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Pathway Fractional Integral Operators of Generalized K-Wright Function and K_4-Function

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abstract: In the present work we introduce a composition formula of the pathway fractional integration operator with finite product of generalized K-Wright function and K_4-function. The obtained results are in terms of generalized Wright function. Certain special cases of the main results given here are also considered to correspond with some known and new (presumably) pathway fractional integral formulas.

Key Words: Pathway fractional integral operator, Generalized K-Wright function, K_4-function, Generalized M-series, Special Function, Fractional calculus

Contents

1. Introduction and Preliminaries

In recent years, the fractional calculus has become one of the most rapidly growing research subject of mathematical analysis due to its numerous applications in various parts of science as well as mathematics.
The importance of the role played by the Wright function in partial differential equation of fractional order is well known and was widely treated in papers by several authors including Gorenflo, Luchko, Mainardi [5], Mainardi [14] and many more.
Throughout this paper $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers respectively, also $\mathbb{R}^+ = (0, \infty), \mathbb{N}_0 = 0, 1, ...$ and $\mathbb{Z}^- = -1, -2, ...$

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Let $\alpha_i, \beta_j \in \mathbb{R}\backslash\{0\}$ and $a_i, b_j \in \mathbb{C}$, $i = \left(\overline{1,p}\right); j = \left(\overline{1,q}\right)$, then the Generalized Wright function is defined for $z \in \mathbb{C}$ by Wright [30] in the following manner:

$$ {}_p\psi_q(z) = {}_p\psi_q\left[\begin{array}{c|c} (a_i,\alpha_i)_{1,p} & \\ (b_j,\beta_j)_{1,q} & z \end{array}\right] := \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+n\alpha_i)}{\prod_{j=1}^{q}\Gamma\left(b_j+n\beta_j\right)}\frac{z^n}{n!}, \tag{1.1}$$

where $\Gamma(z)$ is the Euler gamma function [3]. The conditions for existence of (1.1) together with its representation in terms of the Mellin-Barnes integral and of the H-function were established by Kilbas et al. [8].

Generalized k- Gamma function $\Gamma_k(z)$ defined as [2]

$$\Gamma_k(z) = \lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{z}{k}-1}}{(z)_{n,k}}, \quad k \in \mathbb{R}^+,\ z \in \mathbb{C}\backslash k\mathbb{Z}^-, \tag{1.2}$$

and $(z)_{n,k}$ is the k-Pochammer symbol [2] defined for complex $z \in \mathbb{C}$ and $k \in \mathbb{R}$ by

$$(z)_{n,k} = \begin{cases} 1 & if\ \ n=0 \\ z(z+k)(z+2k)\ldots(z+(n-1)k) & if\ \ n\in\mathbb{N} \end{cases}. \tag{1.3}$$

For $\Re(z) > 0$ and $k \in \mathbb{R}^+$, then $\Gamma_k(z)$ defined as the integral

$$\Gamma_k(z) = \int_0^{\infty} t^{z-1}e^{-\frac{t^k}{k}}dt, \tag{1.4}$$

due to which the following identity holds [2]:

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right). \tag{1.5}$$

Recently, Gehlot and Prajapati [4], introduced the following generalized K-Wright function defined in terms of k-gamma function by the series

$$ {}_p\psi_q^k(z) = {}_p\psi_q^k\left[\begin{array}{c|c} (a_i,\alpha_i)_{1,p} & \\ (b_j,\beta_j)_{1,q} & z \end{array}\right] := \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k(a_i+n\alpha_i)}{\prod_{j=1}^{q}\Gamma_k\left(b_j+n\beta_j\right)}\frac{z^n}{n!}, \tag{1.6}$$

where $k \in \mathbb{R}^+$ and $(a_i+n\alpha_i), \left(b_j+n\beta_j\right) \in \mathbb{C}\backslash k\mathbb{Z}^-\forall n \in \mathbb{N}_0$. Here, $\Gamma_k(z)$ is defined in (1.2). For $k=1$, the generalized K-Wright function reduces to the generalized Wright function. Wright function [30], generalized Mittag-Leffler function [8] and Bessel-Maitland function [15] are some other particular cases.

Here, in the present paper, we aim at establishing a (presumably) new fractional integration formula of pathway type involving the generalized K-Wright function. Some interesting special cases of our main result are also considered.
Recently, Nair [[20], p. 239] introduce a pathway fractional integral operator by using the pathway idea of Mathai [16].
Let $f(x) \in L(a,b)$, $\eta \in \mathbb{C}$, $\Re(\eta) > 0$, $a > 0$ and let us take a "‘pathway parameter"’ $\alpha < 1$. Then the pathway fractional integration operator is defined and represented as follows:

$$\left(P_{0+}^{(\eta,\alpha,a)}f\right)(x) = x^{\eta}\int_0^{\frac{x}{a(1-\alpha)}}\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}}f(t)\,dt, \tag{1.7}$$

where $L(a,b)$ is the set of Lebesgue measurable function defined on (a,b).

If we set $\alpha=0$, $a=1$, and replacing η by $\eta-1$ in (1.7) then we have the following relationship:

$$\left(P_{0+}^{(\eta-1,0,1)}f\right)(x)=\Gamma(\eta)\left[\left(I_{0+}^{\eta}f\right)(x)\right], \tag{1.8}$$

where I_{0+}^{η} is the left-sided Riemann-Liouville fractional integral operator [23].

2. Pathway Fractional Integration of Generalized K-Wright Function

In this section we consider composition of the pathway fractional integral $P_{0+}^{(\eta,\alpha,a)}$ given by (1.7) with the generalized K-Wright function (1.6).

Theorem 2.1. *Let $\alpha<1$, the parameters $z,\eta,\rho\in\mathbb{C}$, $\alpha_i,\beta_j\in\mathbb{R}\backslash\{0\}$, $a_i,b_j\in\mathbb{C}$, $i=\left(\overline{1,p}\right); j=\left(\overline{1,q}\right)$, $\Re(\eta)>0$, $a>0$, $\Re(\rho+\sigma)>0$, $\sigma>0$ and $\Re\left(\frac{\eta}{1-\alpha}\right)>-1$, then there holds the following formula:*

$$P_{0+}^{(\eta,\alpha,a)}\left\{z^{\rho-1}\,{}_p\psi_q^k\left(cz^{\sigma}\right)\right\}=\frac{\prod_{j=1}^{q}k^{\frac{b_j+\beta_j n}{k}-1}\,z^{\rho+\eta}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\prod_{i=1}^{p}k^{\frac{a_i+\alpha_i n}{k}-1}\left[a(1-\alpha)\right]^{\rho}}$$
$$\times\ {}_{p+1}\psi_{q+1}\left[\begin{matrix}\left(\frac{a_i}{k},\frac{\alpha_i}{k}\right)_1^p,(\rho,\sigma);\\ \left(\frac{b_j}{k},\frac{\beta_j}{k}\right)_1^q,\left(1+\rho+\frac{\eta}{1-\alpha},\sigma\right);\end{matrix}\ c\left(\frac{z}{a(1-\alpha)}\right)^{\sigma}\right]. \tag{2.1}$$

Proof: Let the left-hand side of the (2.1) be denoted by I. Applying (1.6) and using the (1.7) to (2.1), we get

$$\begin{aligned}I&=P_{0+}^{(\eta,\alpha,a)}\left\{z^{\rho-1}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k\left(a_i+n\alpha_i\right)}{\prod_{j=1}^{q}\Gamma_k\left(b_j+n\beta_j\right)}\frac{\left(cz^{\sigma}\right)^n}{n!}\right\}\\&=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k\left(a_i+n\alpha_i\right)}{\prod_{j=1}^{q}\Gamma_k\left(b_j+n\beta_j\right)}\frac{c^n}{n!}P_{0+}^{(\eta,\alpha,a)}\left\{z^{\rho+\sigma n-1}\right\}.\end{aligned}$$

By using the known relationship between the Beta function $B(\alpha,\beta)$ and the Gamma function [20,28]

$$P_{0+}^{(\eta,\alpha,a)}\left\{t^{\mu-1}\right\}=\frac{t^{\eta+\mu}}{\left[a(1-\alpha)\right]^{\mu}}\frac{\Gamma(\mu)\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma\left(\frac{\eta}{1-\alpha}+\mu+1\right)},\quad\left(\alpha<1;\Re(\eta)>0;\Re(\mu)>0\right). \tag{2.2}$$

Here, applying (2.2) with μ replaced by $(\rho+\sigma n)$ to the pathway integral and using (1.5), after a little simplification, we obtain the following result

$$I=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}k^{\frac{a_i+\alpha_i n}{k}-1}\Gamma\left(\frac{a_i+\alpha_i n}{k}\right)}{\prod_{j=1}^{q}k^{\frac{b_j+\beta_j n}{k}-1}\Gamma\left(\frac{b_j+\beta_j n}{k}\right)}\frac{\Gamma(\rho+\sigma n)\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma\left(1+\frac{\eta}{1-\alpha}+\rho+\sigma n\right)}\frac{c^n\,z^{\eta+\rho+\sigma n}}{\left[a(1-\alpha)\right]^{\rho+\sigma n}\,n!} \tag{2.3}$$

$$= \frac{\prod_{j=1}^{q} k^{\frac{b_j+\beta_j n}{k}-1} z^{\rho+\eta} \Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\prod_{i=1}^{p} k^{\frac{a_j+\alpha_i n}{k}-1} \left[a\left(1-\alpha\right)\right]^{\rho}}$$

$$\times \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma\left(\frac{a_i}{k}+\frac{\alpha_i}{k}n\right) \Gamma\left(\rho+\sigma n\right)}{\prod_{j=1}^{q} \Gamma\left(\frac{b_j}{k}+\frac{\beta_j}{k}n\right) \Gamma\left(1+\frac{\eta}{1-\alpha}+\rho+\sigma n\right)} \frac{c^n z^{\sigma n}}{\left[a\left(1-\alpha\right)\right]^{\sigma n} n!},$$

then, in view of (1.1), is easy to arrive at the expression of (2.1). This completes the proof. □

Setting $p = 1$ and $q = 1$ in the Theorem 2.1 yields the following result:

Corollary 2.2. *Let $\alpha < 1$, the parameters $z, \eta, \rho, a, b \in \mathbb{C}$, $\alpha, \beta \in \mathbb{R}\setminus\{0\}$, $\Re(\eta) > 0$, $d > 0$, $\Re(\rho+\sigma) > 0$, $\sigma > 0$ and $\Re\left(\frac{\eta}{1-\alpha}\right) > -1$, then there holds the following formula:*

$$P_{0+}^{(\eta,\lambda,d)} \left\{ z^{\rho-1} \, {}_1\psi_1^k \left(cz^{\sigma}\right) \right\} = \frac{k^{\frac{b+\beta n}{k}-1} z^{\rho+\eta} \Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{k^{\frac{a+\alpha n}{k}-1} \left[d\left(1-\lambda\right)\right]^{\rho}}$$

$$\times \, {}_2\psi_2 \left[\begin{matrix} \left(\frac{a}{k}, \frac{\alpha}{k}\right), (\rho, \sigma); \\ \left(\frac{b}{k}, \frac{\beta}{k}\right), \left(1+\rho+\frac{\eta}{1-\lambda}, \sigma\right); \end{matrix} \; c\left(\frac{z}{d\left(1-\lambda\right)}\right)^{\sigma} \right]. \tag{2.4}$$

3. The K_4-function and its relationship with other special functions

The K_4-function introduced by Sharma [26], is defined by the power series

$$K_4^{(\alpha,\beta,\gamma),(c,d):(p,q)}\left(a_1, ..., a_p; b_1, ..., b_q; x\right) = \sum_{n=0}^{\infty} \frac{\left(a_1\right)_n ... \left(a_p\right)_n \left(\gamma\right)_n c^n \left(x-d\right)^{(n+\gamma)\alpha-\beta-1}}{\left(b_1\right)_n ... \left(b_q\right)_n n! \, \Gamma\left(\left(n+\gamma\right)\alpha-\beta\right)}, \tag{3.1}$$

where $\alpha, \beta, \gamma, x \in \mathbb{C}$, $\Re(\alpha\gamma-\beta) > 0$; $(a_i)_n$ $(i = 1, 2, ..., p)$ and $(b_j)_n$ $(j = 1, 2, ..., q)$ are the Pochhammer symbols which are defined as follows:

$$\left(\gamma\right)_n = \frac{\Gamma\left(\gamma+n\right)}{\Gamma\left(\gamma\right)} = \begin{cases} 1, & (n = 0, \gamma \neq 0) \\ \gamma\left(\gamma+1\right)...\left(\gamma+n-1\right), & (n \in \mathbb{N}, \gamma \in \mathbb{C}) \end{cases}. \tag{3.2}$$

The series (3.1) is defined when none of the parameters $b_j s$ is a negative integer or zero. If any numerator parameter a_i is a negative or zero, then the series (3.1) terminates to a polynomial in x. The series is convergent for all x if $p > q+1$. When $r = s+1$ and $|x| = 1$, the series can convergence in some cases.
If we set $\beta = \alpha - \beta$, $\gamma = 1$, $c = 1$ and $d = 0$ in (3.1), then we obtain the following relation:

$$K_4^{(\alpha,\alpha-\beta,1),(1,0):(p,q)}\left(a_1, ..., a_p; b_1, ..., b_q; x\right) = x^{\beta-1} \, {}_pM_q^{\alpha,\beta}\left(a_1, ..., a_p; b_1, ..., b_q; x\right), \tag{3.3}$$

where ${}_pM_q^{\alpha,\beta}$ is the generalized M-series [12,27], and defined as

$$ {}_pM_q^{\alpha,\beta}\left(a_1,...,a_p;\ b_1,...,b_q;\ x\right)=\sum_{n=0}^{\infty}\frac{\left(a_1\right)_n\ ...\ \left(a_p\right)_n}{\left(b_1\right)_n\ ...\ \left(b_q\right)_n}\frac{x^n}{\Gamma\left(\alpha n+\beta\right)}, \tag{3.4}$$

where $\alpha,\beta,x\in\mathbb{C}$, $\Re\left(\alpha\right)>0$ and $\left(a_j\right)_n,\left(b_j\right)_n$ are known Pochhammer symbols. New we state further relations with other Special functions.

(i). If we set $p=1=q$ and $a_i=1$, $b_j=1$ in (3.1), then we arrive at the following relation:

$$K_4^{(\alpha,\beta,\gamma),(c,d):(1,1)}\left(1;\ 1;\ x\right)=G_{\alpha,\beta,\gamma}\left(c,d,x\right)=\sum_{n=0}^{\infty}\frac{\left(\gamma\right)_n\ c^n\ \left(x-d\right)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\left(\left(n+\gamma\right)\alpha-\beta\right)}, \tag{3.5}$$

where $G_{\alpha,\beta,\gamma}\left(c,d,x\right)$ is the G-function (but not the Meijer's G-function) defined by Lorenzo and Hartley [13] (see also, [25]).

(ii). Further, if we put $\gamma=1$ in (3.5), then K_4-function readily yields the following relationship with R-function:

$$K_4^{(\alpha,\beta,1),(c,d):(1,1)}\left(1;\ 1;\ x\right)=R_{\alpha,\beta}\left(c,d,x\right)=\sum_{n=0}^{\infty}\frac{c^n\ \left(x-d\right)^{(n+1)\alpha-\beta-1}}{\Gamma\left(\left(n+1\right)\alpha-\beta\right)}, \tag{3.6}$$

where $x>d\geq 0$, $\alpha\geq 0$, $\Re\left(\alpha-\beta\right)>0$; and $R_{\alpha,\beta}\left(c,d,x\right)$ is the R-function defined by Lorenzo and Hartley [13].

(iii). If we take $d=\beta=0$ in (3.6), we get

$$K_4^{(\alpha,0,1),(c,0):(1,1)}\left(1;\ 1;\ x\right)=F_\alpha\left(c,x\right)=\sum_{n=0}^{\infty}\frac{c^n\ \left(x\right)^{(n+1)\alpha-1}}{\Gamma\left(\left(n+1\right)\alpha\right)}, \tag{3.7}$$

where $F_\alpha\left(c,x\right)$ is the F-function defined by Robotnov and Hartley, for example see [6].

Next, we define some more relationship between K_4-function and various special functions.

Mittag-Leffler function [18,19]:

$$K_4^{(\alpha,\alpha-1,1),(-c,0):(1,1)}\left(1;\ 1;\ x\right)=E_\alpha(-cx^\alpha)\ =\ \sum_{n=0}^{\infty}\frac{(-c)^n\ x^{n\alpha}}{\Gamma\left(n\alpha+1\right)}. \tag{3.8}$$

Agarwal's function [1]:

$$K_4^{(\alpha,\alpha-\beta,1),(1,0):(1,1)}\left(1;\ 1;\ x\right)\ =\ E_{\alpha,\beta}\left(x^\alpha\right)\ =\ \sum_{n=0}^{\infty}\frac{x^{n\alpha+\beta-1}}{\Gamma\left(n\alpha+\beta\right)}. \tag{3.9}$$

Erdélyi's function [29]:

$$K_4^{(\alpha,\alpha-\beta,1),(1,0):(1,1)}(1;\ 1;\ x)=x^{\beta-1}E_{\alpha,\beta}(x^\alpha)=x^{\beta-1}\sum_{n=0}^{\infty}\frac{x^{n\alpha}}{\Gamma\left(n\alpha+\beta\right)},\ \alpha>0,\ \beta>0\,. \tag{3.10}$$

Miller and Ross's function:

$$K_4^{(1,-\beta,1),(c,0):(1,1)}(1;\,1;\,x) = E_x(\beta,c) \; = \; \sum_{n=0}^{\infty} \frac{c^n\;x^{n+\beta}}{\Gamma\left(n+\beta+1\right)}\,. \tag{3.11}$$

Generalized Mittag-Leffler function [7]:

$$K_4^{(\alpha,\beta,1),(c,0):(1,1)}(1;\,1;\,x) = x^{\alpha-\beta-1}E_{\alpha,\alpha-\beta}\left(cx^{\alpha}\right) = \sum_{n=0}^{\infty} \frac{c^n\;x^{(n+1)\alpha-\beta-1}}{\Gamma\left(\left(n+1\right)\alpha-\beta\right)}\,. \tag{3.12}$$

New generalized Mittag-Leffler function [21]:

$$K_4^{(\alpha,\beta,\gamma),(c,0):(1,1)}(1;\,1;\,x) = x^{\alpha\gamma-\beta-1}E^{\gamma}_{\alpha,\alpha\gamma-\beta}(cx^{\alpha}) = x^{\alpha\gamma-\beta-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\,c^n\;x^{\alpha n}}{n!\Gamma\left((n+\gamma)\,\alpha-\beta\right)}. \tag{3.13}$$

Wright function [17]:

$$K_4^{(\alpha,\beta,\gamma),(c,0):(1,1)}(1;\,1;\,x) = \frac{x^{\alpha\gamma-\beta-1}}{\Gamma(\gamma)}\;{}_1\Psi_1\left[\begin{array}{c}(\gamma,1)\\(\alpha\gamma-\beta,\alpha)\end{array};cx^{\alpha}\right], \tag{3.14}$$

where ${}_1\Psi_1(x)$ is special case of the generalized Wright's hypergeometric function ${}_p\Psi_q(x)$ defined in (1.1).

H function [17]:

$$K_4^{(\alpha,\beta,\gamma),(c,0):(1,1)}(1;\,1;\,x) = \frac{x^{\alpha\gamma-\beta-1}}{\Gamma(\gamma)}\;H^{1,1}_{1,2}\left[-cx^{\alpha}\left|\begin{array}{c}(1-\gamma,1)\\(0,1)\,,(1-\alpha\gamma+\beta,\alpha)\end{array}\right.\right]. \tag{3.15}$$

$\overline{H}$ *function* [17]:

$$K_4^{(\alpha,\beta,\gamma),(c,0):(1,1)}(1;\,1;\,x) = \frac{x^{\alpha\gamma-\beta-1}}{\Gamma(\gamma)}\;\overline{H}^{1,1}_{1,2}\left[-cx^{\alpha}\left|\begin{array}{c}(1-\gamma,1;1)\\(0,1)\,,(1-\alpha\gamma+\beta,\alpha;1)\end{array}\right.\right]. \tag{3.16}$$

In the next section, we apply the integral operator (1.7) to the K_4-function defined as in (3.1) and express the image in terms of generalized Wright hypergeometric functions.

4. Pathway Fractional Integration of K_4-function

In this section we consider composition of the pathway fractional integral $P_{0+}^{(\eta,\lambda,c)}$ given by (1.7) with the generalized K_4-function (3.1).

Theorem 4.1. *Let $\lambda < 1$, the parameters $x, \alpha, \beta, \gamma, \eta, \rho \in \mathbb{C}$, $\Re(\alpha\gamma - \beta) > 0$, $\Re(\eta) > 0$, $d > 0$, $\Re(\rho) > 0$ and $\Re\left(\frac{\eta}{1-\lambda}\right) > -1$, then there holds the following formula:*

$$P_{0+}^{(\eta,\lambda,d)}\left\{(x-c)^{\rho} K_4^{(\alpha,\beta,\gamma),(a,c):(p,q)}(x)\right\} = \frac{\left\{\prod_{j=1}^{q}\Gamma(b_j)\right\}(x-c)^{\eta+\gamma\alpha+\rho-\beta}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\left\{\prod_{i=1}^{p}\Gamma(a_i)\right\}\Gamma(\gamma)\,[d(1-\lambda)]^{\gamma\alpha+\rho-\beta}}$$
$$\times\ {}_{p+2}\psi_{q+2}\left[\begin{array}{c}(a_i,1)_1^p,(\gamma,1),(\gamma\alpha+\rho-\beta,\alpha);\\ (b_j,1)_1^q,(\gamma\alpha-\beta,\alpha),\left(1+\gamma\alpha+\rho-\beta+\frac{\eta}{1-\lambda},\alpha\right);\end{array}\ a\left(\frac{x-c}{d(1-\lambda)}\right)^{\alpha}\right]. \tag{4.1}$$

Proof: Let the left-hand sidee of the (4.1) be denoted by J. By applying (3.1) and using (1.7) to left-hand side of (4.1), then we have

$$J = P_{0+}^{(\eta,\lambda,d)}\left\{(x-c)^{\rho}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n\,(\gamma)_n\,a^n\,(x-c)^{(n+\gamma)\alpha-\beta-1}}{(b_1)_n\ldots(b_q)_n\,n!\,\Gamma((n+\gamma)\alpha-\beta)}\right\}$$
$$= \sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n\,(\gamma)_n\,a^n}{(b_1)_n\ldots(b_q)_n\,n!\,\Gamma((n+\gamma)\alpha-\beta)}\,P_{0+}^{(\eta,\lambda,d)}\left\{(x-c)^{(n+\gamma)\alpha+\rho-\beta-1}\right\}.$$

By applying (2.2) and after a little simplification, we arrive at

$$J = \sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n\,(\gamma)_n\,a^n}{(b_1)_n\ldots(b_q)_n\,n!\,\Gamma((n+\gamma)\alpha-\beta)}\,\frac{(x-c)^{\eta+(n+\gamma)\alpha+\rho-\beta}}{[d(1-\lambda)]^{(n+\gamma)\alpha+\rho-\beta}}$$
$$\times\frac{\Gamma((n+\gamma)\alpha+\rho-\beta)\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\Gamma\left(\frac{\eta}{1-\lambda}+(n+\gamma)\alpha+\rho-\beta+1\right)}, \tag{4.2}$$

next we use (3.2), then we obtain

$$= \sum_{n=0}^{\infty}\frac{\Gamma(a_1+n)\ldots\Gamma(a_p+n)\,\Gamma(b_1)\ldots\Gamma(b_q)\,\Gamma(\gamma+n)\,a^n}{\Gamma(b_1+n)\ldots\Gamma(b_q+n)\,\Gamma(a_1)\ldots\Gamma(a_p)\,\Gamma(\gamma)\,n!}\,\frac{(x-c)^{\eta+\gamma\alpha+\rho-\beta}}{[d(1-\lambda)]^{\gamma\alpha+\rho-\beta}}$$
$$\times\frac{(x-c)^{n\alpha}}{[d(1-\lambda)]^{n\alpha}}\,\frac{\Gamma(\gamma\alpha+\rho+n\alpha-\beta)\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\Gamma((n+\gamma)\alpha-\beta)\,\Gamma\left(\frac{\eta}{1-\lambda}+(n+\gamma)\alpha+\rho-\beta+1\right)}$$
$$= \frac{\left\{\prod_{j=1}^{q}\Gamma(b_j)\right\}(x-c)^{\eta+\gamma\alpha+\rho-\beta}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\left\{\prod_{i=1}^{p}\Gamma(a_i)\right\}\Gamma(\gamma)\,[d(1-\lambda)]^{\gamma\alpha+\rho-\beta}}$$
$$\times\sum_{n=0}^{\infty}\frac{\left\{\prod_{i=1}^{p}\Gamma(a_i+n)\right\}\Gamma(\gamma+n)\,\Gamma(\gamma\alpha+\rho+n\alpha-\beta)}{\left\{\prod_{j=1}^{q}\Gamma(b_j+n)\right\}\Gamma((n+\gamma)\alpha-\beta)\,\Gamma\left(\frac{\eta}{1-\lambda}+(n+\gamma)\alpha+\rho-\beta+1\right)}$$

$$\times \frac{a^n (x-c)^{n\alpha}}{[d(1-\lambda)]^{n\alpha} \, n!},$$

whose last summation, in view of (1.1), is easily seen to arrive at the expression in (4.1). This completes the proof. □

Setting $p = 1 = q$ in (4.1) then we obain the following result:

Corollary 4.2. *Let $\lambda < 1$, the parameters $x, \alpha, \beta, \gamma, \eta, \rho \in \mathbb{C}$, $\Re(\alpha\gamma - \beta) > 0$, $\Re(\eta) > 0$, $d > 0$, $\Re(\rho) > 0$ and $\Re\left(\frac{\eta}{1-\lambda}\right) > -1$, then there holds the following formula:*

$$P_{0+}^{(\eta,\lambda,s)} \left\{ (x-c)^{\rho} \, K_4^{(\alpha,\beta,\gamma),(d,c):(1,1)}(x) \right\} = \frac{\Gamma(b)\,(x-c)^{\eta+\gamma\alpha+\rho-\beta}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\Gamma(a)\,\Gamma(\gamma)\,[d(1-\lambda)]^{\gamma\alpha+\rho-\beta}}$$
$$\times \, {}_3\psi_3 \left[\begin{array}{c} (a,1),(\gamma,1)\ (\gamma\alpha+\rho-\beta,\alpha); \\ (b,1),(\gamma\alpha-\beta,\alpha)\ \left(1+\gamma\alpha+\rho-\beta+\frac{\eta}{1-\lambda},\alpha\right); \end{array} s\left(\frac{x-c}{d(1-\lambda)}\right)^{\alpha} \right]. \quad (4.3)$$

5. Special cases of Theorem 4.1

In this section we state certain special cases of the Theorem 4.1, which are also considered to correspond with some known and new (presumably) pathway fractional integral formulas.

(i). If we set $\beta = \alpha - \beta$, $\gamma = 1$, $a = 1$ and $c = 0$ in (4.1), then we obtain the following pathway fractional integral operator associated with generalized M-series.

$$P_{0+}^{(\eta,\lambda,d)} \left\{ x^{\rho} \, {}_pM_q^{\alpha,\beta}(a_1,...,a_p;\, b_1,...,b_q;\, x) \right\} = \frac{x^{\eta+\rho+1}\left\{\prod_{j=1}^{q}\Gamma(b_j)\right\}\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\left\{\prod_{i=1}^{p}\Gamma(a_i)\right\}[d(1-\lambda)]^{\rho+\beta}}$$
$$\times \, {}_{p+2}\psi_{q+2} \left[\begin{array}{c} (a_i,1)_1^p,(1,1),(\rho+\beta,\alpha); \\ (b_j,1)_1^q,(\beta,\alpha),\left(1+\rho+\beta+\frac{\eta}{1-\lambda},\alpha\right); \end{array} \left(\frac{x}{d(1-\lambda)}\right)^{\alpha} \right], \quad (5.1)$$

the existence conditions can easily follow from Theorem 4.1.

(ii). If we set $p = 1 = q$ and $a_i = 1 = b_j$ in (4.1), then we obtain the following pathway fractional integral operator of G-function [13].

$$P_{0+}^{(\eta,\lambda,d)} \left\{ (x-c)^{\rho} \, G_{\alpha,\beta,\gamma}(a,c,x) \right\} = \frac{(x-c)^{\eta+\gamma\alpha+\rho-\beta}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\Gamma(\gamma)\,[d(1-\lambda)]^{\gamma\alpha+\rho-\beta}}$$
$$\times \, {}_2\psi_2 \left[\begin{array}{c} (\gamma,1),(\gamma\alpha+\rho-\beta,\alpha); \\ (\gamma\alpha-\beta,\alpha),\left(1+\gamma\alpha+\rho-\beta+\frac{\eta}{1-\lambda},\alpha\right); \end{array} a\left(\frac{x-c}{d(1-\lambda)}\right)^{\alpha} \right]. \quad (5.2)$$

(iii). If we put $\gamma = 1$ in (5.2), then we have the following result:

$$P_{0+}^{(\eta,\lambda,d)}\left\{(x-c)^{\rho}\, R_{\alpha,\beta}\left(a,c,x\right)\right\} = \frac{(x-c)^{\eta+\alpha+\rho-\beta}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\left[d\left(1-\lambda\right)\right]^{\alpha+\rho-\beta}}$$
$$\times\ {}_2\psi_2\left[\begin{array}{c}(1,1)\,,\,(\alpha+\rho-\beta,\alpha)\,;\\ (\alpha-\beta,\alpha)\,,\,\left(1+\alpha+\rho-\beta+\frac{\eta}{1-\lambda},\alpha\right);\end{array}\ a\left(\frac{x-c}{d\left(1-\lambda\right)}\right)^{\alpha}\right], \tag{5.3}$$

where $R_{\alpha,\beta}\left(a,c,x\right)$ is the R-function [13].

(iv). Further, if we take $c = 0$ and $\beta = 0$ in (5.3), then we arrive at the following result:

$$P_{0+}^{(\eta,\lambda,d)}\left\{x^{\rho}\, F_{\alpha}\left(a,x\right)\right\} = \frac{x^{\eta+\alpha+\rho}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\left[d\left(1-\lambda\right)\right]^{\alpha+\rho}}$$
$$\times\ {}_2\psi_2\left[\begin{array}{c}(1,1)\,,\,(\alpha+\rho,\alpha)\,;\\ (\alpha,\alpha)\,,\,\left(1+\alpha+\rho+\frac{\eta}{1-\lambda},\alpha\right);\end{array}\ a\left(\frac{x}{d\left(1-\lambda\right)}\right)^{\alpha}\right], \tag{5.4}$$

where $F_{\alpha}\left(a,x\right)$ is the F-function [6].

(v). If we set $p = 1 = q$, $a_i = 1 = b_j$, $\beta = \alpha - 1, \gamma = 1$ and $c = 0$ in (4.1), then we obtain the following pathway fractional integral operator of Mittag-Leffler function [18].

$$P_{0+}^{(\eta,\lambda,d)}\left\{x^{\rho}\, E_{\alpha}\left(ax^{\alpha}\right)\right\} = \frac{x^{\eta+\rho+1}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\left[d\left(1-\lambda\right)\right]^{\rho+1}}$$
$$\times\ {}_2\psi_2\left[\begin{array}{c}(1,1)\,,\,(\rho+1,\alpha)\,;\\ (1,\alpha)\,,\,\left(2+\rho+\frac{\eta}{1-\lambda},\alpha\right);\end{array}\ a\left(\frac{x}{d\left(1-\lambda\right)}\right)^{\alpha}\right]. \tag{5.5}$$

(vi). If we set $p = 1 = q$, $a_i = 1 = b_j$, $\beta = \alpha - \beta, \gamma = 1$, $a = 1$ and $c = 0$ in (4.1), then we obtain the following pathway fractional integral operator of generalized Mittag-Leffler function (also known as AgarwalŠs function) [1].

$$P_{0+}^{(\eta,\lambda,d)}\left\{x^{\rho}\, E_{\alpha,\beta}\left(x^{\alpha}\right)\right\} = \frac{x^{\eta+\rho+\beta}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\left[d\left(1-\lambda\right)\right]^{\rho+\beta}}$$
$$\times\ {}_2\psi_2\left[\begin{array}{c}(1,1)\,,\,(\rho+\beta,\alpha)\,;\\ (\beta,\alpha)\,,\,\left(1+\rho+\beta+\frac{\eta}{1-\lambda},\alpha\right);\end{array}\ \left(\frac{x}{d\left(1-\lambda\right)}\right)^{\alpha}\right]. \tag{5.6}$$

(vii). If we set $p = 1 = q$, $a_i = 1 = b_j$, $\alpha = 1, \beta = -\beta, \gamma = 1$ and $c = 0$ in (4.1), then we obtain the following pathway fractional integral operator of Miller and RossŠs function [3].

$$P_{0+}^{(\eta,\lambda,d)}\left\{x^{\rho}\,E_{x}\left(\beta,a\right)\right\}=\frac{x^{1+\eta+\rho+\beta}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\left[d\left(1-\lambda\right)\right]^{1+\rho+\beta}}$$
$$\times\;{}_{2}\psi_{2}\left[\begin{array}{c}(1,1)\,,\,(1+\rho+\beta,1)\,;\\(1+\beta,1)\,,\,\left(2+\rho+\beta+\frac{\eta}{1-\lambda},1\right);\end{array}\;a\left(\frac{x}{d\left(1-\lambda\right)}\right)\right].\qquad(5.7)$$

(viii). If we set $p=1=q$, $a_i=1=b_j$, $\gamma=1$ and $c=0$ in (4.1), then we arrive at the following pathway fractional integral operator of generalized Mittag-Leffler function [7].

$$P_{0+}^{(\eta,\lambda,d)}\left\{x^{\rho}\,E_{\alpha,\alpha-\beta}\left(ax^{\alpha}\right)\right\}=\frac{x^{\eta+\rho+1}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\left[d\left(1-\lambda\right)\right]^{\alpha+\rho-\beta}}$$
$$\times\;{}_{2}\psi_{2}\left[\begin{array}{c}(1,1)\,,\,(\alpha+\rho-\beta,\alpha)\,;\\(\alpha-\beta,\alpha)\,,\,\left(1+\alpha+\rho-\beta+\frac{\eta}{1-\lambda},\alpha\right);\end{array}\;a\left(\frac{x}{d\left(1-\lambda\right)}\right)^{\alpha}\right].\qquad(5.8)$$

(ix). If we set $p=1=q$, $a_i=1=b_j$, and $c=0$ in (4.1), then we arrive at the following pathway fractional integral operator of generalized Mittag-Leffler function [21].

$$P_{0+}^{(\eta,\lambda,d)}\left\{x^{\rho}\,E^{\gamma}_{\alpha,\alpha\gamma-\beta}\left(ax^{\alpha}\right)\right\}=\frac{x^{\eta+\rho+1}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\Gamma\left(\gamma\right)\,\left[d\left(1-\lambda\right)\right]^{\gamma\alpha+\rho-\beta}}$$
$$\times\;{}_{2}\psi_{2}\left[\begin{array}{c}(\gamma,1)\,,\,(\gamma\alpha+\rho-\beta,\alpha)\,;\\(\gamma\alpha-\beta,\alpha)\,,\,\left(1+\gamma\alpha+\rho-\beta+\frac{\eta}{1-\lambda},\alpha\right);\end{array}\;a\left(\frac{x}{d\left(1-\lambda\right)}\right)^{\alpha}\right].\qquad(5.9)$$

(x). If we use relation (3.14) in (4.1) then we have the following interesting result for Wright function.

$$P_{0+}^{(\eta,\lambda,d)}\left\{x^{\rho}\;{}_{1}\psi_{1}\left[\begin{array}{c}(\gamma,1)\,;\\(\alpha\gamma-\beta,\alpha)\,;\end{array}\;ax^{\alpha}\right]\right\}=\frac{x^{\eta+\rho+1}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\left[d\left(1-\lambda\right)\right]^{\gamma\alpha+\rho-\beta}}$$
$$\times\;{}_{2}\psi_{2}\left[\begin{array}{c}(\gamma,1)\,,\,(\gamma\alpha+\rho-\beta,\alpha)\,;\\(\gamma\alpha-\beta,\alpha)\,,\,\left(1+\gamma\alpha+\rho-\beta+\frac{\eta}{1-\lambda},\alpha\right);\end{array}\;a\left(\frac{x}{d\left(1-\lambda\right)}\right)^{\alpha}\right].\qquad(5.10)$$

(xi). If we use relation (3.15) in (4.1) then we have the following pathway fractional integral operator of H-function [22,17].

$$P_{0+}^{(\eta,\lambda,d)}\left\{x^{\rho}\,H^{1,1}_{1,2}\left[-ax^{\alpha}\left|\begin{array}{c}(1-\gamma,1)\\(0,1)\,,(1-\alpha\gamma+\beta,\alpha)\end{array}\right.\right]\right\}=\frac{x^{\eta+\rho+1}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{\left[d\left(1-\lambda\right)\right]^{\gamma\alpha+\rho-\beta}}$$
$$\times\;{}_{2}\psi_{2}\left[\begin{array}{c}(\gamma,1)\,,\,(\gamma\alpha+\rho-\beta,\alpha)\,;\\(\gamma\alpha-\beta,\alpha)\,,\,\left(1+\gamma\alpha+\rho-\beta+\frac{\eta}{1-\lambda},\alpha\right);\end{array}\;a\left(\frac{x}{d\left(1-\lambda\right)}\right)^{\alpha}\right].\qquad(5.11)$$

(xii). If we use relation (3.16) in (4.1) then we have the following pathway fractional integral operator of $\bar{H}$-function [17] (see also, [9,10,24]).

$$P_{0+}^{(\eta,\lambda,d)}\left\{x^{\rho}\,\bar{H}_{1,2}^{1,1}\left[-ax^{\alpha}\left|\begin{array}{c}(1-\gamma,1;1)\\(0,1),(1-\alpha\gamma+\beta,\alpha;1)\end{array}\right.\right]\right\}=\frac{x^{\eta+\rho+1}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{[d\,(1-\lambda)]^{\gamma\alpha+\rho-\beta}}$$
$$\times\ {}_2\psi_2\left[\begin{array}{c}(\gamma,1),(\gamma\alpha+\rho-\beta,\alpha);\\(\gamma\alpha-\beta,\alpha),\left(1+\gamma\alpha+\rho-\beta+\frac{\eta}{1-\lambda},\alpha\right);\end{array}\ a\left(\frac{x}{d\,(1-\lambda)}\right)^{\alpha}\right].\qquad(5.12)$$

6. Concluding Remarks

We conclude this investigation by remarking that the results obtained here are general in character and useful in deriving various integral formulas in the theory of the pathway fractional integration operator. Most of the results obtained here, besides being of a very general character, have been put in a compact form avoiding the occurrence of infinite series and thus making them useful from the point of view of applications. Pathway fractional integral operator of several special functions also given as special cases of our mail result.

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