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Four Dimensional Joint Moments due to Dirichlet Density and Their Applications in Summability of Quadruple Hypergeometric Functions

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abstract: Using the Exton's multiple joint moments in four dimensional spaces due to Dirichlet density and a generalization of Bosanquet and Kestelman theorem , we prove some theorems in summability of the series containing quadruple hypergeometric functions. These theorems generalize some well known generating functions and multiplication theorems involving product of hypergeometric functions of one and more variables. We discuss some other applications and establish several interesting particular cases. Finally, we obtain an approximation formula of the series involving Exton's quadruple hypergeometric function K_{11}.

Key Words: Exton's multiple joint moments, Dirichlet density, summability of the series, Lauricella's triple hypergeometric functions and Exton's quadruple hypergeometric functions.

Contents

1. Introduction

Exton [7, p. 232] has defined the joint moments for k-dimensional random variable $(x_1, ..., x_k)$ and with the density $f(x_1, ..., x_k)$ in the form

$$\mu'_{m_1,...,m_k}(x_1, ..., x_k) = \int_{-\infty}^{\infty} .(k). \int_{-\infty}^{\infty} x_1^{m_1}...x_k^{m_k} f(x_1, ..., x_k)\, dx_1...dx_k \qquad (1.1)$$

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The Dirichlet density is defined by (see Exton [7, p.232] and Mathai and Haubold [12])

$$f(x_1,...,x_k) = \frac{\Gamma(\nu_1+...+\nu_{k+1})}{\Gamma(\nu_1)...\Gamma(\nu_{k+1})} x_1^{\nu_1-1}...x_k^{\nu_k-1}(1-x_1-...-x_k)^{\nu_{k+1}-1},$$
$$\forall x_i \geq 0 (i=1,...,k), x_1+...+x_k \leq 1$$

and

$$f\left(x_1,...x_k\right) = 0 \text{ elsewhere.} \tag{1.2}$$

Here, in Eqn. (1.2) all $\nu_i\,(i=1,...,k+1)$ are real and positive.

In our investigation, we consider the Exton's quadruple hypergeometric function $K_{11}\left(.,.,.,.\right)$ defined by following Euler type integral formula (see Exton [6])

$$K_{11}(a,a,a,a,b_1,b_2,b_3,b_4;c,c,c,d;x,y,z,t)$$
$$= \frac{\Gamma(c)\Gamma(d)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(b_4)\Gamma(c-b_1-b_2-b_3)\Gamma(d-b_4)}$$
$$\int\iiint_{\mathbb{R}} u^{b_1-1}v^{b_2-1}w^{b_3-1}s^{b_4-1}(1-u-v-w)^{c-b_1-b_2-b_3-1}(1-s)^{d-b_4-1}$$
$$(1-ux-vy-wz-st)^{-a}dudvdwds$$

provided that $Re(b_1), Re(b_2), Re(b_3), Re(b_4), Re(c-b_1-b_2-b_3)$, and $Re(d-b_4)$ are positive and the region $\mathbb{R}$ is such that

$$u \geq 0, v \geq 0, w \geq 0, u+v+w \leq 1, 0 \leq s \leq 1. \tag{1.3}$$

For the sake of our present investigation, we define the Dirichlet density (1.2) for four dimensional spaces in the form

$$f(x,y,z,t)$$
$$= \frac{\Gamma(a)(\alpha)^{-(\mu_1+\sigma_1)}(\beta)^{-(\mu_2+\sigma_2)}(\gamma)^{-(\mu_3+\sigma_3)}(\delta)^{-(\mu_4+\sigma_4)}}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)}$$
$$x^{(\mu_1+\sigma_1)-1}y^{(\mu_2+\sigma_2)-1}z^{(\mu_3+\sigma_3)-1}t^{(\mu_4+\sigma_4)-1}$$
$$(1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1})^{\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)-1}$$

in the region $x \geq 0, y \geq 0, z \geq 0, t \geq 0$ and $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}+\frac{t}{\delta} \leq 1$ and otherwise $f\left(x,y,z,t\right)=0$, provided that

$$\left(a-\sum_{i=1}^{4}\left(\mu_i+\sigma_i\right)\right) > 0, \left(\mu_i+\sigma_i\right) > 0\left(i=1,2,3,4\right), \alpha,\beta,\gamma,\delta \text{ are positive real.} \tag{1.4}$$

We also present following generalization of Bosanquet and Kestelman Theorem (see [1]):

Theorem 1.1. *Let $f(x_1,...,x_k)$ is a density function of random variable $(x_1,...,x_k)$ in the region $x_i \in (\alpha_i,\beta_i), \beta_i > \alpha_i, \alpha_i \geq 0\, (\forall i = 1,2,...,k)$ and otherwise, $f(x_1, ...x_k) = 0$. Again, in this region $g_n(x_1,...,x_k)$ be a sequence of multivariable measurable function and if for any constant η there exists*

$$\sum_{n=0}^{\infty}\left|\int_{\alpha_1}^{\beta_1}.(k).\int_{\alpha_k}^{\beta_k} g_n(x_1,...,x_k)\, f(x_1,...,x_k)\, dx_1...dx_k\right| \leq \eta \tag{1.5}$$

Then

$$\sum_{n=0}^{\infty}|g_n(x_1,...,x_k)| \leq \eta \tag{1.6}$$

Proof: Since $f(x_1,...,x_k)$ is a density function of random variable $(x_1,...,x_k)$ in the region $x_i \in (\alpha_i,\beta_i), \beta_i > \alpha_i, \alpha_i \geq 0\, (\forall i = 1,2,...,k)$ and otherwise, $f(x_1,...x_k) = 0$, therefore,

$$\int_{\alpha_1}^{\beta_1}.(k).\int_{\alpha_k}^{\beta_k} f(x_1,...,x_k)\, dx_1...dx_k = 1 \tag{1.7}$$

Further, with help of the Eqns. (1.5) and (1.7) we may write

$$\sum_{n=0}^{\infty}\left|\int_{\alpha_1}^{\beta_1}.(k).\int_{\alpha_k}^{\beta_k} (g_n(x_1,...,x_k) - \eta)\, f(x_1,...,x_k)\, dx_1...dx_k\right| \leq 0 \tag{1.8}$$

Also, $f(x_1,...,x_k) \neq 0$ in the region $x_i \in (\alpha_i,\beta_i), \beta_i > \alpha_i, \alpha_i \geq 0 (\forall i = 1,2,...,k)$, hence from Eqn. (1.8) we easily get Eqn. (1.6). □

Remark 1.2. *When we put $\alpha_i = 0\, (\forall i = 1,2,...,k)$ in Theorem 1.1, it gives us second proof of the Theorem due to Kumar and Yadav [11].*

Here in our work, we introduce and investigate Exton's multiple joint moments in four dimensional spaces due to Dirichlet density and then employ these extended spaces to summability of the series involving quadruple hypergeomtric functions. Relevant connections of the results presented here with those that were obtained in earlier works are also indicated precisely. Finally, in Section 5, we obtain an approximation formula of series involving Exton's quadruple hypergeometric function K_{11}.

2. Four Dimensional Joint Moments for Dirichlet Density and Their Quadruple Summable Series

In this section, on using Eqn. (1.1), we consider four dimensional Exton type joint moments due to Dirichlet density given in Eqn. (1.4) in the following form:

$$\mu'_{m,n,p,q} = \frac{\Gamma(a)(\alpha)^{-(\mu_1+\sigma_1)}(\beta)^{-(\mu_2+\sigma_2)}(\gamma)^{-(\mu_3+\sigma_3)}(\delta)^{-(\mu_4+\sigma_4)}}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)}$$
$$\int\iiint_{\mathbb{R}} x^{(\mu_1+\sigma_1+m)-1}y^{(\mu_2+\sigma_2+n)-1}z^{(\mu_3+\sigma_3+p)-1}t^{(\mu_4+\sigma_4+q)-1}$$
$$(1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1})^{\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)-1}dxdydzdt$$

where, the region $\mathbb{R}$ is such that $x \geq 0, y \geq 0, z \geq 0, t \geq 0$ and $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}+\frac{t}{\delta} \leq 1$, and $\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right) > 0, (\mu_i+\sigma_i) > 0\,(i=1,2,3,4), \alpha,\beta,\gamma,\delta$ are positive real,

$$\forall m,n,p,q \in \mathbb{N}_0\left(\mathbb{N}_0 = \mathbb{N}\bigcup\{0\}\right)(\mathbb{N}\, is\, a\, set\, of\, natural\, numbers) \tag{2.1}$$

Next, we present a theorem to get the sum of the series involving joint moments given in Eqn. (2.1):

Theorem 2.1. *If $b_1, b_2, b_3, b_4, c, d, h_1, h_2, h_3,$ and $h_4 \in \mathbb{C}$ (a set of complex numbers) such that $Re\,(c) > Re\,(b_1+b_2+b_3), Re\,(d) > Re(b_4)$ and $Re(b_i) > 0 \forall i = 1,2,3,4$, also all conditions given in Eqns. (1.4) and (2.1) are satisfied, then the quadruple series*

$$\sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c)_{m+n+p}(d)_q\, m!n!p!q!}\mu'_{m,n,p,q}h_1^m h_2^n h_3^p h_4^q \tag{2.2}$$

is summable for $|h_4\delta| < 1$, $\max\{|h_1\alpha|, |h_2\beta|, |h_3\gamma|\} < 1$ and equal to

$$ {}_2F_1(b_4, \mu_4+\sigma_4; d; h_4\delta) \times F_B^{(3)}(b_1, b_2, b_3, \mu_1+\sigma_1, \mu_2+\sigma_2, \mu_3+\sigma_3; c; h_1\alpha, h_2\beta, h_3\gamma) \tag{2.3}$$

Proof: Make an appeal to Eqn. (2.1) in the Eqn. (2.2) and the definition of the quadruple hypergeometric function $K_{11}(.,.,.,.)$ (see Exton [5]) to get the equality

$$\sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c)_{m+n+p}(d)_q m!n!p!q!} \mu'_{m,n,p,q} h_1^m h_2^n h_3^p h_4^q$$
$$= \frac{\Gamma(a)(\alpha)^{-(\mu_1+\sigma_1)}(\beta)^{-(\mu_2+\sigma_2)}(\gamma)^{-(\mu_3+\sigma_3)}(\delta)^{-(\mu_4+\sigma_4)}}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)}$$
$$\iiiint_{\mathbb{R}} x^{(\mu_1+\sigma_1)-1} y^{(\mu_2+\sigma_2)-1} z^{(\mu_3+\sigma_3)-1} t^{(\mu_4+\sigma_4)-1}$$
$$(1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1})^{\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)-1} \tag{2.4}$$
$$K_{11}(a,a,a,a,b_1,b_2,b_3,b_4;c,c,c,d;h_1x,h_2y,h_3z,h_4t)dxdydzdt$$

Now, in right hand side of Eqn. (2.4) in the hypergeometric integral formula make some manipulations and use the Euler type integral formula (1.3) to get

$$\sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c)_{m+n+p}(d)_q m!n!p!q!} \mu'_{m,n,p,q} h_1^m h_2^n h_3^p h_4^q$$
$$= \frac{\Gamma(a)}{\Gamma(b_4)\Gamma(d-b_4)} \int_0^1 s^{b_4-1}(1-s)^{d-b_4-1}(1-h_4\delta s)^{-(\mu_4+\sigma_4)} ds$$
$$\times \frac{\Gamma(c)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(c-b_1-b_2-b_3)} \int_0^1\int_0^1\int_0^1 u^{b_1-1}v^{b_2-1}w^{b_3-1}$$
$$\times (1-u-v-w)^{(c-b_1-b_2-b_3)-1}(1-h_1\alpha u)^{-(\mu_1+\sigma_1)}$$
$$\times (1-h_2\beta v)^{-(\mu_2+\sigma_2)}(1-h_3\gamma w)^{-(\mu_3+\times\sigma_3)} dudvdw \tag{2.5}$$

Then, making an appeal to the Euler type integral formula of Gaussian hypergeometric function${}_2F_1(.)$ (see Srivastava and Karlsson [16], Srivastava and Manocha [17], Mathai and Haubold [12]) and that of Lauricella's triple hypergeometric function$F_B^{(3)}(.,.,.)$ (see Srivastava and Manocha [17] and Exton [7]) in right hand side of Eqn. (2.5), we get

$$\sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c)_{m+n+p}(d)_q m!n!p!q!} \mu'_{m,n,p,q} h_1^m h_2^n h_3^p h_4^q$$
$$= {}_2F_1(b_4,\mu_4+\sigma_4;d;h_4\delta)$$
$$\times F_B^{(3)}(b_1,b_2,b_3,\mu_1+\sigma_1,\mu_2+\sigma_2,\mu_3+\sigma_3;c;h_1\alpha,h_2\beta,h_3\gamma) \tag{2.6}$$

Again, in right hand side of Eqn. (2.6), the hypergeometric function ${}_2F_1(.)$ converges absolutely for $|h_4\delta| < 1$ and Lauricella's triple hypergeometric function $F_B^{(3)}(.,.,.)$ converges absolutely when $\max\{|h_1\alpha|,|h_2\beta|,|h_3\gamma|\} < 1$, therefore, the series in left hand side of Eqn. (2.6) is summable.

Hence, this is the Theorem. □

3. Applications in Summability of the Series Involving Quadruple Hypergeometric Functions

In this section, we obtain the sum of series involving Exton's quadruple hypergeometric function $K_{11}(.,.,.,.)$ on presenting following theorems:

Theorem 3.1. *If all conditions given in Theorem 2.1 are satisfied, then for* $|T| < 1$ *and the Diritchlet density* (1.4)*, the series*

$$\sum_{r=0}^{\infty} K_{11}\left(a, a, a, a, b_1, b_2, b_3, -r; c, c, c, d; h_1 x, h_2 y, h_3 z, h_4 t\right) \frac{T^r}{r!}$$

is summable and equal to the formula

$$e^T {}_1F_1(\mu_4 + \sigma_4; d; -h_4 \delta T) \times F_B^{(3)} (b_1, b_2, b_3, \mu_1 + \sigma_1, \mu_2 + \sigma_2, \mu_3 + \sigma_3; c; h_1 \alpha, h_2 \beta, h_3 \gamma) \quad (3.1)$$

provided that $\max\{|h_1\alpha|, |h_2\beta|, |h_3\gamma|\} < 1$.

Proof: In Eqns. (2.4) and (2.6) replacing b_4 by $-r$ and then multiplying them by $\frac{T^r}{r!}$, again summing r, from $-\infty$, to ∞, we get

$$\frac{\Gamma(a)(\alpha)^{-(\mu_1+\sigma_1)}(\beta)^{-(\mu_2+\sigma_2)}(\gamma)^{-(\mu_3+\sigma_3)}(\delta)^{-(\mu_4+\sigma_4)}}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma(a - \sum\limits_{i=1}^{4}(\mu_i+\sigma_i))}$$
$$\sum_{r=0}^{\infty} \frac{T^r}{r!} \int \iiint_{\mathbb{R}} x^{(\mu_1+\sigma_1)-1} y^{(\mu_2+\sigma_2)-1} z^{(\mu_3+\sigma_3)-1} t^{(\mu_4+\sigma_4)-1}$$
$$(1 - x\alpha^{-1} - y\beta^{-1} - z\gamma^{-1} - t\delta^{-1})^{(a - \sum\limits_{i=1}^{4}(\mu_i+\sigma_i))-1}$$
$$K_{11}(a, a, a, a, b_1, b_2, b_3, -r; c, c, c, d; h_1x, h_2y, h_3z, h_4t)dxdydzdt$$
$$= \sum_{r=0}^{\infty} \frac{T^r}{r!} {}_2F_1(-r, \mu_4 + \sigma_4; d; h_4\delta)$$
$$\times F_B^{(3)}(b_1, b_2, b_3, \mu_1 + \sigma_1, \mu_2 + \sigma_2, \mu_3 + \sigma_3; c; h_1\alpha, h_2\beta, h_3\gamma) \quad (3.2)$$

Then in right hand side of Eqn. (3.2), using the formula due to Chaundy [3, p.62] (see also Brafman [2, p. 943], Erdélyi et al. [4, p. 367], Srivastava and

Manocha [17, p. 166, for p = 1 and q = 1]), we get

$$\frac{\Gamma(a)(\alpha)^{-(\mu_1+\sigma_1)}(\beta)^{-(\mu_2+\sigma_2)}(\gamma)^{-(\mu_3+\sigma_3)}(\delta)^{-(\mu_4+\sigma_4)}}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)}$$

$$\sum_{r=0}^{\infty}\int\iiint_{\mathbb{R}} x^{(\mu_1+\sigma_1)-1}y^{(\mu_2+\sigma_2)-1}z^{(\mu_3+\sigma_3)-1}t^{(\mu_4+\sigma_4)-1}$$

$$\times K_{11}(a,a,a,a,b_1,b_2,b_3,-r;c,c,c,d;h_1x,h_2y,h_3z,h_4t)\frac{T^r}{r!}$$

$$\left(1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}\right)^{\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)-1}dxdydzdt$$

$$=e^{T}{}_1F_1(\mu_4+\sigma_4;d;-h_4\delta T)$$

$$\times F_B^{(3)}(b_1,b_2,b_3,\mu_1+\sigma_1,\mu_2+\sigma_2,\mu_3+\sigma_3;c;h_1\alpha,h_2\beta,h_3\gamma) \tag{3.3}$$

Finally, make an appeal to the Theorem 1.1 in Eqn. (3.3), we get Eqn. (3.1). Again for $|T|<1$ and $\max\{|h_1\alpha|,|h_2\beta|,|h_3\gamma|\}<1$, it is convergent and hence the series consisting Exton's quadruple hypergeometric function $K_{11}(.,.,.,.)$ is summable. □

In similar manners we can obtain following Theorems:

Theorem 3.2. *If all conditions given in Theorem 2.1 are satisfied, then for Dirichlet density defined in (1.4), $|T|<1$ and for the bounded sequences*

$$\left\langle\frac{\prod_{j=1}^{P}(c_j)_r}{\prod_{j=1}^{Q}(d_j)_r}\right\rangle(\forall r=0,1,2,...),\ \textit{the series}$$

$$\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{P}(c_j)_r}{\prod_{j=1}^{Q}(d_j)_r}K_{11}(a,a,a,a,b_1,b_2,b_3,-r;c,c,c,d;h_1x,h_2y,h_3z,h_4t)\frac{T^r}{r!}$$

is summable and equal to the formula

$$F\begin{matrix}P:0;1\\Q:0;1\end{matrix}\left[\begin{matrix}(c_P):-;\mu_4+\sigma_4;\\(d_Q):-;d;\end{matrix}T,-h_4\delta T\right]$$

$$\times F_B^{(3)}(b_1,b_2,b_3,\mu_1+\sigma_1,\mu_2+\sigma_2,\mu_3+\sigma_3;c;h_1\alpha,h_2\beta,h_3\gamma) \tag{3.4}$$

Proof: In this Theorem, to get thte result (3.4), we use the technique of Theorem 3.1 and make an appeal to the result due to Srivastava [13] (see also Srivastava and Daoust [15] and Srivastava and Manocha [17, p. 165]).

Here in Eqn. (3.4), $F\begin{matrix}P:R;U\\Q:S;V\end{matrix}\left[\begin{matrix}(c_P):(a_R);(e_U);\\(d_Q):(b_S);(f_V);\end{matrix}x,y\right]$ is two variable Kampe' de Fe'riet function [9] (see also Srivastava and Manocha [17, Eqn. (16), p.63]. □

Theorem 3.3. *If all conditions given in Theorem 2.1 are satisfied, then for the Dirichlet density defined in Eqn.* (1.4)*, $|T| < 1$ and any $\lambda \in \mathbb{C}$, the series is summable and equal to*

$$(1-T)^{-\lambda} F_1\left(\mu_4+\sigma_4, \sigma, \lambda; d; h_4\delta, \frac{-h_4\delta T}{1-T}\right)$$
$$\times F_B^{(3)}\left(b_1, b_2, b_3, \mu_1+\sigma_1, \mu_2+\sigma_2, \mu_3+\sigma_3; c; h_1\alpha, h_2\beta, h_3\gamma\right) \tag{3.5}$$

Proof: It is easy to observe that by making an appeal to the result due to Srivastava [14, p. 26 Eqn. (1.2)] (see Srivastava and Manocha [17, p. 150]), we have the result (3.5). □

Further, make an appeal to the result due to Khan [10, p.181] (see Srivastava and Manocha [17, p. 141], to get

Theorem 3.4. *If all conditions given in Theorem 2.1 are satisfied, then for Dirichlet density defined in* (1.4)*, $|T| < 1$ and any $\lambda \in \mathbb{C}$ and for the bounded sequences $\left\langle \frac{\prod_{j=1}^{P}(c_j)_r}{\prod_{j=1}^{Q}(d_j)_r} \right\rangle$ $(\forall r = 0, 1, 2, ...)$, the series*

$$\sum_{r=0}^{\infty} \binom{\lambda}{r} \frac{\prod_{j=1}^{P}(c_j)_r}{\prod_{j=1}^{Q}(d_j)_r} K_{11}(a, a, a, a, b_1, b_2, b_3, -r;$$
$$c, c, c, 1+\lambda-r; h_1x, h_2y, h_3z, h_4t) \frac{T^r}{r!}$$

is summable and equal to

$$F\begin{matrix} P:1;1 \\ Q:0;0 \end{matrix}\left[\begin{matrix} (c_P):\mu_4+\sigma_4;-\lambda \\ (d_Q):-;-; \end{matrix} - h_4\delta T, -T\right]$$
$$\times F_B^{(3)}\left(b_1, b_2, b_3, \mu_1+\sigma_1, \mu_2+\sigma_2, \mu_3+\sigma_3; c; h_1\alpha, h_2\beta, h_3\gamma\right) \tag{3.6}$$

Theorem 3.5. *If all conditions given in Theorem 2.1 are satisfied, then for Dirichlet density defined in* (1.4)*, $|T| < 1$ and any $\lambda \in \mathbb{C}$, the series*

$$\sum_{r=0}^{\infty} \frac{(\lambda)_r T^r}{r!} K_{11}\left(a, a, a, a, \lambda+r, b_2, b_3, \lambda+r; c, c, c, d; h_1x, h_2y, h_3z, h_4t\right)$$

is summable and equal to

$$(1-T)^{-\lambda} \sum_{r=0}^{\infty} \frac{(\lambda)_r (\mu_4+\sigma_4)_r (\mu_1+\sigma_1)_r}{r!\,(d)_r\,(c)_r} \left(\frac{h_1h_4\alpha\delta T}{(1-T)^2}\right)^r$$
$${}_2F_1\left(\lambda+r, \mu_4+\sigma_4+r; d+r; \frac{h_4\delta}{1-T}\right)$$
$$\times F_B^{(3)}\left(\lambda+r, b_2, b_3, \mu_1+\sigma_1+r, \mu_2+\sigma_2, \mu_3+\sigma_3; c+r; \frac{h_1\alpha}{1-T}, h_2\beta, h_3\gamma\right)$$

provided that

$$\left|\frac{h_4\delta}{1-T}\right| < 1,\ \max\left\{\left|\frac{h_1\alpha}{1-T}\right|, |h_2\beta|, |h_3\gamma|\right\} < 1. \tag{3.7}$$

4. Other Applications with Special Cases

With a view to describing and illustrating some special cases involving the results of known and unknown bilinear and bilateral functions, we begin this section by presenting the following Theorem:

Theorem 4.1. (Converse of Theorem 1.1 in four dimensional spaces). *In the region* $\mathbb{R}$ *given by* $x \geq 0, y \geq 0, z \geq 0, t \geq 0$ *and* $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} + \frac{t}{\delta} \leq 1$*, let* $g_n(x, y, z, t)$*be a sequence of four dimensional measurable function due to following Dirichlet measure*

$$dF = \frac{\Gamma(a)(\alpha)^{-(\mu_1+\sigma_1)}(\beta)^{-(\mu_2+\sigma_2)}(\gamma)^{-(\mu_3+\sigma_3)}(\delta)^{-(\mu_4+\sigma_4)}}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)}$$
$$x^{(\mu_1+\sigma_1)-1}y^{(\mu_2+\sigma_2)-1}z^{(\mu_3+\sigma_3)-1}t^{(\mu_4+\sigma_4)-1}$$
$$\left(1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}\right)^{\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)-1}dxdydzdt$$

where,

$$\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right) > 0, (\mu_i+\sigma_i) > 0\,(i=1,2,3,4), \alpha, \beta, \gamma, \delta \text{ are positive real,} \tag{4.1}$$

and if for any constant η *there exists*

$$\sum_{n=0}^{\infty}|g_n(x, y, z, t)| \leq \eta \tag{4.2}$$

Then

$$\sum_{n=0}^{\infty}\left|\int_{\mathbb{R}} g_n(x, y, z, t)\,dF\right| \leq \eta \tag{4.3}$$

Proof: For non-negative measurable functions and in consequences of Lebesgue convergence theorem for Dirichlet measure, which is defined in Eqn. (4.1), and the series (4.2), we get

$$\sum_{n=0}^{\infty}\left|\int_{\mathbb{R}} g_n(x, y, z, t)\,dF\right| \leq \eta\int_{\mathbb{R}} dF \tag{4.4}$$

Finally, the Eqn. (4.4) easily gives us (4.3). □

Now employing Theorems 3.5 and 4.1, we evaluate the bilateral generating relation

$$\sum_{r=0}^{\infty} \frac{(\lambda)_r}{r!} {}_2F_1(\lambda + r, \mu_4 + \sigma_4; d; h_4\delta)$$
$$F_B^{(3)}(\lambda + r, b_2, b_3, \mu_1 + \sigma_1, \mu_2 + \sigma_2, \mu_3 + \sigma_3; c; h_1\alpha, h_2\beta, h_3\gamma)T^r$$
$$= (1 - T)^{-\lambda} \sum_{r=0}^{\infty} \frac{(\lambda)_r(\mu_4 + \sigma_4)_r(\mu_1 + \sigma_1)_r}{r!(d)_r(c)_r} \left(\frac{h_1h_4\alpha\delta T}{(1 - T)^2}\right)^r$$
$${}_2F_1(\lambda + r, \mu_4 + \sigma_4 + r; d + r; \frac{h_4\delta}{1 - T})$$
$$\times F_B^{(3)}(\lambda + r, b_2, b_3, \mu_1 + \sigma_1 + r, \mu_2 + \sigma_2, \mu_3 + \sigma_3; c + r; \frac{h_1\alpha}{1 - T}, h_2\beta, h_3\gamma)$$

provided that

$$\left|\frac{h_4\delta}{1 - T}\right| < 1, \max\left\{\left|\frac{h_1\alpha}{1 - T}\right|, |h_2\beta|, |h_3\gamma|\right\} < 1. \tag{4.5}$$

Take $h_2 \to 0$ in both sides of Eqn. (4.5) to get another bilateral relation

$$\sum_{r=0}^{\infty} \frac{(\lambda)_r}{r!} {}_2F_1(\lambda + r, \mu_4 + \sigma_4; d; h_4\delta) F_3(\lambda + r, b_3, \mu_1 + \sigma_1, \mu_3 + \sigma_3; c; h_1\alpha, h_3\gamma) T^r$$
$$= (1 - T)^{-\lambda} \sum_{r=0}^{\infty} \frac{(\lambda)_r (\mu_4 + \sigma_4)_r (\mu_1 + \sigma_1)_r}{r! (d)_r (c)_r} \left(\frac{h_1h_4\alpha\delta T}{(1 - T)^2}\right)^r$$
$${}_2F_1\left(\lambda + r, \mu_4 + \sigma_4 + r; d + r; \frac{h_4\delta}{1 - T}\right)$$
$$\times F_3\left(\lambda + r, b_3, \mu_1 + \sigma_1 + r, \mu_3 + \sigma_3; c + r; \frac{h_1\alpha}{1 - T}, h_3\gamma\right)$$

provided that

$$\left|\frac{h_4\delta}{1 - T}\right| < 1, \max\left\{\left|\frac{h_1\alpha}{1 - T}\right|, |h_3\gamma|\right\} < 1. \tag{4.6}$$

Further, set $h_3 \to 0$ in both sides of Eqn. (4.6) to get bilinear relation due to Srivastava and Manocha [17, p. 298]

$$\sum_{r=0}^{\infty} \frac{(\lambda)_r}{r!} {}_2F_1(\lambda + r, \mu_4 + \sigma_4; d; h_4\delta) {}_2F_1(\lambda + r, \mu_1 + \sigma_1; c; h_1\alpha) T^r$$
$$= (1 - T)^{-\lambda} \sum_{r=0}^{\infty} \frac{(\lambda)_r (\mu_4 + \sigma_4)_r (\mu_1 + \sigma_1)_r}{r! (d)_r (c)_r} \left(\frac{h_1h_4\alpha\delta T}{(1 - T)^2}\right)^r$$
$${}_2F_1\left(\lambda + r, \mu_4 + \sigma_4 + r; d + r; \frac{h_4\delta}{1 - T}\right)$$
$$\times {}_2F_1\left(\lambda + r, \mu_1 + \sigma_1 + r; c + r; \frac{h_1\alpha}{1 - T}\right)$$

provided that

$$\left|\frac{h_4\delta}{1-T}\right| < 1, \left|\frac{h_1\alpha}{1-T}\right| < 1 \tag{4.7}$$

Again on replacing b_3 by χ and h_3 by $\frac{h_3}{\chi}$ and then take $\chi \to \infty$, in both sides of Eqn. (4.6), we get

$$\sum_{r=0}^{\infty} \frac{(\lambda)_r}{r!} {}_2F_1\left(\lambda+r, \mu_4+\sigma_4; d; h_4\delta\right) \Xi_1\left(\lambda+r, \mu_1+\sigma_1, \mu_3+\sigma_3; c; h_1\alpha, h_3\gamma\right) T^r$$
$$= (1-T)^{-\lambda} \sum_{r=0}^{\infty} \frac{(\lambda)_r (\mu_4+\sigma_4)_r (\mu_1+\sigma_1)_r}{r! (d)_r (c)_r} \left(\frac{h_1 h_4 \alpha\delta T}{(1-T)^2}\right)^r$$
$${}_2F_1\left(\lambda+r, \mu_4+\sigma_4+r; d+r; \frac{h_4\delta}{1-T}\right)$$
$$\times\, \Xi_1\left(\lambda+r, \mu_1+\sigma_1+r, \mu_3+\sigma_3; c+r; \frac{h_1\alpha}{1-T}, h_3\gamma\right)$$

provided that

$$\left|\frac{h_4\delta}{1-T}\right| < 1, \left|\frac{h_1\alpha}{1-T}\right| < 1, |h_3\gamma| < \infty. \tag{4.8}$$

Similarly using Theorems 3.1 and 4.1,we may find the generating function due to Chaundy [3, p.62] (see also Brafman [2, p. 943], Erdélyi et al. [4, p. 367], Srivastava and Manocha [17, p. 166, for p = 1 and q = 1]) as

$$\sum_{r=0}^{\infty} \frac{T^r}{r!} {}_2F_1\left(-r, \mu_4+\sigma_4; d; h_4\delta\right) = e^T {}_1F_1\left(\mu_4+\sigma_4; d; -h_4\delta T\right) \tag{4.9}$$

Again with the help of the Theorem 4.1 in the Theorems 3.2, 3.3 and 3.4, we may find the generating functions of Srivastava [13] (see also Srivastava and Daoust [15]), Srivastava [14, p. 26 Eqn. (1.2)], Khan [10, p. 181] respectively.

5. Approximation Formula

In this section, we obtain an approximation formula for the summation of the series consisting Exton's quadruple hypergeometric function $K_{11}(.,.,.,.)$.

To obtain this formula we make an appeal to the following theorems due to T. M. Flett [Proc. Edinburgh, Math. Soc. (2) 18 (1972), p. 31-34] (see Joshi and Arya [8]):

Theorem 5.1. (Theorem due to Flett (see Joshi and Arya [8]))
Let $c > a$, $a > (c-b) > 0$, $b > 0$, $0 < x < 1$, then the approximation formula of Gaussian hypergeometric function ${}_2F_1(.)$ is given by

$$(1-x)^{c-a-b} < {}_2F_1(a, b; c; x) < \frac{\Gamma(a+b-c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)} (1-x)^{c-a-b}. \tag{5.1}$$

Theorem 5.2. (Theorem due to Joshi and Arya [8]) *Let* $2 \geq c \geq$ *max* $\{1, 2b_1, 2b_2, 2a_1 - 1\}$, $a_2 \leq 1$, $c > a_1$, $a_1 > (c - b_1) > 0$, $b_1 > 0$, $c > a_2$, $a_2 > (c - b_2) > 0$, $b_2 > 0$, $0 < x < 1$, $0 < y < 1$, *then the approximation formula for the Appell's function* $F_3[.,.]$ *(see Srivastava and Manocha [17, p. 53]) is given by*

$$(1-x)^{c-a_1-b_1}(1-y)^{c-a_2-b_2}\left[1-\frac{c+1}{c}+\frac{c+1}{c}\left\{1+\frac{a_1a_2b_1b_2xy}{c(c+1)^2}\right\}^{-1}\right]$$
$$< F_3[a_1,a_2,b_1,b_2;c;x,y]$$
$$< (1-x)^{c-a_1-b_1}(1-y)^{c-a_2-b_2}\frac{\Gamma(a_1+b_1-c)\Gamma(a_2+b_2-c)(\Gamma(c))^2}{\Gamma(a_1)\Gamma(a_2)\Gamma(b_1)\Gamma(b_2)}$$
$$\times\left[1-\frac{2(c-1)}{c}+\frac{2(c-1)}{c}\left\{1+\frac{c(c+1)xy}{2(c-1)}\right\}^{-1}\right] \quad (5.2)$$

We also use the Lauricella's triple hypergeometric function $F_B^{(3)}(.,.,.)$ *defined by*

$$F_B^{(3)}[a_1,a_2,a_3,b_1,b_2,b_3;c;x,y,z] = \sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_n(a_3)_p(b_1)_m(b_2)_n(b_3)_p}{(c)_{m+n+p}m!n!p!}x^my^nz^p \quad (5.3)$$

provided that $\max\{|x|,|y|,|z|\} < 1$.

The asymptotic estimates of Kummer's confluent hypergeometric function is given as (see Srivastava and Manocha [17, p.38])

When $d - a$ *and* z *are bounded and* $d \to \infty$, *then*

$$_1F_1(a;d;z) \simeq e^z \quad (5.4)$$

Theorem 5.3. (The Approximation Formula of Lauricella's Triple Hypergeometric Function $F_B^{(3)}(.,.,.)$)

Let $2 \geq c \geq \max\{1, 2b_1, 2b_2, 2b_3, 2a_1 - 1\}$, $a_2 \leq 1$, $a_3 \leq 1$, $c > a_1$, $a_1 > (c - b_1) > 0$, $b_1 > 0$, $c < a_2$, $a_2 > (c - b_2) > 0$, $b_2 > 0$, $c > a_3$, $a_3 > (c - b_3) > 0$, $b_3 > 0$, $0 < x < 1$, $0 < y < 1$, $0 < z < 1$, *then*

$$F_B^{(3)}[a_1,a_2,a_3,b_1,b_2,b_3;c;x,y,z]$$
$$< (1-x)^{c-a_1-b_1}(1-y)^{c-a_2-b_2}(1-z)^{c-a_3-b_3}$$
$$\times\frac{\Gamma(a_1+b_1-c)\Gamma(a_2+b_2-c)\Gamma(a_3+b_3-c)(\Gamma(c))^3}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}$$
$$\times\left[1-\frac{2(c-1)}{c}+\frac{2(c-1)}{c}\left\{1+\frac{c(c+1)xy}{2(c-1)}\right\}^{-1}\right] \quad (5.5)$$

Proof: The Eqn. (5.3) may be written as

$$F_B^{(3)}\left[a_1,a_2,a_3,b_1,b_2,b_3;c;x,y,z\right]=\sum_{p=0}^{\infty}\frac{(a_3)_p\,(b_3)_p}{(c)_p\,p!}z^p F_3\left[a_1,a_2,b_1,b_2;c+p;x,y\right] \tag{5.6}$$

Again, *all* $c+p\geq c, \forall p\in\mathbb{N}_0$, therefore the Eqn. (5.6) gives us

$$F_B^{(3)}\left[a_1,a_2,a_3,b_1,b_2,b_3;c;x,y,z\right]<{}_2F_1\left(a_3,b_3;c;z\right)F_3\left[a_1,a_2,b_1,b_2;c;x,y\right] \tag{5.7}$$

Now, using the results (5.1) and (5.2) in Eqn.(5.7), we find the formula (5.5). □

Theorem 5.4. (The Approximation Formula of Series Consisting Quadruple Hypergeometric Function $K_{11}(.,.,.,.)$)
Let for any $a, 2\geq c\geq, \max\{1,2(\mu_1+\sigma_1),2(\mu_2+\sigma_2),2(\mu_3+\sigma_3),2b_1-1\}$, $b_2\leq 1$, $b_3\leq 1, c>b_1$, $b_1>(c-\mu_1-\sigma_1)>0$, $\mu_1+\sigma_1>0, c<b_2, b_2>(c-\mu_2-\sigma_2)>0$, $\mu_2+\sigma_2>0, c<b_3, b_3>(c-\mu_3-\sigma_3)>0$, $\mu_3+\sigma_3>0, d\to\infty, d$ *and* $\mu_4+\sigma_4$ *are such that* $(d-\mu_4+\sigma_4)$ *are bounded,* $0<h_1\alpha<1$, $0<h_2\beta<1, 0<h_3\gamma<1$, $0<h_4\delta<1$, $h_i>0\ (\forall i=1,2,3,4)$, $0<T<1$, *then in the region* $\mathbb{R}$ *given by* $x\geq 0, y\geq 0, z\geq 0, t\geq 0, \frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}+\frac{t}{\delta}\leq 1$, *we have*

$$\begin{aligned}
&\sum_{r=0}^{\infty}K_{11}\left(a,a,a,a,b_1,b_2,b_3,-r;c,c,c,d;h_1x,h_2y,h_3z,h_4t\right)\frac{T^r}{r!}\\
&<e^{T(1-h_4\delta)}\left(1-h_1\alpha\right)^{c-b_1-(\mu_1+\sigma_1)}\left(1-h_2\beta\right)^{c-b_2-(\mu_2+\sigma_2)}\left(1-h_3\gamma\right)^{c-b_3-(\mu_3+\sigma_3)}\\
&\times\tfrac{\Gamma(b_1+(\mu_1+\sigma_1)-c)\Gamma(b_2+(\mu_2+\sigma_2)-c)\Gamma(b_3+(\mu_3+\sigma_3)-c)(\Gamma(c))^3}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)}\\
&\times\left[1-\tfrac{2(c-1)}{c}+\tfrac{2(c-1)}{c}\left\{1+\tfrac{c(c+1)h_1h_2\alpha\beta}{2(c-1)}\right\}^{-1}\right]
\end{aligned} \tag{5.8}$$

Proof: Make an appeal to the Eqn. (5.4) and the Theorem 5.3 in the Eqn. (3.1) of Theorem 3.1, to obtain the result (5.8). □

Example 5.5. *Let* $h_1=h_2=h_3=h_4=\frac{1}{10},\alpha=7,\beta=5,\gamma=6,\delta=8,T=0.4, d=10000.9, \mu_4+\sigma_4=10000, b_1=1.1, b_2=0.9, b_3=1.2, \mu_1+\sigma_1=0.6, \mu_2+\sigma_2=0.6, \mu_3+\sigma_3=0.5, c=1.3$, *then in the region* $\mathbb{R}$ *given by* $x\geq 0, y\geq 0, z\geq 0, t\geq 0, \frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}+\frac{t}{\delta}\leq 1$, *with help of Eqn. (5.8), we have*

$$\sum_{r=0}^{\infty}K_{11}\left(a,a,a,a,b_1,b_2,b_3,-r;c,c,c,d;h_1x,h_2y,h_3z,h_4t\right)\frac{T^r}{r!}<9.138062885. \tag{5.9}$$

Example 5.6. *In above example 5.5 for the given all values, take* $h_4\to 0$, *in that region* $\mathbb{R}$, *to get approximation formula of Lauricella's triple hypergeometric function*$F_D^{(3)}\left(.,.,.\right)$

$$F_D^{(3)}\left(a,b_1,b_2,b_3;c;h_1x,h_2y,h_3z\right) < 8.435493835. \tag{5.10}$$

Again, an appeal to the Theorem 4.1 in Eqn. (5.10)*, gives us*

$$F_B^{(3)}\left(b_1,b_2,b_3,\mu_1+\sigma_1,\mu_2+\sigma_2,\mu_3+\sigma_3;c;h_1\alpha,h_2\beta,h_3\gamma\right)\ < 8.435493835 \tag{5.11}$$

Example 5.7. *In example 5.5 for the given values, take $h_3 \to 0\, and\, h_4 \to 0$, in that region $\mathbb{R}$, to get approximation formula for Appell's double hypergeometric function $F_1\left(.,.\right)$*

$$F_1\left(a,b_1,b_2;c;h_1x,h_2y\right) < 4.779896696. \tag{5.12}$$

Again, making an appeal to the Theorem 4.1 in Eqn. (5.12), gives us the equivalent value obtained by Joshi and Arya [8, Eqn. (3.9)] for $F_3(.)$

$$F_3\left(b_1,b_2,\mu_1+\sigma_1,\mu_2+\sigma_2;c;h_1\alpha,h_2\beta\right)\ < 4.779896696 \tag{5.13}$$

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