m,n) and $Ay <^P y$.*

Proof: Let $x = (x_{m,n}) \in {}_2\overline{S}_0^B$. Then there exists a subset $B_1 \subset \mathbb{N} \times \mathbb{N}$ with $\delta_2(B_1) = 1$ such that

$$P - \lim\ x_{m,n} = 0 \text{ over } B_1.$$

Let $(x_{m_k,n_l}) \in {}_2S_0^B$. Then there exists a subset $B_2 \subset \mathbb{N} \times \mathbb{N}$ with $\delta_2(B_2) = 1$ such that

$$P - \lim\ x_{m_k,n_l} = 0 \text{ over } B_2.$$

Since $Ax <^{st_2} x$, we have

$$st_2 - \lim\ \frac{|x_{m_k,n_l}|}{|x_{m,n}|} = 0.$$

Then there exists a subset $B_3 \subset \mathbb{N} \times \mathbb{N}$ with $\delta_2(B_3) = 1$ such that

$$P - \lim\ \frac{|x_{m_k,n_l}|}{|x_{m,n}|} = 0 \text{ over } B_3.$$

Let $D = B_1 \cap B_2 \cap B_3$. Then clearly $\delta_2(D) = 1$.
For $r \neq m_k, s \neq n_l, (k,l) \in \mathbb{N} \times \mathbb{N}$, let us define the sequence $y = (y_{m,n})$ as follows:

$$y_{r,s} = \begin{cases} x_{r,s} & \text{if } (r,s) \in D; \\ (rs)^{-3} & \text{otherwise} \end{cases}$$

and

$$y_{m_k,n_l} = \begin{cases} x_{m_k,n_l} & \text{if } (k,l) \in D; \\ y_{m,n}\ (mn)^{-3} & \text{otherwise} \end{cases}$$

Then we have $y = (y_{m,n}) \in {}_2S_0^B$ such that $x_{m,n} = y_{m,n}$ *a.a.(m,n)* and this implies $Ay <^P y$. □

Remark 3.13. *The converse of the above theorem is not always true. This follows from the following example.*

Example 1. Let $A = (a_{k,l,m,n})$ be a subsequence transformation defined by $a_{m,n,m_k,n_l} = 1$, and $a_{p,q,r,s} = 0$, otherwise. Let $D = \{(m_i, n_j) \in \mathbb{N} \times \mathbb{N} : m_i = i^2, n_j = j^2, i, j \in \mathbb{N}\}$. Let us define $y = (y_{m,n})$ as $y_{m,n} = (mn)^{-3}$ for all $m, n \in \mathbb{N}$ and $x = (x_{m,n})$ as

$$x_{m,n} = \begin{cases} 1 & \text{for all } (m,n) \in D; \\ (mn)^{-3} & \text{for all} (m,n) \notin D \end{cases}$$

Then $Ay <^P y$ but $Ax <^{st_2} x$ does not hold.

Theorem 3.14. *Let $x = (x_{m,n}) \in {}_2\overline{S}_0^B$ and A be a subsequence transformation. Then the following are equivalent.*

(a) $Ax <^{st_2} x$.

(b) Then there exists a subset $K = \{(m_i, n_j) : i, j \in \mathbb{N}\}$ *of* $\mathbb{N} \times \mathbb{N}$ *such that* $\delta_2(K) = 1$ *and* $Az <^P z$, *where* $z = (x_{m_i,n_j})$.

Proof: (a)$\Rightarrow$ (b) is obvious in view of the Theorem 3.9.

(b)$\Rightarrow$ (a) is obvious from the definition.

This completes the proof. □

Theorem 3.15. *Let $x = (x_{m,n}) \in {}_2\overline{S}_0^B$ and A be a subsequence transformation such that $Ax <^{st_2} x$. Then there exists $y = (y_{m,n}) \in {}_2\overline{S}_0^B$ such that $x <^{st_2} y$ and $Ay <^P y$.*

Proof: Consider the sequence

$$y_{m,n} = |x_{m,n}|^{\frac{1}{2}} \text{ for all } m, n \in \mathbb{N}.$$

Then clearly $y = (y_{m,n}) \in {}_2\overline{S}_0^B$ such that $x <^{st_2} y$ and $Ay <^P y$. This establishes the theorem. □

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Bipan Hazarika
Department of Mathematics, Rajiv Gandhi University,
Rono Hills, Doimukh-791 112, Arunachal Pradesh, India
E-mail address: `bh_rgu@yahoo.co.in`