Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) v. 35 2 (2017): 223–234.
ISSN-00378712 in press
doi:10.5269/bspm.v35i2.21993

Subordination and superordination results of $p-$valent analytic functions involving a linear operator

T. M. Seoudy

abstract: In this paper we derive some subordination and superordination results for certain $p-$valent analytic functions in the open unit disc, which are acted upon by a class of a linear operator. Some of our results improve and generalize previously known results.

Key Words: Analytic function, Hadamard product, differential subordination, superordination, linear operator.

Contents

1. Introduction

Let $H(U)$ denotes the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a,p]$ denotes the subclass of the functions $f \in H(U)$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + ...(a \in \mathbb{C}; p \in \mathbb{N} = \{1, 2, ..\}).$$

Also, let $\mathcal{A}(p)$ be the subclass of the functions $f \in H(U)$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N}), \tag{1.1}$$

and set $\mathcal{A} \equiv \mathcal{A}(1)$. For functions $f(z) \in \mathcal{A}(p)$, given by (1.1), and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \qquad (p \in \mathbb{N}), \tag{1.2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

2000 *Mathematics Subject Classification:* 30C45.
Submitted September 25, 2013. Published March 02, 2016

For $f, g \in H(U)$, we say that the function f is subordinate to g, if there exists a Schwarz function w, i.e, $w \in H(U)$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. This subordination is usually denoted by $f(z) \prec g(z)$. It is well-known that, if the function g is univalent in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [6] and [11]).

Supposing that h and k are two analytic functions in U, let

$$\phi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}.$$

If h and $\varphi(h(z), zh^{'}(z), z^2h^{''}(z); z)$ are univalent functions in U and if h satisfies the second-order superordination

$$k(z) \prec \varphi(h(z), zh^{'}(z), z^2h^{''}(z); z), \tag{1.4}$$

then h is called to be a solution of the differential superordination (1.4). A function $q \in H(U)$ is called a subordinant of (1.4), if $q(z) \prec h(z)$ for all the functions h satisfying (1.4). A univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all of the subordinants q of (1.4), is said to be the best subordinant.

Recently, Miller and Mocanu [12] obtained sufficient conditions on the functions k, q and φ for which the following implication holds:

$$k(z) \prec \varphi(h(z), zh^{'}(z), z^2h^{''}(z); z) \Rightarrow q(z) \prec h(z).$$

Using these results, Bulboaca [4] considered certain classes of first-order differential superordinations, as well as superordination-preserving integral operators [5]. Ali et al. [1], using the results from [4], obtained sufficient conditions for certain normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf^{'}(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent normalized functions in U.

For complex parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ ($\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}; j = 1, 2, ..., s$), we now define the generalized hypergeometric function ${}_qF_s(\alpha_1,..., \alpha_q; \beta_1, ..., \beta_s; z)$ by (see, for example, [18, p.19])

$${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k} . \frac{z^k}{k!} \tag{1.5}$$

$$(q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U),$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}), \\ \theta(\theta + 1)....(\theta + \nu - 1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases} \tag{1.6}$$

Let

$$\begin{aligned} h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) &= z^p{}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) \\ &= z^p+\sum_{k=p+1}^{\infty}\Gamma_{p,q,s}\left(\alpha_1\right)z^k, \end{aligned}$$

where

$$\Gamma_{p,q,s}\left(\alpha_1\right)=\frac{(\alpha_1)_{k-p}...(\alpha_q)_{k-p}}{(\beta_1)_{k-p}...(\beta_s)_{k-p}(1)_{k-p}}, \tag{1.7}$$

and using the Hadamard product, El-Ashwah and Aouf [8] defined the following operator

$$I^{m,\ell}_{p,\lambda}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s):A(p)\to A(p)$$

by

$$\begin{aligned} I^{0,\ell}_{p,\lambda}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z) &= f(z)*h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z); \\ I^{1,\ell}_{p,\lambda}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z) &= (1-\lambda)(f(z)*h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)) \\ &\quad +\frac{\lambda}{(p+\ell)z^{\ell-1}}(z^{\ell}f(z)*h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z))'; \end{aligned}$$

and

$$I^{m,\ell}_{p,\lambda}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z)=I^{1,\ell}_{p,q,s,\lambda}(I^{m-1,\ell}_{p,q,s,\lambda}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z)). \tag{1.8}$$

If $f\in A(p)$, then from (1.1) and (1.8), we can easily see that

$$I^{m,\ell}_{p,\lambda}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z)=z^p+\sum_{k=p+1}^{\infty}\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^m\Gamma_{p,q,s}\left(\alpha_1\right)a_kz^k. \tag{1.9}$$

$$(p\in\mathbb{N};\ m\in\mathbb{N}_0=\mathbb{N}\cup\{0\};\ell\geq 0;\lambda\geq 0;z\in U)$$

It can be easily verified from the definition (1.9) that:

$$z(I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z))'=\alpha_1 I^{m,\ell}_{p,q,s,\lambda}(\alpha_1+1)f(z)-(\alpha_1-p)I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z), \tag{1.10}$$

where

$$I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)=I^{m,\ell}_{p,\lambda}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z).$$

It should be remarked that the linear operator $I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)$ is a generalization of many other linear operators considered earlier. In particular, we have

$$I^{0,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)=H_{p,q,s}(\alpha_1)f(z),$$

where the linear operator $H_{p,q,s}(\alpha_1)$ was investigated by Dziok and Srivastava [9] (see also [13], [10] and [2]), and also we have

$$I^{0,\ell}_{p,2,1,\lambda}(a,1;c)f(z)=L_p(a,c)f(z)(a\in\mathbb{R};c\in\mathbb{R}\backslash\mathbb{Z}_0^-),$$

where the linear operator $L_p(a,c)$ was studied by Saitoh [16] which yields the operator $L(a,c)f(z)$ introduced by Carlson and Shaffer [7] for $p=1$.

2. Preliminaries

In order to prove our subordination and superordination results, we make use of the following known definition and results.

Definition 2.1. [12] *Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U}\backslash E(f)$, where*

$$E(f)=\left\{\zeta:\zeta\in\partial \text{ and } \lim_{z\to\zeta}f(z)=\infty\right\} \tag{2.1}$$

and are such that $f^{'}(\zeta)\neq 0$ for $\zeta\in\partial U\backslash E(f)$.

Lemma 2.2. [11] *Let the function $q(z)$ be univalent in the unit disc U and let θ and φ be analytic in a domain D containing $q(U)$ with $\varphi(w)\neq 0$ when $w\in q(U)$. Set $Q(z)=zq^{'}(z)\varphi(q(z))$ and $h(z)=\theta(q(z))+Q(z)$. Suppose that*

(i) $Q(z)$ is starlike univalent in U,

(ii) $\Re\left(\dfrac{zh^{'}(z)}{Q(z)}\right)>0$ for $z\in U$.

If p is analytic with $p(0)=q(0)$, $p(U)\subseteq D$ and

$$\theta(p(z))+zp^{'}(z)\varphi(p(z))\prec\theta(q(z))+zq^{'}(z)\varphi(q(z)), \tag{2.2}$$

then $p(z)\prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2.3. [6] *Let $q(z)$ be convex univalent in the unit disc U and let θ and φ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\Re\left\{\dfrac{\theta^{'}(q(z))}{\varphi(q(z))}\right\}>0$ for $z\in U$;

(ii) $zq^{'}(z)\varphi(q(z))$ is starlike univalent in U.

If $p(z)\in H[q(0),1]\cap Q$, with $p(U)\subseteq D$, and $\theta(p(z))+zp^{'}(z)\varphi(p(z))$ is univalent in U, and

$$\theta(q(z))+zq^{'}(z)\varphi(q(z))\prec\theta(p(z))+zp^{'}(z)\varphi(p(z)), \tag{2.4}$$

then $q(z)\prec p(z)$ and $q(z)$ is the best subordinant.

The following lemma gives us a necessary and sufficient condition for the univalence of a special function which will be used in some particular case.

Lemma 2.4. [15] *The function $q(z)=(1-z)^{-2ab}\,(a,b\in\mathbb{C}^*)$ is univalent in the unit disc U if and only if $|2ab-1|\leq 1$ or $|2ab+1|\leq 1$.*

3. Main Results

Unless otherwise mentioned, we assume throughout this paper that $p\in\mathbb{N}, m\in\mathbb{N}_0, \ell\geq 0, \lambda\geq 0$ and the power understood as principal values.

Theorem 3.1. *Let $q(z)$ be univalent in U such that $q(0) = 1$, $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike in U. Let $f \in A(p)$ and suppose that f and q satisfy the next conditions:*

$$\left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}{z^p}\right]^{\mu} \neq 0 \quad (\mu \in \mathbb{C}^*; z \in U), \tag{3.1}$$

and

$$\Re\left\{1 + \frac{\zeta}{\gamma}q(z) + \frac{2\delta}{\gamma}[q(z)]^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0 \quad (\zeta, \delta \in \mathbb{C}; \gamma \in \mathbb{C}^*; z \in U). \tag{3.2}$$

If

$$\Psi(z) \prec \chi + \zeta q(z) + \delta[q(z)]^2 + \gamma\frac{zq'(z)}{q(z)}, \tag{3.3}$$

where

$$\Psi(z) = \chi + \zeta\left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}{z^p}\right]^{\mu} + \delta\left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}{z^p}\right]^{2\mu}$$

$$+ \gamma\mu\alpha_1\left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1+1)f(z)}{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)} - 1\right], \tag{3.4}$$

then

$$\left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}{z^p}\right]^{\mu} \prec q(z),$$

and q is the best dominant of (3.3).

Proof: Let

$$h(z) = \left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}{z^p}\right]^{\mu} \ (z \in U). \tag{3.5}$$

According to (3.1) the function $h(z)$ is analytic in U, and differentiating (3.5) logarithmically with respect to z, we obtain

$$\frac{zh'(z)}{h(z)} = \mu\left[\frac{z(I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z))'}{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)} - p\right].$$

By using the identity (1.10), we obtain

$$\frac{zh'(z)}{h(z)} = \mu\alpha_1\left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1+1)f(z)}{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)} - 1\right].$$

In order to prove our result we will use Lemma 2.2. In this lemma consider

$$\theta(w)=\chi+\zeta w+\delta w^{2} \quad \textit{and} \quad \varphi(w)=\frac{\gamma}{w},$$

then θ is analytic in $\mathbb{C}$ and $\varphi(w)\neq 0$ is analytic in $\mathbb{C}^{*}$.Also, if we let

$$Q(z)=zq^{'}(z)\varphi(q(z))=\gamma\frac{zq^{'}(z)}{q(z)},$$

and

$$g(z)=\theta(q(z))+Q(z)=\chi+\zeta q\left(z\right)+\delta\left[q\left(z\right)\right]^{2}+\gamma\frac{zq^{'}(z)}{q(z)}.$$

We see that $Q(z)$ is starlike function in U. From (3.2), we also have

$$\Re\left\{\frac{zg^{'}(z)}{Q(z)}\right\}=\Re\left\{1+\frac{\zeta}{\gamma}q\left(z\right)+\frac{2\delta}{\gamma}\left[q\left(z\right)\right]^{2}-\frac{zq^{'}(z)}{q(z)}+\frac{zq^{''}(z)}{q^{'}(z)}\right\}>0\quad(z\in U),$$

and then, by using Lemma 2.2 we deduce that the subordination (3.3) implies $h(z)\prec q(z)$, and the function q is the best dominant of (3.3).

Putting $q=2, s=p=1, m=0, \alpha_{1}=a+1\,(a\in\mathbb{C}), \alpha_{2}=1$ and $\beta_{1}=c$ $(c\in\mathbb{C}\backslash\mathbb{Z}_{0}^{-})$ in Theorem 3.1, we obtain the following result which improves the corresponding work of Shammugam et al. [17, Theorem 3]. □

Corollary 3.2. *Let $q(z)$ be univalent in U such that $q(0)=1$, $q(z)\neq 0$ and $\frac{zq^{'}(z)}{q(z)}$ is starlike in U. Let $f\in\mathcal{A}$ such that*

$$\left[\frac{L\left(a+1,c\right)f(z)}{z}\right]^{\mu}\neq 0\quad(\mu\in\mathbb{C}^{*};z\in U),\tag{3.6}$$

and suppose that q satisfies (3.2).If

$$\Lambda\left(z\right)\prec\chi+\zeta q\left(z\right)+\delta\left[q\left(z\right)\right]^{2}+\gamma\frac{zq^{'}(z)}{q(z)},\tag{3.7}$$

where

$$\Lambda\left(z\right)=\chi+\zeta\left[\frac{L\left(a+1,c\right)f(z)}{z}\right]^{\mu}+\delta\left[\frac{L\left(a+1,c\right)f(z)}{z}\right]^{2\mu}$$
$$+\gamma\mu\left(a+1\right)\left[\frac{L\left(a+2,c\right)f(z)}{L\left(a+1,c\right)f(z)}-1\right],\tag{3.8}$$

then

$$\left[\frac{L\left(a+1,c\right)f(z)}{z}\right]^{\mu}\prec q\left(z\right),$$

and q is the best dominant of (3.7).

Putting $q(z) = \frac{1+Az}{1+Bz}\,(-1 \leq B < A \leq 1)$ in Corollary 3.2, we obtain the following result which improves the corresponding work of Shammugam et al. [17, Corollary 1].

Corollary 3.3. *Assume that*

$$\Re\left\{\frac{1-ABz^2}{(1+Az)(1+Bz)}+\frac{\zeta}{\gamma}\left[\frac{1+Az}{1+Bz}\right]+\frac{2\delta}{\gamma}\left[\frac{1+Az}{1+Bz}\right]^2\right\}>0 \quad (\zeta,\delta\in\mathbb{C};\gamma\in\mathbb{C}^*;z\in U)$$

holds. Let $f \in \mathcal{A}$ such that (3.6) *holds. If*

$$\Lambda(z) \prec \chi + \zeta\frac{1+Az}{1+Bz} + \delta\left[\frac{1+Az}{1+Bz}\right]^2 + \frac{\gamma(A-B)z}{(1+Az)(1+Bz)}, \tag{3.9}$$

where $\Lambda(z)$ is given by (3.8), *then*

$$\left[\frac{L(a+1,c)\,f(z)}{z}\right]^{\mu} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (3.9).

Putting $q(z) = \left(\frac{1+z}{1-z}\right)^{\vartheta}\,(0 < \vartheta \leq 1)$ in Corollary 3.2, we obtain the following result which improves the corresponding work of Shammugam et al. [17, Corollary 2].

Corollary 3.4. *Assume that*

$$\Re\left\{\frac{1-3z^2}{1-z^2}+\frac{\zeta}{\gamma}\left[\frac{1+z}{1-z}\right]^{\vartheta}+\frac{2\delta}{\gamma}\left[\frac{1+z}{1-z}\right]^{2\vartheta}\right\}>0 \quad (\zeta,\delta\in\mathbb{C};\gamma\in\mathbb{C}^*;z\in U)$$

holds. Let $f \in \mathcal{A}$ such that (3.6) holds. If

$$\Lambda(z) \prec \chi + \zeta\left(\frac{1+z}{1-z}\right)^{\vartheta} + \delta\left(\frac{1+z}{1-z}\right)^{2\vartheta} + \frac{2\gamma\vartheta z}{(1-z^2)} \quad (0<\vartheta\leq 1), \tag{3.10}$$

where $\Lambda(z)$ is given by (3.8), *then*

$$\left[\frac{L(a+1,c)\,f(z)}{z}\right]^{\mu} \prec \left(\frac{1+z}{1-z}\right)^{\vartheta},$$

and $\left(\frac{1+z}{1-z}\right)^{\vartheta}$ is the best dominant of (3.10).

Putting $q(z) = e^{\mu Az}\,(|\mu A| < \pi)$ in Corollary 3.2, we obtain the following result which improves the corresponding work of Shammugam et al. [17, Corollary 3].

Corollary 3.5. *Assume that*

$$\Re\left\{1+\frac{\zeta}{\gamma}e^{\mu Az}q(z)+\frac{2\delta}{\gamma}e^{2\mu Az}\right\}>0 \quad (\zeta,\delta\in\mathbb{C};\gamma\in\mathbb{C}^{*};z\in U)$$

holds. Let $f\in A$ such that (3.6) holds. If

$$\Lambda(z)\prec\chi+\zeta e^{\mu Az}+\delta e^{2\mu Az}+\gamma A\mu z \qquad (|\mu A|<\pi), \tag{3.11}$$

where $\Lambda(z)$ is given by (3.8), *then*

$$\left[\frac{L(a+1,c)\,f(z)}{z}\right]^{\mu}\prec e^{\mu Az},$$

and $e^{\mu Az}$ is the best dominant of (3.11).

Putting $q=s+1,\alpha_i=1(i=1,..,s+1),\beta_j=1(j=1,..,s),m=\zeta=\delta=0,\chi=p=1,\gamma=\frac{1}{ab}(a,b\in\mathbb{C}^{*}),\mu=a$, and $q(z)=(1-z)^{-2ab}$ in Theorem 3.1, then combining this together with Lemma 2.4 we obtain the next result due to Obradovic et al. [14, Theorem 1].

Corollary 3.6. [14] *Let $a,b\in\mathbb{C}^{*}$ such that $|2ab-1|\leq 1$ or $|2ab+1|\leq 1$. Let $f\in A$ and suppose that $\frac{f(z)}{z}\neq 0$ for all $z\in U$. If*

$$1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+z}{1-z},$$

then

$$\left(\frac{f(z)}{z}\right)^{a}\prec(1-z)^{-2ab} \tag{3.12}$$

and $(1-z)^{-2ab}$ is the best dominant of (3.12).

Remark 3.7. *For $a=1$, Corollary 3.6 reduces to the recent result of Srivastava and Lashin* [19].

Putting $q=s+1,\alpha_i=1(i=1,..,s+1),\beta_j=1(j=1,..,s),m=\zeta=\delta=0,\chi=p=\gamma=1$, and $q(z)=(1+Bz)^{\frac{\mu(A-B)}{B}}$ in Theorem 3.10, and using Lemma 2.3 we obtain the next result.

Corollary 3.8. *Let $-1\leq A<B\leq 1$ with $B\neq 0$, and suppose that $\left|\frac{\mu(A-B)}{B}-1\right|\leq 1$ or $\left|\frac{\mu(A-B)}{B}+1\right|\leq 1$. Let $f\in A$ such that $\frac{f(z)}{z}\neq 0$ for all $z\in U$, and let $\mu\in\mathbb{C}^{*}$. If*

$$1+\mu\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+[B+\mu(A-B)]z}{1+Bz},$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu}\prec(1+Bz)^{\frac{\mu(A-B)}{B}}, \tag{3.13}$$

and $(1+Bz)^{\frac{\mu(A-B)}{B}}$ is the best dominant of (3.13).

Putting $q = s+1, \alpha_i = 1 (i = 1, .., s+1), \beta_j = 1 (j = 1, .., s), m = \zeta = \delta = 0, \chi = p = 1, \gamma = \frac{e^{i\tau}}{ab\cos\tau} (a, b \in \mathbb{C}^*; |\tau| < \frac{\pi}{2}), \mu = a$, and $q(z) = (1-z)^{-2ab\cos\tau e^{-i\tau}}$ in Theorem 3.1, we obtain the following result due to Aouf et al. [3, Theorem 1].

Corollary 3.9. [3] *Let* $a, b \in \mathbb{C}^*, |\tau| < \frac{\pi}{2}$ *and let* $\left|2ab\cos\tau e^{-i\tau} - 1\right| \leq 1$ *or* $\left|2ab\cos\tau e^{-i\tau} + 1\right| \leq 1$. *Let* $f \in \mathcal{A}$ *and suppose that* $\frac{f(z)}{z} \neq 0$ *for all* $z \in U$. *If*

$$1 + \frac{e^{i\tau}}{b\cos\tau}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+z}{1-z}$$

then

$$\left(\frac{f(z)}{z}\right)^a \prec (1-z)^{-2ab\cos\tau e^{-i\tau}} \tag{3.14}$$

and $(1-z)^{-2ab\cos\tau e^{-i\tau}}$ *is the best dominant of (3.14).*

Theorem 3.10. *Let* q *be convex in* U *such that* $q(0) = 1$ *and* $\frac{zq'(z)}{q(z)}$ *is starlike in* U. *Further assume that*

$$\Re\left\{(\zeta + 2\delta q(z)) \frac{q(z) q'(z)}{\gamma}\right\} > 0 \quad (\zeta, \delta \in \mathbb{C}; \gamma \in \mathbb{C}^*). \tag{3.15}$$

Let $f \in \mathcal{A}(p)$ *such that*

$$0 \neq \left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1) f(z)}{z^p}\right]^\mu \in H[q(0), 1] \cap Q. \tag{3.16}$$

If $\Psi(z)$ *given by* (3.4) *is univalent in* U *and satisfies the following superordination condition*

$$\chi + \zeta q(z) + \delta [q(z)]^2 + \gamma \frac{zq'(z)}{q(z)} \prec \Psi(z), \tag{3.17}$$

then

$$q(z) \prec \left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1) f(z)}{z^p}\right]^\mu,$$

and q *is the best subordinant of* (3.17).

Putting $q = 2, s = p = 1, m = 0, \alpha_1 = a + 1 (a \in \mathbb{C}), \alpha_2 = 1$ and $\beta_1 = c$ $(c \in \mathbb{C} \backslash \mathbb{Z}_0^-)$ in Theorem 3.10, we obtain the following result which improves the corresponding work of Shammugam et al. [17, Theorem 4].

Corollary 3.11. *Let* q *be convex in* U *such that* $q(0) = 1$ *and* $\frac{zq'(z)}{q(z)}$ *is starlike in* U. *Further assume that*

$$\Re\left\{(\zeta + 2\delta q(z)) \frac{q(z) q'(z)}{\gamma}\right\} > 0 \quad (\zeta, \delta \in \mathbb{C}; \gamma \in \mathbb{C}^*). \tag{3.18}$$

Let $f \in \mathcal{A}$ such that

$$0 \neq \left[\frac{L\left(a+1, c\right) f(z)}{z}\right]^{\mu} \in H[q(0), 1] \cap Q. \tag{3.19}$$

If $\Lambda\left(z\right)$ given by (3.8) *is univalent in U and satisfies the following superordination condition*

$$\chi + \zeta q\left(z\right) + \delta\left[q\left(z\right)\right]^{2} + \gamma \frac{zq^{'}\left(z\right)}{q(z)} \prec \Lambda\left(z\right), \tag{3.20}$$

then

$$q(z) \prec \left[\frac{L\left(a+1, c\right) f(z)}{z}\right]^{\mu},$$

and q is the best subordinant of (3.20).

Combining Theorems 3.1 and 3.10, we obtain the following two sandwich results:

Theorem 3.12. *Let q_i be two convex functions in U such that $q_i(0) = 1$ and $\frac{zq_i^{'}(z)}{q_i(z)}$ $(i = 1, 2)$ is starlike in U. Suppose that $q_1(z)$ satisfies* (3.18) *and $q_2(z)$ satisfies* (3.2)*. Let $f \in \mathcal{A}(p)$ and suppose that $\left[\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1) f(z)}{z^p}\right]^{\mu} \in H[q(0), 1] \cap Q$. If $\Psi\left(z\right)$ given by* (3.4) *is univalent in U, and*

$$\chi + \zeta q_1\left(z\right) + \delta\left[q_1\left(z\right)\right]^{2} + \gamma \frac{zq_1^{'}(z)}{q_1(z)} \prec \Psi\left(z\right) \prec \chi + \zeta q_2\left(z\right) + \delta\left[q_2\left(z\right)\right]^{2} + \gamma \frac{zq_2^{'}(z)}{q_2(z)}, \tag{3.21}$$

then

$$q_1\left(z\right) \prec \left[\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1) f(z)}{z^p}\right]^{\mu} \prec q_2\left(z\right),$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant of (3.21).

Putting $q = 2, s = p = 1, m = 0, \alpha_1 = a + 1\left(a \in \mathbb{C}\right), \alpha_2 = 1$ and $\beta_1 = c$ $(c \in \mathbb{C} \backslash \mathbb{Z}_0^-)$ in Theorem 3.12, we obtain the following result which improves the corresponding work of Shammugam et al. [17, Theorem 5].

Corollary 3.13. *Let q_i be two convex functions in U such that $q_i(0) = 1$ and $\frac{zq_i^{'}(z)}{q_i(z)}$ $(i = 1, 2)$ is starlike in U. Suppose that $q_1(z)$ satisfies* (3.18) *and $q_2(z)$ satisfies* (3.2)*. Let $f \in \mathcal{A}$ and suppose that $\left[\frac{L(a+1,c) f(z)}{z}\right]^{\mu} \in H[q(0), 1] \cap Q$. If $\Lambda\left(z\right)$ given by* (3.8) *is univalent in U, and*

$$\chi + \zeta q_1\left(z\right) + \delta\left[q_1\left(z\right)\right]^{2} + \gamma \frac{zq_1^{'}(z)}{q_1(z)} \prec \Lambda\left(z\right) \prec \chi + \zeta q_2\left(z\right) + \delta\left[q_2\left(z\right)\right]^{2} + \gamma \frac{zq_2^{'}(z)}{q_2(z)}, \tag{3.22}$$

then

$$q_1\left(z\right) \prec \left[\frac{L\left(a+1, c\right) f(z)}{z}\right]^{\mu} \prec q_2\left(z\right),$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant of (3.22).

Acknowledgments

The author is grateful to the referees for their valuable suggestions.

References

1. R. M. Ali, V. Ravichandran, M. H. Khan and K. G. Subramanian, Differential Sandwich theorems for certain analytic functions, Far East J. Math. Sci. 15(2004), 87-94.
2. A. L. Alina, Differential sandwich theorems using a multiplier transformation and Ruscheweyh derivative, Advances in mathematics: Scientific J. 4 (2015), no.2, 195–207.
3. M. K. Aouf, F. M. Al-Oboudi and M. M. Haidan, On some results for $\lambda-$spirallike and $\lambda-$Robertson functions of complex order, Publ. Instit. Math. Belgrade, 77(2005), no. 91, 93-98.
4. T. Bulboaca, Classes of frist order differential superordinations, Demonstratio Math. 35(2002), no.2, 287-292.
5. T. Bulboaca, A class of superordination-preserving integral operators, Indeg. Math. (N.S.) 13(2002), no.3, 301-311.
6. T. Bulboaca, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.
7. B. C. Carlson and D.B. Shaffer, Starlike and prestarlike hypergeometric functions, SIAM J. Math. Anal., 15(1984), 737–745.
8. R. M. El-Ashwah and M.K. Aouf, Differential subordination and superordination for certain subclasses of p-valent functions, Math. Comput. Modelling 51(2010), no. 5-6, 349-360.
9. J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput., 103(1999), 1–13.
10. N. Magesh, J. Jothibasu and S. Murthy, Subordination and superordination results associated with the generalized hypergeometric function, Math. Slovaca 64 (2014), No. 5, 1197–1216.
11. S. S. Miller and P. T. Mocanu, Differential Subordinations : Theory and Applications, Series on Monographs and Texbooks in Pure and Appl. Math. No. 225 Marcel Dekker, Inc. New York, 2000.
12. S. S. Miller and P. T. Mocanu, Subordinant of differential superordinations, Complex Variables 48 (2003), no. 10, 815-826.
13. G.Murugusundaramoorthy and N. Magesh, Differential subordinations and superordinations for analytic functions defined by Dziok – Srivastava linear operator, J. Inequal. Pure Appl. Math. 7. (2006), no. 4, Art. 152, 1-9.
14. M. Obradovic, M. K. Aouf and S. Owa, On some results for starlike functions of complex order, Publ. Inst. Math. (Beograd) (N.S.) 46 (60), (1989), 79-85.
15. W. C. Royster, On the univalence of a certain integral, Michigan Math. J. 12 (1965), 385-387.
16. H. Saitoh, A linear operator and its applications of first order differential subordinations, Math. Japon., 44(1996), 31–38.
17. T. N. Shanmugam, V. Ravichandran M. Darus and S. Sivasubramanian, Differential sandwich theorems for some subclasses of analytic functions involving a linear operator, Acta Math. Univ. Comenianae, 74(2007), no. 2, 287-294.
18. H. M. Srivastava and P. W. Karlsson, Multiple Gaussion Hypergeometric Series, Halsted Press, Ellis Horwood Limited, Chichester, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1985.
19. H. M. Srivastava and A. Y. Lashin, Some applications of the Briot-Bouquet differential subordination, J. Inequal. Pure. Appl. Math. 6 (2005), no.2, Art. 41, 1-7.

T. M. Seoudy
Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt.
Current address: The University College at Al-Jamoom, Umm Al-Qura University, Makkah, Saudi Arabia.
E-mail address: tms00@fayoum.edu.eg