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Regular Matrix Transformation on Triple Sequence Spaces

S.Debnath, B.C.Das, D.Bhattacharya and J.Debnath

abstract: The main aim of this paper is to introduce the necessary and sufficient conditions for a particular type of transformation of the form $A : (a_{l,m,n,p,q,r})$ be regular from a triple sequence space to another triple sequence space.

Key Words: Triple sequence, regular matrix transformation,divergent triple series.

Contents

1. Introduction and Preliminaries

Several definitions for giving a value to a divergent simple series , as for example the Cesaro's and Holder's means, can be expressed by means of a linear transformation defined by infinite matrix of numbers. Two types of transformations are given as follows, one by a triangular and the other by a square matrix.

$$T: \begin{pmatrix} a_{1,1} & & & & & & \\ a_{2,1} & a_{2,2} & & & & & \\ a_{3,1} & a_{3,2} & a_{3,3} & & & & \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$S: \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & \cdots & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & \cdots & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & \cdots & \cdots \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

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For any given sequence (x_n) a new sequence (y_n) is defined as follows:

$y_n = \sum_{k=1}^{n} a_{n,k}x_k$ for the matrix T,

$y_n = \sum_{k=1}^{\infty} a_{n,k}x_k$ for the matrix S,

provided in the later case y_n has a meaning. If to any matrix of type T we adjoin the elements $a_{n,k} = 0, k > n$ (all n), we obtain a matrix of type S. Since this addition does not affect the transformation, any transformation of the type T may be considered as a special case of a transformation of type S. If for either transformation $\lim_{n\to\infty} y_n$ exists, the limit is called the generalized value of the sequence x_n by the transformation. When x_n converges, y_n converges to the same value, then the transformation is said to be regular. The criterion for regularity of these transformations is stated as follows:

For the following results, one may refer to Robison [9].

Theorem 1.1. *A necessary and sufficient condition that the transformation T be regular is that*

(a) $\lim_{n\to\infty} a_{n,k} = 0$ *,for every k*

(b) $\lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k} = 1$

(c) $\sum_{k=1}^{n} |a_{n,k}| < A$ *, for all n*

Theorem 1.2. *A necessary and sufficient condition that the transformation S be regular is that*

(a) $\lim_{n\to\infty} a_{n,k} = 0$ *,for every k*

(b) $\sum_{k=1}^{\infty} |a_{n,k}|$ *converge for each n*

(c) $\sum_{k=1}^{\infty} |a_{n,k}| < A$ *, for all n*

(d) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1$

Corresponding to these definitions of summability for a single series, G. M. Robison [9] have given definitions for giving a value to a divergent double series by considering the double sequence for the series and established the conditions of regularity of linear transformations on double sequence spaces.

A triple sequence (real or complex) can be defined as a function $x : N\times N\times N \to R(C)$, where N, R and C denote the set of natural numbers, real numbers and complex numbers respectively. Different types of notions of triple sequences was introduced and investigated at the initial stage by Sahiner et. al. [10], Dutta et. al. [1], Debnath and Das [2] and many others.

2. Main Result

We can define the following definitions for giving a value to a divergent triple series. Let the series be represented as follows:

$u_{1,1,1} + u_{1,1,2} + u_{1,1,3} + u_{1,1,4} + u_{1,1,5} + \ldots\ldots\ldots\ldots$
$+u_{1,2,1} + u_{1,2,2} + u_{1,2,3} + u_{1,2,4} + \ldots\ldots\ldots\ldots\ldots\ldots$
$+u_{1,3,1} + u_{1,3,2} + u_{1,3,3} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$+u + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$+u_{2,1,1} + u_{2,1,2} + u_{2,1,3} + u_{2,1,4} + u_{2,1,5} + \ldots\ldots\ldots\ldots$
$+u_{2,2,1} + u_{2,2,2} + u_{2,2,3} + u_{2,2,4} + \ldots\ldots\ldots\ldots\ldots\ldots$
$+u_{2,3,1} + u_{2,3,2} + u_{2,3,3} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$+u + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$+u_{3,1,1} + u_{3,1,2} + u_{3,1,3} + u_{3,1,4} + u_{3,1,5} + \ldots\ldots\ldots\ldots$
$+u_{3,2,1} + u_{3,2,2} + u_{3,2,3} + u_{3,2,4} + \ldots\ldots\ldots\ldots\ldots\ldots$
$+u_{3,3,1} + u_{3,3,2} + u_{3,3,3} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$+u + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$+u_{4,1,1} + u_{4,1,2} + u_{4,1,3} + u_{4,1,4} + u_{4,1,5} + \ldots\ldots\ldots\ldots$
$+u_{4,2,1} + u_{4,2,2} + u_{4,2,3} + u_{4,2,4} + \ldots\ldots\ldots\ldots\ldots\ldots$
$+u_{4,3,1} + u_{4,3,2} + u_{4,3,3} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$+u + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Then the triple sequence $(x_{l,m,n})$ for this series is given by the following expression:

$x_{l,m,n} = \sum_{p=1,q=1,r=1}^{l,m,n} u_{p,q,r}$

Thus we have the following recurrence relations

$u_{l,m,n} = (x_{l,m,n} + x_{l,m-1,n-1} - x_{l,m,n-1} - x_{l,m-1,n}) - (x_{l-1,m,n} + x_{l-1,m-1,n-1} - x_{l-1,m,n-1} - x_{l-1,m-1,n}),\ (l,m,n > 1);$

$u_{l,m,1} = (x_{l,m,1} - x_{l,m-1,1}) - (x_{l-1,m,1} - x_{l-1,m-1,1});\ (l,m > 1)$

$u_{l,1,n} = (x_{l,1,n} - x_{l,1,n-1}) - (x_{l-1,1,n} - x_{l-1,1,n-1});\ (l,n > 1)$

$u_{1,m,n} = (x_{1,m,n} - x_{1,m,n-1}) - (x_{1,m-1,n} - x_{1,m-1,n-1})$; $(m, n > 1)$

$u_{l,1,1} = x_{l,1,1} - x_{l-1,1,1}$; $(l > 1)$

$u_{1,m,1} = x_{1,m,1} - x_{1,m-1,1}$; $(m > 1)$

$u_{1,1,n} = x_{1,1,n} - x_{1,1,n-1}$; $(n > 1)$

$u_{1,1,1} = x_{1,1,1}$;

Now we define a new sequence by the relation

$y_{l,m,n} = \sum_{p=1}^{l} \sum_{q=1}^{m} \sum_{r=1}^{n} a_{l,m,n,p,q,r} x_{p,q,r}$

We call this transformation and its matrix $A : (a_{l,m,n,p,q,r})$ of type T, here $p \leq l, q \leq m, r \leq n$. Again we may write

$y_{l,m,n} = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{r=1}^{\infty} a_{l,m,n,p,q,r} x_{p,q,r}$

We call this transformation and its matrix $A : (a_{l,m,n,p,q,r})$ of type S, here p, q and r take all positive integral values. Any transformation of type T may be considered as a special case of a transformation of type S, for by adding the elements

(i) $a_{l,m,n,p,q,r} = 0$, $1 \leq p \leq l, 1 \leq q \leq m, n < r$, for all l,m and n,

(ii) $a_{l,m,n,p,q,r} = 0$, $1 \leq p \leq l, m < q, 1 \leq r \leq n$, for all l,m and n,

(iii) $a_{l,m,n,p,q,r} = 0$, $1 \leq p \leq l, m < q, n < r$, for all l,m and n,

(iv) $a_{l,m,n,p,q,r} = 0$, $l < p, 1 \leq q \leq m, 1 \leq r \leq n$, for all l,m and n,

(v) $a_{l,m,n,p,q,r} = 0$, $l < p, 1 \leq q \leq m, n < r$, for all l,m and n,

(vi) $a_{l,m,n,p,q,r} = 0$, $l < p, m < q, 1 \leq r \leq n$, for all l,m and n,

(vii) $a_{l,m,n,p,q,r} = 0$, $l < p, m < q, n < r$, for all l,m and n,

For a given matrix of type T we obtain a matrix of type S such that the resulting transformation is identical with the original one. If for either transformation $y_{l,m,n}$ possesses a limit, the limit is called the generalized value of the sequence $x_{l,m,n}$ by the transformation.

It is well known fact that if a simple series converges, the corresponding sequence is bounded. Like as double series [9] the above result does not hold for a triple series. To proof this we consider the following example:

Consider the series

$u_{1,1,n} = 1; u_{2,1,n} = -1$ and $u_{l,m,n} = 0$, $(l > 2, m > 1)$

This series converges, but the corresponding sequence is not bounded. Thus convergent triple series may be divided into two classes according to whether the corresponding sequences are bounded or not. The following definition of regularity of a transformation is constructed with regard to convergent bounded sequence; thus even if a transformation is regular it need not give to an unbounded convergent sequence the value to which it converges.

A transformation on triple sequence is regular if whenever $x_{l,m,n}$ is a bounded convergent sequence to L then $y_{l,m,n}$ also converges to L.

Theorem 2.1. *A necessary and sufficient condition that the transformation T be regular is that*

(a) $lim_{l,m,n\to\infty} a_{l,m,n,p,q,r} = 0$, *for each p,q and r ,*

(b) $lim_{l,m,n\to\infty} \Sigma_{p=1,q=1,r=1}^{l,m,n} a_{l,m,n,p,q,r} = 1$,

(c) $lim_{l,m,n\to\infty} \Sigma_{p=1}^{l} \mid a_{l,m,n,p,q,r} \mid = 0$, *for each q and r ,*

(d) $lim_{l,m,n\to\infty} \Sigma_{q=1}^{m} \mid a_{l,m,n,p,q,r} \mid = 0$, *for each p and r ,*

(e) $lim_{l,m,n\to\infty} \Sigma_{r=1}^{n} \mid a_{l,m,n,p,q,r} \mid = 0$, *for each p and q ,*

(f) $\Sigma_{p=1,q=1,r=1}^{l,m,n} \mid a_{l,m,n,p,q,r} \mid \leq A$, *where A is some constant,*

Proof of necessity:

(a) To show the necessity of condition (a), consider a sequence $(x_{l,m,n})$ as follows:

$$x_{l,m,n} = \begin{cases} 1 & \text{where } l = i, m = j, n = k \\ 0 & \text{otherwise} \end{cases}$$

Then $\lim_{l,m,n\to\infty} x_{l,m,n} = 0$ and $y_{l,m,n} = a_{l,m,n,i,j,k}$

Therefore for $\lim_{l,m,n\to\infty} y_{l,m,n} = 0$, it is necessary that $\lim_{l,m,n\to\infty} a_{l,m,n,i,j,k} = 0$, for each i, j and k. Thus condition (a) is necessary.

(b) For condition (b) we consider the sequence $(x_{l,m,n})$ defined as follows:

$x_{l,m,n} = 1$

Then the sequence $(y_{l,m,n})$ becomes

$y_{l,m,n} = \Sigma_{p=1,q=1,r=1}^{l,m,n} a_{l,m,n,p,q,r}$. Since $\lim_{l,m,n\to\infty} y_{l,m,n} = 1$, hence the condition (b) is necessary.

(c) To proof the necessity of condition (c), we assume that the condition (a) is satisfied and (c) is not, and obtained a contradiction. Since we are assuming that for $r = r_0$ (some fixed integer) the sequence $\Sigma_{p=1}^{l}|a_{l,m,n,p,q,r}|$ does not approach to zero, for some pre-assigned constant $k > 0$ there must exist a sub-sequence of this sequence, such that each element of it is greater than k. We choose l_1, m_1 and n_1 such that

$\Sigma_{p=1}^{l_1}|a_{l_1,m_1,n_1,p,q,r_0}| > k$

Now choose $l_2 > l_1$, $m_2 > m_1$ and $n_2 > n_1$ thus

$\Sigma_{p=1}^{l_1}|a_{l_2,m_2,n_2,p,q,r_0}| \leq k/2$, $\Sigma_{p=1}^{l_2}|a_{l_2,m_2,n_2,p,q,r_0}| > k/2$

In general choose $l_t > l_{t-1}$, $m_t > m_{t-1}$ and $n_t > n_{t-1}$ such that

$\Sigma_{p=1}^{l_{t-1}}|a_{l_t,m_t,n_t,p,q,r_0}| < k/2^{t-1}$, $\Sigma_{p=1}^{l_t}|a_{l_t,m_t,n_t,p,q,r_0}| > k$ ... ... (1)

Equality (1) gives

$\Sigma_{p=1_{t-1}+1}^{l_t}|a_{l_t,m_t,n_t,p,q,r_0}| > k - k/2^{t-1} = k(1 - 1/2^{t-1})$ ... ... (2)

Consider the sequence $(x_{l,m,n})$ defined as follows:

$x_{l,m,n} = 0$, $n \neq r_0$

$x_{l,m,n} =$sgn a_{l_1,m_1,n_1,p,q,r_0}, $l \leq l_1$;

$x_{l,m,n} =$sgn a_{l_2,m_2,n_2,p,q,r_0}, $l_1 < l \leq l_2$;

....
....

$x_{l,m,n} =$sgna_{l_t,m_t,n_t,p,q,r_0}, $l_{t-1} < l \leq l_t$;

....

... ... (3)

$\lim_{l,m,n\to\infty} x_{l,m,n} = 0$, for the sequence $(x_{l,m,n})$ we have

$y_{l_t,m_t,n_t} = \Sigma_{p=1}^{l_t} a_{l_t,m_t,n_t,p,q,r_0} x_{p,q,r_0}$

$= \Sigma_{p=1}^{l_{t-1}} a_{l_t,m_t,n_t,p,q,r_0} x_{p,q,r_0} + \Sigma_{p=l_{t-1+1}}^{l_t} a_{l_t,m_t,n_t,p,q,r_0} x_{p,q,r_0}$

Now from (1), (2) and (3) it follows that

$\mid \Sigma_{p=1}^{l_{t-1}} a_{l_t,m_t,n_t,p,q,r_0} x_{p,q,r_0} \mid \leq \mid \Sigma_{p=1}^{l_{t-1}} a_{l_t,m_t,n_t,p,q,r_0} \mid \leq k/2^{t-1}$

$\Sigma_{p=l_{t-1+1}}^{l_t} a_{l_t,m_t,n_t,p,q,r_0} x_{p,q,r_0} = \Sigma_{p=l_{t-1+1}}^{l_t} \mid a_{l_t,m_t,n_t,p,q,r_0} \mid \geq k(1 - 1/2^{t-1})$

Therefore

$y_{l_t,m_t,n_t} \geq k(1 - 1/2^{t-1}) - k/2^{t-1} = k(1 - 1/2^{t-2})$

Thus $y_{l,m,n}$ does not have the limit zero, from which the condition (c) is necessary.

(d) The above proof can be used for showing the necessity of condition (d).

(e) In a similar way we can proof the necessity of condition (e).

(f) We consider that the conditions (a) and (b) are satisfied and the conditions (f) are not. Choose l_1, m_1 and n_1 such that

$\Sigma_{p=1,q=1,r=1}^{l_1,m_1,n_1} |a_{l_1,m_1,n_1,p,q,r}| \leq 1$

choose $l_2 > l_1$, $m_2 > m_1$ and $n_2 > n_1$ thus

$\Sigma_{p=1,q=1,r=1}^{l_1,m_1,n_1} |a_{l_2,m_2,n_2,p,q,r}| \leq 2$, $\Sigma_{p=1,q=1,r=1}^{l_2,m_2,n_2} |a_{l_2,m_2,n_2,p,q,r}| \geq 2^4$

In general we choose $l_t > l_{t-1}$, $m_t > m_{t-1}$ and $n_t > n_{t-1}$, such that

$\Sigma_{p=1,q=1,r=1}^{l_{t-1},m_{t-1},n_{t-1}} |a_{l_t,m_t,n_t,p,q,r}| \leq 2^{t-1}$, $\Sigma_{p=1,q=1,r=1}^{l_t,m_t,n_t} |a_{l_t,m_t,n_t,p,q,r}| \geq 2^{2t}$ ....(4)

Now from equations (4) we get

$$\Sigma_{p=1,q=1,r=n_{t-1}+1}^{l_{t-1},m_{t-1},n_t} |a_{l_t,m_t,n_t,p,q,r}| + \Sigma_{p=l_{t-1}+1,q=m_{t-1}+1,r=1}^{l_t,m_t,n_{t-1}} |a_{l_t,m_t,n_t,p,q,r}|$$
$$+ \Sigma_{p=l_{t-1}+1,q=m_{t-1}+1,r=n_{t-1}+1}^{l_t,m_t,n_t} |a_{l_t,m_t,n_t,p,q,r}| \geq 2^{2t} - 2^{t-1} \geq 2^{2t} - 2^{2t-1} = 2^{2t-1}$$

We now consider three sequences of integers as follows:

$l_1 < l_2 < l_3 < l_4 < l_5$ ....

$m_1 < m_2 < m_3 < m_4 < m_5$ .... and

$n_1 < n_2 < n_3 < n_4 < n_5$ ....

$\Sigma_{p=1,q=1,r=1}^{l_{t-1},m_{t-1},n_{t-1}} |a_{l_t,m_t,n_t,p,q,r}| \leq 2^{t-1}$,$t > 1$

$\Sigma_{p=1,q=1,r=n_{t-1}+1}^{l_{t-1},m_{t-1},n_t} |a_{l_t,m_t,n_t,p,q,r}| + \Sigma_{p=l_{t-1}+1,q=m_{t-1}+1,r=1}^{l_t,m_t,n_{t-1}} |a_{l_t,m_t,n_t,p,q,r}|$

$+ \Sigma_{p=l_{t-1}+1,q=m_{t-1}+1,r=n_{t-1}+1}^{l_t,m_t,n_t} |a_{l_t,m_t,n_t,p,q,r}| \leq 2^{2t-1}$ ....(5)

Consider the sequence $(x_{l,m,n})$ defined as follows:

$x_{l,m,n} = sgn\ a_{l_1,m_1,n_1,p,q,r}$, $p \leq l_1, q \leq m_1, r \leq n_1;$

$x_{l,m,n} = 1/2 sgn\ a_{l_2,m_2,n_2,p,q,r},$

$1 \leq p \leq l_1, 1 \leq q \leq m_1, 1 \leq r \leq n_1, l_1 < p \leq l_2, m_1 < q \leq m_2, n_1 < r \leq n_2;$

....

....

$x_{l,m,n} = 1/2^{t-1} sgn\ a_{l_t,m_t,n_t,p,q,r},$

$1 \leq p \leq l_{t-1}, 1 \leq q \leq m_{t-1}, 1 \leq r \leq n_{t-1}, l_{t-1}$

$< p \leq l_t, m_{t-1} < q \leq m_t, n_{t-1} < r \leq n_t;$

....

.................................(6)

Here $lim_{l,m,n\to\infty} x_{l,m,n} = 0$, now we consider

$y_{l_t,m_t,n_t} = \Sigma_{p=1,q=1,r=1}^{l_t,m_t,n_t} a_{l_t,m_t,n_t,p,q,r} x_{p,q,r}$

$= \Sigma_{p=1,q=1,r=1}^{l_{t-1},m_{t-1},n_{t-1}} a_{l_t,m_t,n_t,p,q,r} x_{p,q,r} + \Sigma_{p=1,q=1,r=n_{t-1}+1}^{l_{t-1},m_{t-1},n_t} a_{l_t,m_t,n_t,p,q,r} x_{p,q,r}$

$+ \Sigma_{p=l_{t-1}+1,q=m_{t-1}+1,r=1}^{l_t,m_t,n_{t-1}} a_{l_t,m_t,n_t,p,q,r} x_{p,q,r}$

$+ \Sigma_{p=l_{t-1}+1,q=m_{t-1}+1,r=n_{t-1}+1}^{l_t,m_t,n_t} a_{l_t,m_t,n_t,p,q,r} x_{p,q,r}$

From (5) and (6) we get the following result

$|\Sigma_{p=1,q=1,r=1}^{l_{t-1},m_{t-1},n_{t-1}} a_{l_t,m_t,n_t,p,q,r} x_{p,q,r}| \leq \Sigma_{p=1,q=1,r=1}^{l_{t-1},m_{t-1},n_{t-1}} |a_{l_t,m_t,n_t,p,q,r} x_{p,q,r}| \leq 2^{t-1}$

$\Sigma_{p=1,q=1,r=n_{t-1}+1}^{l_{t-1},m_{t-1},n_t} a_{l_t,m_t,n_t,p,q,r} x_{p,q,r} + \Sigma_{p=l_{t-1}+1,q=m_{t-1}+1,r=1}^{l_t,m_t,n_{t-1}} a_{l_t,m_t,n_t,p,q,r} x_{p,q,r}$

$+ \Sigma_{p=l_{t-1}+1,q=m_{t-1}+1,r=n_{t-1}+1}^{l_t,m_t,n_t} a_{l_t,m_t,n_t,p,q,r} x_{p,q,r}$

$= 1/2^{t-1}\ [\Sigma_{p=1,q=1,r=n_{t-1}+1}^{l_{t-1},m_{t-1},n_t} \mid a_{l_t,m_t,n_t,p,q,r} \mid$

$+ \Sigma_{p=l_{t-1}+1,q=m_{t-1}+1,r=1}^{l_t,m_t,n_{t-1}} \mid a_{l_t,m_t,n_t,p,q,r} \mid$

$+ \Sigma_{p=l_{t-1}+1,q=m_{t-1}+1,r=n_{t-1}+1}^{l_t,m_t,n_t} \mid a_{l_t,m_t,n_t,p,q,r} \mid] \geq (1/2^{t-1})2^{2t-1} = 2^t$

Hence $|y_{l_t,m_t,n_t}| \geq 2^t - 2^{t-1} = 2^{t-1}$

Therefore $lim_{t\to\infty}|y_{l_t,m_t,n_t}| = \infty$

Since this sub-sequence of the sequence $(y_{l,m,n})$ does not converge, so the sequence $(y_{l,m,n})$ does not have limit. Hence the necessity of condition (f) is established.

Proof of sufficiency:

Let the limit of the convergent sequence $(x_{l,m,n})$ be x, then

$y_{l,m,n} - x = \Sigma_{p=1,q=1,r=1}^{l,m,n} a_{l,m,n,p,q,r} x_{p,q,r} - x$

Using condition (b) we can write

$\Sigma_{p=1,q=1,r=1}^{l,m,n} a_{l,m,n,p,q,r} x_{p,q,r} + z_{l,m,n} = 1$ ....(7)

Where
$lim_{l,m,n\to\infty} z_{l,m,n} = 0$

Therefore

$y_{l,m,n} - x = \Sigma_{p=1,q=1,r=1}^{l,m,n} a_{l,m,n,p,q,r}(x_{p,q,r} - x) - z_{l,m,n} x$

Or,

$| y_{l,m,n} - x | \leq | \Sigma_{p=1,q=1,r=1}^{u,v,w} a_{l,m,n,p,q,r}(x_{p,q,r} - x) |$

$+ | \Sigma_{p=1,q=1,r=w+1}^{u,v,n} a_{l,m,n,p,q,r}(x_{p,q,r} - x) |$

$+ | \Sigma_{p=1,q=v+1,r=1}^{u,m,w} a_{l,m,n,p,q,r}(x_{p,q,r} - x) |$

$+ | \Sigma_{p=u+1,q=1,r=1}^{l,v,w} a_{l,m,n,p,q,r}(x_{p,q,r} - x) |$

$+ | \Sigma_{p=1,q=v+1,r=w+1}^{u,m,n} a_{l,m,n,p,q,r}(x_{p,q,r} - x) |$

$+ | \Sigma_{p=u+1,q=1,r=w+1}^{l,v,n} a_{l,m,n,p,q,r}(x_{p,q,r} - x) |$

$+ | \Sigma_{p=u+1,q=v+1,r=1}^{l,m,w} a_{l,m,n,p,q,r}(x_{p,q,r} - x) |$

$+ | \Sigma_{p=u+1,q=v+1,r=w+1}^{l,m,n} a_{l,m,n,p,q,r}(x_{p,q,r} - x) |$

$+ | z_{l,m,n}x |$ ....(8)

Consider $x_{l,m,n} \to x$, and choose u, v and w so large that for any pre-assigned small constant ϵ such that

$| x_{l,m,n} - x | < \epsilon/9A$ whenever $p \geq u$, $q \geq v$, $r \geq w$

Now we consider H be the greatest of the numbers $| x_{l,m,n} - x |$ for all p, q and r.We choose L, M and N such that whenever $l \geq L$, $m \geq M$, $n \geq N$, the following inequalities are satisfied:

$\Sigma_{p=1,q=1,r=1}^{u,v,w} | a_{l,m,n,p,q,r} | < \epsilon/(9uvwH)$ .... (from condition (a))

$\Sigma_{p=1}^{l} | a_{l,m,n,p,q,r} | < \epsilon/(9vwH)$, $q = 1, 2, ..., v; r = 1, 2, ..., w$ (from condition (c))

$\Sigma_{q=1}^{m} | a_{l,m,n,p,q,r} | < \epsilon/(9uwH)$, $p = 1, 2, ..., u; r = 1, 2, ..., w$ (from condition (d))

$\Sigma_{r=1}^{n} | a_{l,m,n,p,q,r} | < \epsilon/(9uvH)$, $p = 1, 2, ..., u; q = 1, 2, ..., v$ (from condition (e))

$z_{l,m,n} < \epsilon/(9 | x |)$ ....(from condition (7))

.................................(9)

Hence whenever $l \geq L$, $m \geq M$, $n \geq N$ we get

$y_{l,m,n} - x \leq \epsilon$ using results (8) and (9) we have

$\lim_{l,m,n\to\infty} y_{l,m,n} - x = 0$ or, $y_{l,m,n} \to x$

Hence the theorem.

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S.Debnath (Corresponding Author)
Department of Mathematics
Tripura University (A Central University)
Suryamaninagar-799022, Agartala,India.
E-mail address: shyamalnitamath@gmail.com

and

B.C.Das
Department of Mathematics
Govt. Degree College, Kamalpur-799285
Dhalai, Tripura,India.
E-mail address: bcdas3744@gmail.com

and

D.Bhattacharya and J.Debnath
Department of Mathematics
National Institute of Technology, Agartala-799055, INDIA.
E-mail address: dbhattacharya_2007@gmail.com
E-mail address: mailme_jdebnath@rediffmail.com