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Renormalized solutions for nonlinear parabolic problems with L^1-data in orlicz-sobolev spaces

Abdelmoujib Benkirane, Youssef El Hadfi, Mostafa El Moumni

ABSTRACT: In this work, we prove an existence result of renormalized solutions in Orlicz-Sobolev spaces for a class of nonlinear parabolic equations with two lower order terms and L^1-data.

Key Words: Nonlinear parabolic equations, Renormalized solutions, Orlicz-Sobolev spaces

Contents

1. Introduction

We consider the following nonlinear parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\Big(a(x,t,u,\nabla u) + \Phi(u)\Big) + \ g(x,t,u,\nabla u) = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, $T > 0$ and Q_T is the cylinder $\Omega \times (0,T)$. The operator $A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray-Lions operator defined in $W_0^{1,x}L_M(Q_T)$.

In the case where A is a Leary-Lions operator defined on $L^p(0,T;W^{1,p}(\Omega))$, Dall'aglio-Orsina [18] and Porretta [28] proved the existence of solutions for the problem (1.1), where g is a nonlinearity with the following "natural" growth condition (of order p)

$$|g(x,t,s,\xi)| \leq d(|s|)\big(c_1(x,t) + |\xi|^p\big),$$

and which satisfies the classical sign condition $g(x,t,s,\xi)s \geq 0$. The right hand side f is assumed to belong to $L^1(Q)$. This result generalizes analogous one of Boccardo-Gallouët [13], see also [12] and [14] for related topics. In all of these results, the

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function a is supposed to satisfy a polynomial growth condition with respect to u and ∇u. In the case where a and g satisfy a more general growth condition with respect to u and ∇u, it is shown in [19] that the appropriate space in which (1.1) can be studied is the inhomogeneous Orlicz-Sobolev space $W^{1,x}L_M(Q)$, where the N-function M is related to the actual growth of a and g. The solvability of (1.1) in this setting is only proved in the variational case i.e. where f belongs to the Orlicz space $W^{-1,x}E_{\overline{M}}(Q)$, see Donaldson [19] for $g \equiv 0$ and Robert [29] for $g \equiv g(x,t,u)$ when A is monotone, $t^2 \ll M(t)$ and $\overline{M}$ satisfies a Δ_2-condition and also Elmahi [20] for $g = g(x,t,u,\nabla u)$ when M satisfies a $\Delta'-$condition and $M(t) \ll t^{\frac{N}{N-1}}$ and finally the recent work Elmahi-Meskine [23] for the general case. A large number of papers was devoted to the study the existence of renormalized solution of parabolic problems under various assumptions and in different contexts: for a review on classical results see [6,7,9,10,15,16,17,28].

In the case where $\Phi = 0$, the existence of entropy solutions for parabolic problems of the form (1.1) in the setting of Orlicz spaces has been proved in A. Elmahi and D. Meskine [23] in the case where f belongs to $L^1(Q)$ and g be a carathéodory function satisfying

$$|g(x,t,s,\xi)| \leq b(|s|)\big(c(x,t) + M(|\xi|)\big)$$

$$g(x,t,s,\xi)s \geq 0.$$

It is our purpose, in this article, to prove the existence of renormalized solution for the problem (1.1) in the setting of the Orlicz Sobolev space $W^{1,x}L_M(Q)$, the nonlinearity g satisfying the sign condition and the function Φ is just assumed to be continuous on $\mathbb{R}$.

Let us briefly summarize the contents of this article. In section 2 we give some preliminaries and gives the definition of N-function and the Orlicz-Sobolev space. Section 3 is devoted to specifying the assumptions on a, Φ, g, f and the definition of a renormalized solution of (1.1). In Section 4 we establish (Theorem 4.1) the existence of such a solution.

2. Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ with the segment property. Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e., M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. Equivalently, M admits the representation : $M(t) = \int_0^t m(s)ds$ where $m : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with $m(0) = 0$, $\quad m(t) > 0 \quad \forall t > 0$ and $m(t) \to \infty$ as $t \to \infty$. The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \overline{m}(s)ds$, where $\overline{m} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\overline{m}(t) = \sup\{s : m(s) \leq t\}$ (see [1,5,26]). We will extend these N-functions into even functions on all $\mathbb{R}$. The N-function M is said to satisfy the Δ_2 condition if, for some $k > 0$:

$$M(2t) \leq kM(t) \quad \forall t \geq 0. \tag{2.1}$$

When this inequality holds only for $t \geq t_0 > 0$, M is said to satisfy the Δ_2 condition near infinity.

Let P, Q be two N-functions, $P \ll Q$ means that P grows essentially less rapidly than Q; i.e. for each $\varepsilon > 0$

$$P(t)/Q(\varepsilon t) \to 0 \quad \text{as } t \to \infty. \tag{2.2}$$

This is the case if and only if

$$\lim_{t\to\infty} Q^{-1}(t)/P^{-1}(t) = 0. \tag{2.3}$$

Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that :

$$\int_\Omega M(u(x))dx < +\infty \quad (resp. \int_\Omega M(\frac{u(x)}{\lambda})dx < +\infty \text{ for some } \lambda > 0). \tag{2.4}$$

Not that $L_M(\Omega)$ is a Banach space under the norm:

$$\|u\|_{M,\Omega} = \inf\{\lambda > 0 : \int_\Omega M(\frac{u(x)}{\lambda})dx \leqslant 1\} \tag{2.5}$$

and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the $\Delta_2$condition, for all t or for t large according to whether Ω has infinite measure or not. The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$. The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2-condition, for all t or for t large, according to whether Ω has infinite measure or not. We now turn to the Orlicz-Sobolev space. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). This is a Banach space under the norm

$$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\leq 1} \|\nabla^\alpha u\|_{M,\Omega}. \tag{2.6}$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\Pi L_M(\Omega)$, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $D(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$,

$$\int_\Omega M\Big(\frac{\nabla^\alpha u_n - \nabla^\alpha u}{\lambda}\Big)dx \to 0 \quad \text{for all } |\alpha| \leq 1.$$

This implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. If M satisfies the Δ_2-condition on $\mathbb{R}^+$ (near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence. Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (cf. [24,25]). Consequently, the action of a distribution T in $W^{-1}L_{\overline{M}}(\Omega)$ on an element u of $W_0^1L_M(\Omega)$ is well defined. It will be denoted by $\langle T, u\rangle$. For $k > 0$, we define the truncation at height $k, T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(s) = \min(k, \max(s, -k)). \tag{2.7}$$

The following lemmas can be found in [4].

Lemma 2.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let M be an N-function, $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$).* Moreover, if the set D of discontinuity points of F' *is finite, then*

$$\frac{\partial(F \circ u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \ in \ \{x \in \Omega : u(x) \notin D\} \\ 0 & a.e. \ in \ \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2.2. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. I suppose that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $F : W^1L_M(\Omega) \to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak $*$ topology $\sigma(\Pi L_M, \Pi E_{\bar{M}})$.*

Let Ω be a bounded open subset of $\mathbb{R}^N, T > 0$ and set $Q = \Omega \times (0, T)$. Let M be an N-function. For each $\alpha \in \mathbb{N}^N$, denote by ∇_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Orlicz-Sobolev spaces of order 1 are defined as follows

$$W^{1,x}L_M(Q) = \{u \in L_M(Q) : \quad \nabla_x^\alpha u \in L_M(Q) \quad \forall \quad |\alpha| \leq 1\} \text{ and} \tag{2.8}$$
$$W^{1,x}E_M(Q) = \{u \in E_M(Q) : \quad \nabla_x^\alpha u \in E_M(Q) \quad \forall \quad |\alpha| \leq 1\}. \tag{2.9}$$

The latter space is a subspace of the former. Both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\leq 1} \|\nabla_x^\alpha u\|_{M,Q}. \tag{2.10}$$

We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_M(Q)$ which has $(N + 1)$ copies. We shall also consider the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. if $u \in W^{1,x}L_M(Q)$ then the function: $t \to u(t) = u(., t)$ is defined on $(0, T)$ with values in $W^1L_M(\Omega)$. If, further,

$u \in W^{1,x}E_M(Q)$ then $u(.,t)$ is a $W^1E_M(\Omega)$-valued and is strongly measurable. Furthermore the following continuous imbedding holds:

$$W^{1,x}E_M(Q) \subset L^1(0,T;W^1E_M(\Omega)).$$

The space $W^{1,x}L_M(Q)$ is not in general separable, if $u \in W^{1,x}L_M(Q)$, we can not conclude that the function $u(t)$ is measurable from $(0,T)$ into $W^1L^M(\Omega)$. However, the scalar function $t \to \|\nabla_x^\alpha u(t)\|_{M,\Omega}$ is in $L^1(0,T)$ for all $|\alpha| \le 1$. The space $W_0^{1,x}E_M(Q)$ is defined as the (norm) closure in $W^{1,x}E_M(Q)$ of $\mathcal{D}(Q)$. We can easily show as in [25] (see the proof of theorem 3 below) that when Ω has the segment property then each element u of the closure of $D(Q)$ with respect to the weak $*$ topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ is limit, in $W^{1,x}L_M(Q)$; of some sequence $(u_n) \subset \mathcal{D}(Q)$ for the modular convergence i.e. there exists $\lambda > 0$ such that, for all $|\alpha| \leqslant 1$,

$$\int_Q M\Big(\frac{\nabla_x^\alpha u_n - \nabla_x^\alpha u}{\lambda}\Big)\,dx\,dt \to 0 \quad \text{ as } n \to \infty. \tag{2.11}$$

This implies that (u_n) converges to$u$ in $W^{1,x}L_M(Q)$ for the weak topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. Consequently,

$$\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_M, \Pi E_{\overline{M}})} = \overline{D(Q)}^{\sigma(\Pi L_M, \Pi L_{\overline{M}})}, \tag{2.12}$$

this space will be denoted by $W_0^{1,x}L_M(Q)$. Furthermore,

$$W_0^{1,x}E_M(Q) = W_0^{1,x}L_M(Q) \cap \Pi E_M.$$

Poincaré's inequality also holds in $W_0^{1,x}L_M(Q)$ and then there is a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q)$ one has

$$\sum_{|\alpha| \le 1} \|\nabla_x^\alpha u\|_{M,Q} \le C \sum_{|\alpha|=1} \|\nabla_x^\alpha u\|_{M,Q}, \tag{2.13}$$

thus both sides of the last inequality are equivalent norms on $W_0^{1,x}L_M(Q)$. We have then the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_M(Q) & F \\ & \\ W_0^{1,x}E_M(Q) & F_0 \end{pmatrix}, \tag{2.14}$$

F being the dual space of $W_0^{1,x}E_M(Q)$. It is also, up to an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q)^\perp$, and will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q)$ and it is shown that

$$W^{-1,x}L_{\overline{M}}(Q) = \{f = \sum_{|\alpha| \le 1} \nabla_x^\alpha f_\alpha : \quad f_\alpha \in L_{\overline{M}}(Q)\}. \tag{2.15}$$

This space will be equipped with the usual quotient norm:

$$\|f\| = \inf \sum_{|\alpha| \le 1} \|f_\alpha\|_{\overline{M},Q} \tag{2.16}$$

where the inf is taken over all possible decompositions

$$f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha : \quad f_\alpha \in L_{\overline{M}}(Q). \tag{2.17}$$

The space F_0 is then given by

$$F_0 = \{f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha : \quad f_\alpha \in E_{\overline{M}}(Q)\} \tag{2.18}$$

and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q)$.

Remark 2.3. *We can easily check, using lemma 2.1, that each uniformly lipschitzian mapping F, with $F(0) = 0$, acts in inhomogeneous Orlics-Sobolev spaces of order 1: $W^{1,x}L_M(Q)$ and $W_0^{1,x}L_M(Q)$.*

Corollary 2.4. *Let M be an N-function. Let (u_n) be a sequence of $W^{1,x}L_M(Q)$ such that*

$$u_n \rightharpoonup u \text{ weakly in } W^{1,x}L_M(Q) \text{ for } (\Pi L_M, \Pi E_{\overline{M}})$$

and

$$\frac{\partial u_n}{\partial t} = h_n + k_n \text{ in } \mathcal{D}'(Q)$$

with (h_n) bounded in $W^{-1,x}L_M(Q)$ and (k_n) bounded in the space $\mathcal{M}(Q)$ of measures on Q. Then

$$u_n \longrightarrow u \text{ strongly in } L^1_{loc}(Q).$$

If further $u_n \in W_0^{1,x}L_M(Q)$ then $u_n \longrightarrow u$ strongly in $L^1(Q)$.

Proof: (See [22]) □

3. Basic assumptions, definition of a renormalized solution

Throughout the paper, we assume that the following assumptions hold true: Let Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, $T > 0$ is given and we set $Q_T = \Omega\times(0, T)$. Let M and P be two $N-$functions such that $P \ll Q$.

Consider a second order operator $A : D(A) \subset W^{1,x}L_M(Q) \to W^{-1,x}L_{\overline{M}}(Q)$ in divergence form

$$A(u) = -\text{ div } a(x, t, u, \nabla u)$$

where $a : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying for almost every $(x, t) \in \Omega \times [0, T]$ and all $s \in \mathbb{R}, \xi \neq \eta \in \mathbb{R}^N$:

$$|a(x, t, s, \xi)| \leqslant b(|s|)\big(c(x, t) + \overline{M}^{-1} M(\nu|\xi|)\big) \tag{3.1}$$

$$\big[a(x, t, s, \xi) - a(x, t, s, \eta)\big]\,[\xi - \eta] > 0 \tag{3.2}$$

$$a(x, t, s, \xi)\xi \geqslant \alpha M(|\xi|) \tag{3.3}$$

where $c(x,t) \in E_{\overline{M}}(Q), c \geqslant 0, b : [0,+\infty) \to [0,+\infty)$ a continuous and nondecreasing function;$\eta, \alpha > 0$ Note that, (3.3) written for $\xi = \epsilon\zeta \quad (\epsilon > 0)$, and the fact that a is a Carathéodory function, imply that $a(x,t,s,0) = 0$ for almost every $(x,t) \in Q$ and every $s \in \mathbb{R}$. Let $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function satisfying for a.e. $(x,t) \in \Omega \times [0,T]$ and forall $s \in \mathbb{R}, \xi \in \mathbb{R}^N$:

$$|g(x,t,s,\xi)| \leqslant d(|s|)(c_2(x,t) + M(|\xi|)) \tag{3.4}$$

$$g(x,t,s,\xi)s \geq 0 \tag{3.5}$$

where $c_2(x,t) \in L^1(Q)$ and $d : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function. Furthermore let

$$\Phi : \mathbb{R} \to \mathbb{R}^N \quad \text{ is a continuous function} \tag{3.6}$$

$$f \text{ is an element of } L^1(Q_T). \tag{3.7}$$

$$u_0 \text{ is an element of } L^1(\Omega). \tag{3.8}$$

Remark 3.1. *As already mentioned in the introduction, problem (1.1) does not admit a weak solution under assumptions 3.1- 3.7 since the growths of $a(x,t,u,\nabla u)$ and $\Phi(u)$ are not controlled with respect to u (so that these fields are not in general defined as distributions, even when u belongs to $W^{1,x}L_M(Q_T)$.*

Throughout this paper $\langle , \rangle$ means for either the pairing between $W_0^{1,x}L_M(Q_T) \cap L^\infty(Q_T)$ and $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ or between $W^{1,x}L_M(Q_T)$ and $W^{-1,x}L_{\overline{M}}(Q_T)$ and $Q_\tau = \Omega \times (0,\tau)$ for $\tau \in [0,T]$.

The definition of a renormalized solution for problem (1.1) can be stated as follows.

Definition 3.2. *A measurable function u defined on Q_T is a renormalized solution of problem (1.1) if:*

$$u \in L^\infty(0,T;L^1(\Omega)) \text{ and } T_k(u) \in W_0^{1,x}L_M(Q_T) \text{ for every } k \geq 0, \tag{3.9}$$

$$\int_{\{(x,t)\in Q_T : n \leq |u(x,t)| \leq n+1\}} a(x,t,u,\nabla u)\nabla u \to 0 \qquad \text{as } n \to +\infty, \tag{3.10}$$

and if, for every function $S \in W^{2,\infty}(\mathbb{R})$, which is piecewise C^1 and such that S' has a compact support, we have

$$\begin{aligned} &\frac{\partial S(u)}{\partial t} - \ div \big(S'(u)a(x,t,u,\nabla u)\big) + S''(u)a(x,t,u,\nabla u)\nabla u \\ &- \ div \big(S'(u)\Phi(u)\big) + S''(u)\Phi(u)\nabla u + g(x,t,u,\nabla u)S'(u) = fS'(u) \in \mathcal{D}'(Q_T) \end{aligned} \tag{3.11}$$

Remark 3.3. *Equation (3.11) is formally obtained through pointwise multiplication of (1.1) by $S'(u)$. However, while $a(x,t,u,\nabla u), \Phi(u)$ and $g(x,t,u,\nabla u)$ does not in general make sense in 1.1, all the terms in (3.11) have a meaning in $D'(\Omega)$ and $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$.*
Indeed, if K is such that $suppS' \subset [-K,K]$, the following identifications are made in (3.11):

- *$S'(u)a(x,t,u,\nabla u)$ identifies with $S'(u)a(x,t,T_K(u),\nabla T_K(u))$ a.e in Q_T. Since indeed $T_K(u)\le K$ a.e in Q_T. Since $S'(u)\in L^\infty(Q_T)$ and with (3.1), (3.9) we obtain that*

$$S'(u)a(x,t,T_K(u),\nabla T_K(u))\in(L_{\overline{M}}(Q_T))^N.$$

- *$S''(u)a(x,t,u,\nabla u)\nabla u$ identifies with $S''(u)a(x,t,T_K(u),\nabla T_K(u))\nabla T_K(u)$ and by the same arguments as above we get*

$$S''(u)a(x,t,T_K(u),\nabla T_K(u))\nabla T_K(u)\in L^1(Q_T).$$

- *$S'(u)\Phi(u)$ and $S''(u)\Phi(u)\nabla u$ respectively identify with $S'(u)\Phi(T_K(u))$ and $S''(u)\Phi(T_K(u))\nabla T_K(u)$. Due to the properties of S and Φ is a continuous function, the functions S', S'' and $\Phi\circ T_K$ are bounded on $\mathbb{R}$ so that (3.9) implies that $S'(u)\Phi(T_K(u))\in(L^\infty(Q_T))^N$, and $S''(u)\Phi(T_K(u))\nabla T_K(u)\in(L_M(Q_T))^N$.*
- *$S'(u)g(x,t,u,\nabla u)\in L^1(Q_T)$ by (3.4).*
- *$S^{'}(u)f\in L^1(Q_T)$ by (3.7).*

4. Statements of results

This section is devoted to establish the following existence theorem:

Theorem 4.1. *Assume that (3.1)-(3.7) hold true. Then the problem (1.1) admits at least a renormalized solution.*

Proof: The proof of Theorem 4.1 is done in 6 steps.

Step 1: Approximate problem.

For $n\in\mathbb{N}^*$, let us define the following approximation of a, g, Φ and f:

$$g_n(x,t,s,\xi)=\frac{g(x,t,s,\xi)}{1+\frac{1}{n}|g(x,t,s,\xi)|} \tag{4.1}$$

$$a_n(x,t,s,\xi)=a_n(x,t,T_n(s),\xi)\qquad\text{a.e in } Q_T,\forall s\in\mathbb{R},\forall\xi\in\mathbb{R}^N, \tag{4.2}$$

$$\Phi_n \text{ is a Lipschitz continuous bounded function from } \mathbb{R} \text{ into } \mathbb{R}^N, \tag{4.3}$$

such that Φ_n uniformly converges to Φ on any compact subest of $\mathbb{R}$ as $n\longrightarrow+\infty$.

$$u_{0n}\in C_0^\infty(\Omega):\|u_{0n}\|_{L^1(\Omega)}\le\|u_0\|_{L^1(\Omega)} \text{ and } u_{0n}\to u_0 \text{ in } L^1(\Omega) \text{ as } n \text{ tends to } +\infty. \tag{4.4}$$

$$f_n\in C_0^\infty(Q_T)\,/\,f_n\to f \text{ in } L^1(Q_T) \text{ as } n \text{ tends to } +\infty \text{ and } \|f_n\|_{L^1(Q_T)}\le\|f\|_{L^1(Q_T)}. \tag{4.5}$$

Let us now consider the approximate problem:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \operatorname{div}\Big(a_n(x,t,u_n,\nabla u_n) + \Phi_n(u_n)\Big) + g_n(x,t,u_n,\nabla u_n) = f_n & \text{in } \Omega \times (0,T), \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(x,0) = u_{0n}(x) & \text{in } \Omega. \end{cases} \tag{4.6}$$

Note that $g_n(x,t,u_n,\nabla u_n)$ satisfies the following conditions

$$|\, g_n(x,t,s,\xi)| \leq \; g(x,t,s,\xi) \text{ and } |\, g_n(x,t,s,\xi)| \leq n. \tag{4.7}$$

Since g_n is bounded for any fixed n, as a consequence, proving of a weak solution $u_n \in W_0^{1,x}L_M(Q_T)$ of (4.6) is an easy task (see e.g. [21,27]).

Step 2: A priori estimates.

The estimates derived in this step rely on usual techniques for problems of the type (4.6).

Proposition 4.2. *Assume that (3.1)-(3.7) are satisfied, and let u_n be a solution of the approximate problem (4.6). Then for all ℓ, $n > 0$, we have*

i) $\|T_\ell(u_n)\|_{W_0^{1,x}L_M(Q_T)} \leq \ell(\|f\|_{L^1(Q_T)} + C_g + \|u_0\|_{L^1(\Omega)}) \equiv C\ell,$

ii) $\lim\limits_{\ell\to+\infty} (meas\{(x,t) \in Q_T : |u_n| > \ell\}) = 0$ *uniformly with respect to* n

iii) $\displaystyle\int_{Q_T} g_n(x,t,u_n,\nabla u_n) \leq C_g$

where C_g *is a positive constant not depending on* n.

Proof: We take $T_\ell(u_n)\chi_{(0,\tau)}$ as test function in (4.6), we get for every $\tau \in (0,T)$

$$\begin{aligned} &\langle \frac{\partial u_n}{\partial t}, T_\ell(u_n)\chi_{(0,\tau)}\rangle + \int_{Q_\tau} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)dx\,dt \\ &+ \int_{Q_\tau} \Phi_n(u_n)\nabla T_\ell(u_n)dx\,dt + \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)T_\ell(u_n)dx\,dt \\ &= \int_{Q_\tau} f_n T_\ell(u_n)dx\,dt, \end{aligned} \tag{4.8}$$

which implies that

$$\begin{aligned} &\int_\Omega \widehat{T}_\ell(u_n(\tau))dx + \int_{Q_\tau} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)dx\,dt \\ &+ \int_{Q_\tau} \Phi_n(u_n)\nabla T_\ell(u_n)dx\,dt \\ &= \int_{Q_\tau} f_n T_\ell(u_n)dx\,dt - \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)T_\ell(u_n)dx\,dt + \int_\Omega \widehat{T}_\ell(u_{0n})dx \end{aligned} \tag{4.9}$$

where

$$\widehat{T}_\ell(s) = \int_0^s T_\ell(t)dt = \begin{cases} \frac{s^2}{2} & \text{if } |s| \leq \ell, \\ \ell|s| - \frac{\ell^2}{2} & \text{if } |s| \geq \ell. \end{cases}$$

The Lipshitz character of Φ_n, Stokes formula together with the boundary condition $u_n = 0$ on $(0,T)\times\Omega$, make it possible to obtain

$$\int_{Q_\tau} \Phi_n(u_n)\nabla T_\ell(u_n)dx\,dt = 0. \tag{4.10}$$

Due to the definition of $\widehat{T}_\ell$ and (4.4) we have

$$0 \leq \int_\Omega \widehat{T}_\ell(u_{0n})dx \leq \ell \int_\Omega |u_{0n}|dx \leq \ell\|u_0\|_{L^1(\Omega)}. \tag{4.11}$$

Consider now for $\theta, \epsilon > 0$ a function $\varrho_\theta^\epsilon \in C^1(\mathbb{R})$ such that

$$\varrho_\theta^\epsilon(s) = \begin{cases} 0 & \text{if } |s| \leq \theta, \\ \text{sign}(s) & \text{if } |s| \geq \theta + \epsilon \end{cases} \tag{4.12}$$

and

$$(\varrho_\theta^\epsilon)'(s) \geq 0 \quad \forall s \in \mathbb{R},$$

then, by using $\varrho_\theta^\epsilon(u_n)$ as a test function in (4.6) and following [28], we can see that

$$\int_{\{|u_n|>\theta\}} |g_n(x,t,u_n,\nabla u_n)|dx\,dt \leq \int_{\{|u_n|>\theta\}} |f_n|dx\,dt + \int_{\{|u_n|>\theta\}} |u_{0n}|dx\,dt \tag{4.13}$$

and so by letting $\theta \longrightarrow 0$ and using Fatou's lemma, we deduce that $g_n(x,t,u_n,\nabla u_n)$ is a bounded sequence in $L^1(Q_T)$, then we obtain iii). By using (4.5), (4.10), (4.11) and iii), permit to deduce from (4.9) that

$$\begin{aligned} \int_\Omega & \widehat{T}_\ell(u_n(\tau))dx + \int_{Q_\tau} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)dx\,dt \\ & = \int_{Q_\tau} f_n T_\ell(u_n)dx\,dt - \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)T_\ell(u_n)dx\,dt + \int_\Omega \widehat{T}_\ell(u_{0n})dx \\ & \leq |\int_{Q_\tau} f_n T_\ell(u_n)dx\,dt| + |\int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)T_\ell(u_n)dx\,dt| + \ell\|u_0\|_{L^1(\Omega)} \\ & \leq \ell\|f_n\|_{L^1(Q_\tau)} + \ell C_g + \ell\|u_0\|_{L^1(\Omega)}. \\ & \leq (\|f\|_{L^1(Q_T)} + C_g + \|u_0\|_{L^1(\Omega)})\ell \\ & \leq C_0\ell, \end{aligned} \tag{4.14}$$

where here and below C_i denote positive constants not depending on n and ℓ. By using (4.14) and the fact that $\widehat{T}_\ell(u_n) \geq 0$, permit to deduce that

$$\int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)dx\,dt \leq C_0\ell, \tag{4.15}$$

which implies by virtue of (3.3) that

$$\int_{Q_T} M(\nabla T_\ell(u_n))dx\,dt \le C_1\ell. \tag{4.16}$$

We deduce from that above inequality (4.14) that

$$\int_\Omega \widehat{T}_\ell(u_n(\tau))dx \le C_0\ell, \quad \text{for almost any } \tau \text{ in } (0,T). \tag{4.17}$$

On the other hand, thanks to Lemma 5.7 of [25], there exists two positive constants δ, λ such that

$$\int_{Q_T} M(v)dx\,dt \le \delta \int_{Q_T} M(\lambda|\nabla v|)dx\,dt \quad \text{for all } v \in W_0^{1,x}L_M(Q_T). \tag{4.18}$$

Taking $v = \dfrac{T_\ell(u_n)}{\lambda}$ in (4.18) and using (4.16), one has

$$\int_{Q_T} M\big(\frac{T_\ell(u_n)}{\lambda}\big)dx\,dt \le C_1\ell, \tag{4.19}$$

which implies that

$$meas\{(x,t) \in Q_T : |u_n| > \ell\} \le \frac{C_2\ell}{M(\frac{\ell}{\lambda})} \tag{4.20}$$

so that

$$\lim_{\ell\to+\infty} \big(meas\{(x,t) \in Q_T : |u_n| > \ell\}\big) = 0 \quad \text{uniformly with respect to } n, \tag{4.21}$$

□

Which completes the proof. We prove the following proposition:

Proposition 4.3. *Let u_n be a solution of the approximate problem (4.6), then we have the following properties:*

$$u_n \longrightarrow u \quad a.\ e.\ in\ Q_T, \tag{4.22}$$

$$a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) \rightharpoonup \varphi_\ell \text{ weakly in } (L_{\overline{M}}(Q_T))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M) \tag{4.23}$$

for some $\varphi_\ell \in (L_{\overline{M}}(Q_T))^N$.

Proof: We have from (4.19) that $T_\ell(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$ for every $\ell > 0$. Consider now a nondecreasing function $\zeta_\ell(s) = s$ for $|s| \le \dfrac{\ell}{2}$ and $\zeta_\ell(s) = \ell$ sign (s). Multiplying the approximating equation by $\zeta'_\ell(u_n)$, we obtain

$$\begin{aligned}\frac{\partial(\zeta_\ell(u_n))}{\partial t} &= \mathrm{div}\big(a_n(x,t,u_n,\nabla u_n)\zeta'_\ell(u_n)\big) - a_n(x,t,u_n,\nabla u_n)\zeta''_\ell(u_n)\nabla u_n \\ &+\mathrm{div}(\zeta'_\ell(u_n)\Phi_n(u_n)) - g_n(x,t,u_n,\nabla u_n)\zeta'_\ell(u_n) + f_n\zeta'_\ell(u_n),\end{aligned} \tag{4.24}$$

in the sense of distributions. This implies, thanks to (4.19) and the fact that $\zeta_\ell^{'}$ has compact support, that $\zeta_\ell(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$ while its time derivative $\frac{\partial(\zeta_\ell(u_n))}{\partial t}$ is bounded in $W_0^{-1,x}L_M(Q_T)+L^1(Q_T)$, hence Corolloray 2.4 allows us to conclude that $\zeta_\ell(u_n)$ is compact in $L^1(Q_T)$. Due to the choice of ζ_ℓ, we conclude that for each ℓ, the sequence $T_\ell(u_n)$ converges almost everywhere in Q_T, which implies that u_n converges almost everywhere to some measurable function u in Q_T . Therfore, following [7,8,10,11,28], we can see that there exists a measurable function $u \in L^\infty(0,T;L^1(\Omega))$ such that for every $\ell > 0$ and a subsequence, not relabeled,

$$u_n \longrightarrow u \quad \text{a. e. in } Q_T,$$

and

$$T_\ell(u_n) \rightharpoonup T_\ell(u) \text{ weakly in } W_0^{1,x}L_M(Q_T) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}), \tag{4.25}$$

$$\text{strongly in } L^1(Q_T) \text{ and a. e. in } Q_T.$$

We prove that $a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))$ is bounded sequence in $L_{\overline{M}}(Q_T)$. Let $\varphi \in (E_M(Q_T))^N$ with $\|\varphi\|_{M,Q_T} = 1$. In view of the monotonicity of a one easily has,

$$\int_{Q_T} \big[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a_n(x,t,T_\ell(u_n),\varphi)\big]\big[\nabla T_\ell(u_n) - \varphi\big]dx\,dt \geq 0, \tag{4.26}$$

which gives

$$\begin{aligned}\int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\varphi dx\,dt \leq \int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n) dx\,dt \\ + \int_{Q_T} a_n(x,t,T_\ell(u_n),\varphi)\big[\nabla T_\ell(u_n) - \varphi\big]dx\,dt,\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}-\int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\varphi dx\,dt \leq \int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n) dx\,dt \\ - \int_{Q_T} a_n(x,t,T_\ell(u_n),-\varphi)\big[\nabla T_\ell(u_n) + \varphi\big]dx\,dt.\end{aligned} \tag{4.28}$$

On the other hand, using (3.1), we see that

$$\overline{M}\left(\frac{|a_n(x,t,T_\ell(u_n),\varphi)|}{2b(\ell)}\right) \leq \overline{M}(c(x,t)) + M(\nu|\varphi|). \tag{4.29}$$

Then, by (4.15) and (4.29) we get that $a_n(x,t,T_\ell(u_n),\varphi)$ is bounded in $(L_{\overline{M}}(Q_T))^N$, implying that, since $T_\ell(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$

$$\left|\int_{Q_T} a_n(x,t,T_\ell(u_n),\varphi)[\nabla T_\ell(u_n) - \varphi]dx\,dt\right| \leq C_4. \tag{4.30}$$

And so, by using the dual norm of $(L_{\overline{M}}(Q_T))^N$ we conclude that $a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))$ is a bounded sequence in $(L_{\overline{M}}(Q_T))^N$, and we obtain (4.23). $\square$

Lemma 4.4. *Let u_u be a solution of the approximate problem (4.6). Then*

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\;=\;0. \tag{4.31}$$

Proof: Considering the following function $\varphi = T_1(u_n - T_m(u_n))$ as test function in (4.6) we obtain,

$$\begin{aligned}
&\langle\frac{\partial u_n}{\partial t}, T_1(u_n - T_m(u_n))\rangle + \int_{\{m\le|u_n|\le m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\
&+\int_{Q_T}\mathrm{div}\Big[\int_0^{u_n}\Phi(r)T_1^{'}(r-T_m(r))dr\Big]dx\,dt\;=\int_{Q_T} f_nT_1(u_n-T_m(u_n))dx\,dt\\
&-\int_{Q_T} g_n(x,t,u_n,\nabla u_n)T_1(u_n-T_m(u_n))dx\,dt.
\end{aligned} \tag{4.32}$$

Using the fact that $\displaystyle\int_0^{u_n}\Phi(r)T_1^{'}(r-T_m(r))dr \in W_0^{1,x}L_M(Q_T)$ and Stokes formula, we get

$$\begin{aligned}
&\int_\Omega U_n^m(u_n(T))dx + \int_{\{m\le|u_n|\le m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\
&\le \int_{Q_T}|f_nT_1(u_n-T_m(u_n))|dx\,dt\;+\int_{Q_T}|g_n(x,t,u_n,\nabla u_n)T_1(u_n-T_m(u_n))|dx\,dt\\
&+\int_\Omega U_n^m(x,u_{0n})dx \le \int_{Q_T}(|f_n|+|g_n(x,t,u_n,\nabla u_n)|)|T_1(u_n-T_m(u_n))|dx\,dt\\
&+\int_\Omega U_n^m(u_{0n})dx,
\end{aligned} \tag{4.33}$$

where $U_n^m(r) = \displaystyle\int_0^r \frac{\partial u_n}{\partial t}T_1(s-T_m(s))ds$. In order to pass to the limit as n tends to $+\infty$ in (4.33), we us $U_n^m(u_n(T)) \ge 0$, $iii)$ and (4.5) we obtain that,

$$\begin{aligned}
&\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n dx\,dt\\
&\le \int_{\{|u|>m\}}(|f|+C_g)dx\,dt + \int_{\{|u_0|>m\}}|u_0|dx.
\end{aligned} \tag{4.34}$$

Finally by (3.7), (3.8) and (4.34) we obtain (4.31). $\square$

Step 3: Almost everywhere convergence of the gradients.

Fix $\ell > 0$ and let $\varphi(r) = r\exp^{\delta r^2}, \delta > 0$. It is well known that when $\delta \ge (\frac{b(\ell)}{2\alpha})^2$ one has

$$\varphi'(r) - \frac{b(\ell)}{\alpha}|\varphi(r)| \ge \frac{1}{2} \quad \text{for all } r\in\mathbb{R}. \tag{4.35}$$

Proposition 4.5. *Let u_n be a solution of the approximate problem (4.6). Then, for any $\ell \geq 0$*

$$\nabla T_\ell(u_n) \longrightarrow \nabla T_\ell(u) \qquad a.\ e.\ in\ Q_T, \tag{4.36}$$

$$a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) \rightharpoonup a(x,t,T_\ell(u),\nabla T_\ell(u)) \quad weakly\ in\ (L_{\overline{M}(Q_T)})^N, \tag{4.37}$$

$$M(|\nabla T_\ell(u_n)|) \to M(|\nabla T_\ell(u)|) \quad strongly\ in\ L^1(Q_T), \tag{4.38}$$

as n tends to $+\infty$.

Let use give the following lemma which will be needed later:

Lemma 4.6. *Assume that (3.1)-(3.7) are satisfied, and let z_n be a sequence in $W_0^{1,x}L_M(Q_T)$ such that,*

$$z_n \rightharpoonup z \qquad in\ W_0^{1,x}L_M(Q_T)\ for\ \sigma(\Pi L_M(Q_T), PiE_{\overline{M}}(Q_T)), \tag{4.39}$$

$$(a_n(x,t,z_n,\nabla z_n))_n\ is\ bounded\ in\ (L_{\overline{M}}(Q_T)))^N, \tag{4.40}$$

$$\int_{Q_T}\big[a_n(x,t,z_n,\nabla z_n) - a_n(x,t,z_n,\nabla z\chi_s)\big]\big[\nabla z_n - \nabla z\chi_s\big]dx\,dt \to 0, \tag{4.41}$$

as n and s tend to $+\infty$, and where χ_s is the characteristic function of

$$Q_s = \{(x,t) \in Q_T; |\nabla z| \leq s\}.$$

Then

$$\nabla z_n \longrightarrow \nabla z\ a.\ e.\ in\ Q_T, \tag{4.42}$$

$$\lim_{n\to+\infty}\int_{Q_T} a_n(x,t,z_n,\nabla z_n)\nabla z_n dx\,dt = \int_{Q_T} a(x,t,z,\nabla z)\nabla z dx\,dt, \tag{4.43}$$

$$M(|\nabla z_n|) \longrightarrow M(|\nabla z|)\ in\ L^1(Q_T). \tag{4.44}$$

Proof: See [23]. □

Proof: (Proposition 4.5).The proof is almost identical of the one given in, e.g. [23]. where the result is established for the growth of $a(x,t,u,Du)$ is controlled with respect to u. This proof is devoted to introduce for $\ell \geq 0$ fixed, a time regularization of the function $T_\ell(u)$, this notion, introduced by R. Landes (see Lemma 6 and Proposition 3, p. 230 and Proposition 4, p. 231 in [27]). More recently, it has been exploited in [13] and [18] to solve a few nonlinear evolution problems with L^1 or measure data.

Let $v_j \in D(Q_T)$ be a sequence such that $v_j \to u$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence and let ψ_i be a sequence which converges strongly to u_0 in $L^1(\Omega)$.

Let $\omega^\beta_{i,j} = T_\ell(v_j)_\beta + \exp^{-\beta t} T_\ell(\psi_i)$ where $T_\ell(v_j)_\beta$ is the mollification with respect to time of $T_\ell(v_j)$, note that $\omega^\beta_{i,j}$ is a smooth function having the following properties:

$$\frac{\partial \omega^\beta_{i,j}}{\partial t} = \beta(T_\ell(v_j) - \omega^\beta_{i,j}), \omega^\beta_{i,j}(0) = T_\ell(\psi_i), \quad |\omega^\beta_{i,j}| \leq \ell, \tag{4.45}$$

$$\omega_{\imath,\jmath}^{\beta} \to T_\ell(u)_\beta + \exp^{-\beta t} T_\ell(\psi_\imath) \text{ in } W_0^{1,x} L_M(Q_T), \tag{4.46}$$

for the modular convergence as $\jmath \to \infty$,

$$T_\ell(u)_\beta + \exp^{-\beta t} T_\ell(\psi_\imath) \to T_\ell(u) \text{ in } W_0^{1,x} L_M(Q_T), \tag{4.47}$$

for the modular convergence as $\imath \to \infty$. Let now the function ρ_m defined on $\mathbb{R}$ with $m \geq \ell$ by:

$$\rho_m(r) = \begin{cases} 1 & \text{if } |r| \leq m, \\ m+1-|r| & \text{if } m \leq |r| \leq m+1, \\ 0 & \text{if } |r| \geq m+1. \end{cases}$$

Let $\theta_{\imath,\jmath}^{\beta,n} = T_\ell(u_n) - \omega_{\imath,\jmath}^{\beta}$ and $\varphi_{\imath,\jmath,n}^{\beta,m} = \varphi(\theta_{\imath,\jmath}^{\beta,n})\rho_m(u_n)$.

Using the admissible test function $\varphi_{\imath,\jmath,n}^{\beta,m}$ as test function in (4.6) leads to

$$\begin{aligned}
&\langle \frac{\partial u_n}{\partial t}, \varphi_{\imath,\jmath,n}^{\beta,m} \rangle + \int_{Q_T} a_n(x,t,u_n,\nabla u_n)(\nabla T_\ell(u_n) - \nabla \omega_{\imath,\jmath}^{\beta})\varphi'(\theta_{\imath,\jmath}^{\beta,n})\rho_m(u_n)dx\,dt \\
&+ \int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla u_n \varphi(\theta_{\imath,\jmath}^{\beta,n})\rho_m'(u_n)dx\,dt \\
&+ \int_{\{m\leq|u_n|\leq m+1\}} \Phi_n(u_n)\nabla u_n \rho_m'(u_n)\varphi(\theta_{\imath,\jmath}^{\beta,n})dx\,dt \\
&+ \int_{Q_T} \Phi_n(u_n)\rho_m(u_n)(\nabla T_\ell(u_n) - \nabla\omega_{\imath,\jmath}^{\beta})\varphi'(\theta_{\imath,\jmath}^{\beta,n})dx\,dt \\
&+ \int_{Q_T} g_n(x,t,u_n,\nabla u_n)\varphi_{\imath,\jmath,n}^{\beta,m} dx\,dt \\
&= \int_{Q_T} f_n \varphi_{\imath,\jmath,n}^{\beta,m} dx\,dt.
\end{aligned} \tag{4.48}$$

Which implies, since $g_n(x,t,u_n,\nabla u_n)\varphi(T_\ell(u_n) - \omega_{\imath,\jmath}^{\beta})\rho_m(u_n) \geq 0$ on $\{|u_n| > \ell\}$:

$$\begin{aligned}
&\langle \frac{\partial u_n}{\partial t}, \varphi_{\imath,\jmath,n}^{\beta,m} \rangle + \int_{Q_T} a_n(x,t,u_n,\nabla u_n)(\nabla T_\ell(u_n) - \nabla \omega_{\imath,\jmath}^{\beta})\varphi'(\theta_{\imath,\jmath}^{\beta,n})\rho_m(u_n)dx\,dt \\
&+ \int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla u_n \varphi(\theta_{\imath,\jmath}^{\beta,n})\rho_m'(u_n)dx\,dt \\
&+ \int_{\{m\leq|u_n|\leq m+1\}} \Phi_n(u_n)\nabla u_n \rho_m'(u_n)\varphi(\theta_{\imath,\jmath}^{\beta,n})dx\,dt \\
&+ \int_{Q_T} \Phi_n(u_n)\rho_m(u_n)(\nabla T_\ell(u_n) - \nabla\omega_{\imath,\jmath}^{\beta})\varphi'(\theta_{\imath,\jmath}^{\beta,n})dx\,dt \\
&+ \int_{\{|u_n|\leq \ell\}} g_n(x,t,u_n,\nabla u_n)\varphi(T_\ell(u_n) - \omega_{\imath,\jmath}^{\beta})\rho_m(u_n) dx\,dt \\
&\leq \int_{Q_T} f_n \varphi_{\imath,\jmath,n}^{\beta,m} dx\,dt.
\end{aligned} \tag{4.49}$$

Denoting by $\epsilon(\imath,\jmath,\beta,n)$ any quantity such that,

$$\lim_{\imath\to\infty} \lim_{\jmath\to\infty} \lim_{\beta\to\infty} \lim_{m\to\infty} \epsilon(\imath,\jmath,\beta,n) = 0.$$

The very definition of the sequence $\omega^{\beta}_{\imath,\jmath}$ makes it possible to establish the following lemma.

Lemma 4.7. *Let $\varphi^{\beta,m}_{\imath,\jmath,n} = \varphi(T_\ell(u_n) - \omega^{\beta}_{\imath,\jmath})\rho_m(u_n)$, we have for any $\ell \geq 0$:*

$$\langle \frac{\partial u_n}{\partial t}, \varphi^{\beta,m}_{\imath,\jmath,n} \rangle \geq \epsilon(\imath,\jmath,\beta,n), \tag{4.50}$$

where $\langle,\rangle$ denotes the duality pairing between $L^1(Q_T)+W^{-1,x}L_{\overline{M}}(Q_T)$ and $L^\infty(Q_T)$ $\cap W^{1,x}_0 L_M(Q_T))$.

Proof: See ([22]). □

Now, we turn to complete the proof of Proposition 4.5. First, it is easy to see that

$$\int_{Q_T} f_n \varphi^{\beta,m}_{\imath,\jmath,n} = \epsilon(\jmath,\beta,n) \tag{4.51}$$

Indeed, by the almost everywhere convergence of u_n, we have that $\varphi(T_\ell(u_n) - \omega^{\beta}_{\imath,\jmath})\rho_m(u_n)$ converges to $\varphi(T_\ell(u) - \omega^{\beta}_{\imath,\jmath})\rho_m(u)$ weakly $*$ in $L^\infty(Q_T)$ and then

$$\lim_{n\longrightarrow\infty} \int_{Q_T} f_n \varphi(T_\ell(u_n) - \omega^{\beta}_{\imath,\jmath})\rho_m(u_n)dx\,dt = \int_{Q_T} f\varphi(T_\ell(u) - \omega^{\beta}_{\imath,\jmath})\rho_m(u)dx\,dt,$$

so that

$$\varphi(T_\ell(u) - \omega^{\beta}_{\imath,\jmath})\rho_m(u) \rightharpoonup \varphi(T_\ell(u) - T_\ell(u)_\beta - \exp^{-\beta t} T_\ell(\psi_\imath))\rho_m(u) \text{ weakly } * \text{ in } L^\infty(Q_T)$$

as $\jmath \to \infty$, also

$$\varphi(T_\ell(u) - T_\ell(u)_\beta - \exp^{-\beta t} T_\ell(\psi_\imath))\rho_m(u) \rightharpoonup 0 \text{ weakly } * \text{ in } L^\infty(Q_T) \text{ as } \beta \longrightarrow \infty. \tag{4.52}$$

Then we deduce that

$$\int_{Q_T} f_n \varphi(T_\ell(u_n) - \omega^{\beta}_{\imath,\jmath})\rho_m(u_n)dx\,dt = \epsilon(\jmath,\beta,n). \tag{4.53}$$

Similarly, Lebesgue's convergence theorem shows that

$$\Phi_n(u_n)\rho_m(u_n) \to \Phi(u)\rho_m(u) \text{ strongly in } (E_{\overline{M}}(Q_T)^N) \text{ as } n \to +\infty$$

and

$$\Phi_n(u_n)\chi_{\{m\leq|u_n|\leq m+1\}}\varphi'(T_\ell(u_n) - \omega^{\beta}_{\imath,\jmath}) \to \Phi(u)\chi_{\{m\leq|u|\leq m+1\}}\varphi'(T_\ell(u) - \omega^{\beta}_{\imath,\jmath})$$

strongly in $(E_{\overline{M}}(Q_T)^N)$ as $n \to +\infty$. Then by virtue of $\nabla T_\ell(u_n) \rightharpoonup \nabla T_\ell(u)$ weakly in $(L_M(Q_T))^N$ and $\nabla u_n \chi_{\{m\leq|u_n|\leq m+1\}} = \nabla T_{m+1}(u_n)\chi_{\{m\leq|u_n|\leq m+1\}}$ a. e. in Q_T, one has

$$\begin{aligned}&\int_{Q_T} \Phi_n(u_n)\rho_m(u_n)(\nabla T_\ell(u_n) - \nabla\omega^{\beta}_{\imath,\jmath})\varphi'(T_\ell(u_n) - \omega^{\beta}_{\imath,\jmath})dx\,dt\\ &\quad\to \int_{Q_T} \Phi(u)\rho_m(u)(\nabla T_\ell(u) - \nabla\omega^{\beta}_{\imath,\jmath})\varphi'(T_\ell(u) - \omega^{\beta}_{\imath,\jmath})dx\,dt\end{aligned}$$

as $n \to +\infty$, and

$$\int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \varphi(T_\ell(u_n) - \omega^\beta_{\imath,\jmath}) dx\, dt$$

$$\longrightarrow \int_{\{m \leq |u| \leq m+1\}} \Phi(u) \nabla u \varphi(T_\ell(u) - \omega^\beta_{\imath,\jmath}) dx\, dt$$

as $n \to +\infty$. On the other hand, by using the modular convergence of $\omega^\beta_{\imath,\jmath}$ as $\jmath \to +\infty$ and letting β tend to infinity, we get

$$\int_{Q_T} \Phi_n(u_n) \rho_m(u_n)(\nabla T_\ell(u_n) - \nabla \omega^\beta_{\imath,\jmath}) \varphi'(T_\ell(u_n) - \omega^\beta_{\imath,\jmath}) dx\, dt = \epsilon(\jmath, \beta, n) \tag{4.54}$$

and

$$\int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \varphi(T_\ell(u_n) - \omega^\beta_{\imath,\jmath}) dx\, dt = \epsilon(\jmath, \beta, n). \tag{4.55}$$

Concerning the third term of the right hand side of (4.48) we obtain that

$$\begin{aligned} &\left| \int_{Q_T} a_n(x,t,u_n,\nabla u_n) \varphi(T_\ell(u_n) - \omega^\beta_{\imath,\jmath}) \nabla u_n \rho'_m(u_n) dx\, dt \right| \\ &\leq \varphi(2\ell) \int_{\{m \leq |u_n| \leq m+1\}} a_n(x,t,u_n,\nabla u_n) \nabla u_n dx\, dt. \end{aligned} \tag{4.56}$$

Then by (4.31) we deduce that,

$$\left| \int_{Q_T} a_n(x,t,u_n,\nabla u_n) \varphi(T_\ell(u_n) - \omega^\beta_{\imath,\jmath}) \nabla u_n \rho'_m(u_n) dx\, dt \right| \leq \epsilon(n, \beta, m). \tag{4.57}$$

We now turn to the fourth term of the left hand side of (4.49). We can write

$$\begin{aligned} &\left| \int_{\{|u_n| \leq \ell\}} g_n(x,t,u_n,\nabla u_n) \varphi(T_\ell(u_n) - \omega^\beta_{\imath,\jmath}) \rho_m(u_n) dx\, dt \right| \\ &\leq d(\ell) \int_{Q_T} c_2(x,t) |\varphi(T_\ell(u_n) - \omega^\beta_{\imath,\jmath})| dx\, dt \\ &+ \tfrac{d(\ell)}{\alpha} \int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) \nabla T_\ell(u_n)(T_\ell(u_n) - \omega^\beta_{\imath,\jmath}) \\ &\times \rho_m(u_n) |\varphi(T_\ell(u_n) - \omega^\beta_{\imath,\jmath})| dx\, dt. \end{aligned} \tag{4.58}$$

Since $c_2(x,t) \in L^1(Q_T)$ it is easy to see that

$$d(\ell) \int_{Q_T} c_2(x,t) |\varphi(T_\ell(u_n) - \omega^\beta_{\imath,\jmath})| dx\, dt = \epsilon(n, \beta, \jmath).$$

On the other hand, the second term of the right hand side of (4.58) reads as

$$\begin{aligned}
&\tfrac{d(\ell)}{\alpha}\int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)|\varphi(T_\ell(u_n)-\omega^\beta_{i,\jmath})|\rho_m(u_n)dx\,dt\\
&=\tfrac{d(\ell)}{\alpha}\int_{Q_T}\big[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))-a_n(x,t,T_\ell(u_n),\nabla T_\ell(v_\jmath)\chi^s_\jmath)\big]\\
&\times\big[\nabla T_\ell(u_n)-\nabla T_\ell(v_\jmath)\chi^s_\jmath\big]|\varphi(T_\ell(u_n)-\omega^\beta_{i,\jmath})|dx\,dt\\
&+\tfrac{d(\ell)}{\alpha}\int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(v_\jmath)\chi^s_\jmath)[\nabla T_\ell(u_n)-\nabla T_\ell(v_\jmath)\chi^s_\jmath]\\
&\times|\varphi(T_\ell(u_n)-\omega^\beta_{i,\jmath})|dx\,dt\\
&+\tfrac{d(\ell)}{\alpha}\int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(v_\jmath)\chi^s_\jmath|\varphi(T_\ell(u_n)-\omega^\beta_{i,\jmath})|dx\,dt,
\end{aligned}\tag{4.59}$$

where $\chi^s_\jmath$ denotes the characteristic function of the subset

$$Q^\jmath_s=\{(x,t)\in Q_T:|\nabla T_\ell(v_\jmath)|\le s\}\text{ for } s>0.$$

And, as above, by letting first n then $\jmath,\beta$ and finally s go to infinity, we can easily see that each one of last two integrals is of the form $\epsilon(n,\beta,\jmath)$. This implies that

$$\begin{aligned}
&\Big|\int_{\{|u_n|\le\ell\}} g_n(x,t,u_n,\nabla u_n)\varphi(T_\ell(u_n)-\omega^\beta_{i,\jmath})\rho_m(u_n)dx\,dt\Big|\\
&\le\tfrac{d(\ell)}{\alpha}\int_{Q_T}[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))-a_n(x,t,T_\ell(u_n),\nabla T_\ell(v_\jmath)\chi^s_\jmath)]\\
&\times[\nabla T_\ell(u_n)-\nabla T_\ell(v_\jmath)\chi^s_\jmath]|\varphi(T_\ell(u_n)-\omega^\beta_{i,\jmath})|dx\,dt+\epsilon(n,\beta,\jmath).
\end{aligned}\tag{4.60}$$

Splitting the first integral on the left hand side of (4.57) where $|u_n|\le\ell$ and $|u_n|>\ell$, we can write,

$$\begin{aligned}
&\int_{Q_T} a_n(x,t,u_n,\nabla u_n)(\nabla T_\ell(u_n)-\nabla\omega^\beta_{i,\jmath})\varphi'(T_\ell(u_n)-\omega^\beta_{i,\jmath})\rho_m(u_n)dx\,dt\\
&=\int_{\{|u_n|\le\ell\}} a_n(x,t,u_n,\nabla u_n)(\nabla T_\ell(u_n)-\nabla\omega^\beta_{i,\jmath})\varphi'(T_\ell(u_n)-\omega^\beta_{i,\jmath})\rho_m(u_n)dx\,dt\\
&-\int_{\{|u_n|>\ell\}} a_n(x,t,u_n,\nabla u_n)\nabla\omega^\beta_{i,\jmath}\varphi'(T_\ell(u_n)-\omega^\beta_{i,\jmath})\rho_m(u_n)dx\,dt.
\end{aligned}\tag{4.61}$$

where we have used the fact that, since $m>\ell,\rho_m(u_n)=1$ on $|u_n|\le\ell$. Since $\rho_m(u_n)=0$ if $|u_n|\ge m+1$, one has

$$\begin{aligned}
&\int_{Q_T} a_n(x,t,u_n,\nabla u_n)(\nabla T_\ell(u_n)-\nabla\omega^\beta_{i,\jmath})\varphi'(T_\ell(u_n)-\omega^\beta_{i,\jmath})\rho_m(u_n)dx\,dt\\
&=\int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))(\nabla T_\ell(u_n)-\nabla\omega^\beta_{i,\jmath})\varphi'(T_\ell(u_n)-\omega^\beta_{i,\jmath})dx\,dt\\
&-\int_{\{|u_n|>\ell\}} a_n(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla\omega^\beta_{i,\jmath}\varphi'(T_\ell(u_n)-\omega^\beta_{i,\jmath})\rho_m(u_n)dx\,dt.\\
&=I_1+I_2.
\end{aligned}\tag{4.62}$$

By letting $n \to +\infty$

$$I_2 = -\int_{\{|u|>\ell\}} \varphi_{m+1} \nabla \omega_{\imath,\jmath}^{\beta} \varphi'(T_\ell(u) - \omega_{\imath,\jmath}^{\beta}) \rho_m(u) dx\, dt + \epsilon(n)$$

which implies that, by letting $\jmath \to +\infty$

$$\begin{aligned} I_2 = -\int_{\{|u|>\ell\}} & \varphi_{m+1} \big[\nabla T_\ell(u)_\beta - \exp^{-\beta t} \nabla T_\ell(\psi_\imath)\big] \\ & \times \varphi'(T_\ell(u) - T_\ell(u)_\beta - \exp^{-\beta t} \nabla T_\ell(\psi_\imath)) \rho_m(u) dx\, dt + \epsilon(n, \jmath) \end{aligned}$$

so that, by letting $\beta \to +\infty$

$$I_2 = -\int_{Q_T} \varphi_{m+1} \nabla T_\ell(u) \chi_{\{|u|>\ell\}} + \epsilon(n, \jmath, \beta) \tag{4.63}$$

Using now the term I_1 of (4.62), we conclude that, it is easy to show that,

$$\begin{aligned} &\int_{Q_T} a_n(x,t,T_\ell(u_n), \nabla T_\ell(u_n))(\nabla T_\ell(u_n) - \nabla \omega_{\imath,\jmath}^{\beta}) \varphi'(T_\ell(u_n) - \omega_{\imath,\jmath}^{\beta}) dx\, dt \\ &= \int_{Q_T} \big[a_n(x,t,T_\ell(u_n), \nabla T_\ell(u_n)) - a_n(x,t,T_\ell(u_n), \nabla T_\ell(v_\jmath)\chi_\jmath^s)\big] \\ &\times \big[\nabla T_\ell(u_n) - \nabla T_\ell(v_\jmath)\chi_\jmath^s)\big] \varphi'(T_\ell(u_n) - \omega_{\imath,\jmath}^{\beta}) dx\, dt \\ &+ \int_{Q_T} a_n(x,t,T_\ell(u_n), \nabla T_\ell(v_\jmath)\chi_\jmath^s) \big[\nabla T_\ell(u_n) - \nabla T_\ell(v_\jmath)\chi_\jmath^s)\big] \varphi'(T_\ell(u_n) - \omega_{\imath,\jmath}^{\beta}) dx\, dt \\ &+ \int_{Q_T} a_n(x,t,T_\ell(u_n), \nabla T_\ell(u_n)) \nabla T_\ell(v_\jmath)\chi_\jmath^s \varphi'(T_\ell(u_n) - \omega_{\imath,\jmath}^{\beta}) dx\, dt \\ &- \int_{Q_T} a_n(x,t,T_\ell(u_n), \nabla T_\ell(u_n)) \nabla \omega_{\imath,\jmath}^{\beta} \varphi'(T_\ell(u_n) - \omega_{\imath,\jmath}^{\beta}) dx\, dt \\ &= J_1 + J_2 + J_3 + J_4, \end{aligned} \tag{4.64}$$

As before, in the following we pass to the limit in (4.64): first we let n tends to $+\infty$, then $\jmath$ then β then m tends tends to $+\infty$. Starting with J_2, observe first that

$$\begin{aligned} J_2 = &\int_{Q_T} a_n(x,t,T_\ell(u_n), \nabla T_\ell(v_\jmath)\chi_\jmath^s) \nabla T_\ell(u_n) \varphi'(T_\ell(u_n) - \omega_{\imath,\jmath}^{\beta}) dx\, dt \\ - &\int_{Q_T} a_n(x,t,T_\ell(u_n), \nabla T_\ell(v_\jmath)\chi_\jmath^s) \nabla T_\ell(v_\jmath)\chi_\jmath^s \varphi'(T_\ell(u_n) - \omega_{\imath,\jmath}^{\beta}) dx\, dt. \end{aligned}$$

Since $a_n(x,t,T_\ell(u_n), \nabla T_\ell(v_\jmath)\chi_\jmath^s) \to a(x,t,T_\ell(u), \nabla T_\ell(v_\jmath)\chi_\jmath^s)$ strongly in $(E_{\overline{M}}(Q_T))^N$ and $\nabla T_\ell(u_n)$
$\rightharpoonup \nabla T_\ell(u)$ weakly in $(L_M(Q_T))^N$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$. Moreover, it is easy to show that

$$\begin{aligned} &\int_{Q_T} a_n(x,t,T_\ell(u_n), \nabla T_\ell(v_\jmath)\chi_\jmath^s) \nabla T_\ell(v_\jmath)\chi_\jmath^s \varphi'(T_\ell(u_n) - \omega_{\imath,\jmath}^{\beta}) dx\, dt \\ \to &\int_{Q_T} a(x,t,T_\ell(u), \nabla T_\ell(v_\jmath)\chi_\jmath^s) \nabla T_\ell(v_\jmath)\chi_\jmath^s \varphi'(T_\ell(u) - \omega_{\imath,\jmath}^{\beta}) dx\, dt \end{aligned}$$

as n tends to $+\infty$. We get

$$J_2 = \int_{Q_T} a(x,t,T_\ell(u), \nabla T_\ell(v_\jmath)\chi^s_\jmath)\big[\nabla T_\ell(u) - \nabla T_\ell(v_\jmath)\chi^s_\jmath)\big]\varphi'(T_\ell(u) - \omega^\beta_{\imath,\jmath})dx\,dt + \epsilon(n),$$

denoting by χ^s the characteristic function of the subset

$$Q_s = \{(x,t) \in Q_T : |\nabla T_\ell(u)| \le s\} \text{ for } s > 0.$$

Since $\nabla T_\ell(v_\jmath)\chi^s_\jmath \to \nabla T_\ell(u)\chi^s$ strongly in $(E_M(Q_T))^N$ as $\jmath \to +\infty$ and $a(x,t,T_\ell(u), \nabla T_\ell(v_\jmath)\chi^s_\jmath) \to a(x,t,T_\ell(u), \nabla T_\ell(u)\chi^s)$ strongly in $(L_{\overline{M}}(Q_T))^N$ as $\jmath$ goes to $+\infty$, we have

$$J_2 = \epsilon(n,\jmath). \tag{4.65}$$

By letting $n \to +\infty$ and since $a(x,t,T_\ell(u_n), \nabla T_\ell(u_n)) \rightharpoonup \varphi_\ell$ weakly in $(L_{\overline{M}}(Q_T))^N$ we have

$$J_3 = \int_{Q_T} \varphi_\ell \nabla T_\ell(v_\jmath)\chi^s_\jmath \varphi'(T_\ell(u) - \omega^\beta_{\imath,\jmath})dx\,dt + \epsilon(n),$$

which gives by letting $\jmath \to +\infty$ and since $v_\jmath \to T_\ell(u)$ in $W^{1,x}_0 L_M(Q_T)$ for the modular convergence, we have

$$J_3 = \int_{Q_T} \varphi_\ell \nabla T_\ell(u)\chi^s \varphi'(T_\ell(u) - T_\ell(u)_\beta + \exp^{-\beta t} T_\ell(\psi_\imath))dx\,dt + \epsilon(n,\jmath), \tag{4.66}$$

implying that, by letting $\beta \to +\infty$, $J_3 = \int_{Q_T} \varphi_\ell \nabla T_\ell(u)\chi^s dx\,dt + \epsilon(n,\jmath,\beta)$, and thus

$$J_3 = \int_{Q_T} \varphi_\ell \nabla T_\ell(u) dx\,dt + \epsilon(n,\jmath,\beta,s). \tag{4.67}$$

Concerning J_4 we can write

$$\begin{aligned} J_4 &= -\int_{\{|u_n| \le \ell\}} a_n(x,t,T_\ell(u_n), \nabla T_\ell(u_n))\nabla\omega^\beta_{\imath,\jmath}\varphi'(T_\ell(u_n) - \omega^\beta_{\imath,\jmath})dx\,dt \\ &= -\int_{\{|u| \le \ell\}} \varphi_\ell \nabla\omega^\beta_{\imath,\jmath}\varphi'(T_\ell(u) - T_\ell(u)_\beta dx\,dt + \epsilon(n), \end{aligned}$$

which implies that, by letting $\jmath \to +\infty$,

$$J_4 = \int_{Q_T} \varphi_\ell \big[\nabla T_\ell(u)_\beta - \exp^{-\beta t} \nabla T_\ell(\psi_\imath)\big]$$
$$\times \varphi'(T_\ell(u) - T_\ell(u)_\beta - \exp^{-\beta t} T_\ell(\psi_\imath))\chi_{\{|u| \le \ell\}} dx\,dt + \epsilon(n,\jmath).$$

By letting $\beta \to +\infty$ we obtain

$$J_4 = -\int_{Q_T} \varphi_\ell \nabla T_\ell(u)\chi_{\{|u| \le \ell\}} dx\,dt + \epsilon(n,\jmath,\beta,s). \tag{4.68}$$

In view of (4.62), (4.63), (4.64), (4.65), (4.67) and (4.68), we conclude then that

$$\begin{aligned}
&\int_{Q_T} a_n(x,t,u_n,\nabla u_n)\big[\nabla T_\ell(u_n) - \nabla \omega^\beta_{\imath,\jmath}\big]\varphi'(T_\ell(u_n)-\omega^\beta_{\imath,\jmath})\rho_m(u_n)dx\,dt \\
&= \int_{Q_T} \big[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a_n(x,t,T_\ell(u_n),\nabla T_\ell(v_\jmath)\chi^s_\jmath)\big] \\
&\times\big[\nabla T_\ell(u_n) - \nabla T_\ell(v_\jmath)\chi^s_\jmath)\big]\varphi'(T_\ell(u_n)-\omega^\beta_{\imath,\jmath})dx\,dt + \epsilon(n,\jmath,\beta,s).
\end{aligned} \tag{4.69}$$

Combining (4.49), (4.50), (4.51),(4.57), (4.60) and (4.69) we obtain

$$\begin{aligned}
&\int_{Q_T} \big[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a_n(x,t,T_\ell(u_n),\nabla T_\ell(v_\jmath)\chi^s_\jmath)\big] \\
&\times\big[\nabla T_\ell(u_n) - \nabla T_\ell(v_\jmath)\chi^s_\jmath\big]\big[\varphi'(T_\ell(u_n)-\omega^\beta_{\imath,\jmath}) - \tfrac{b(\ell)}{\alpha}|\varphi(T_\ell(u_n)-\omega^\beta_{\imath,\jmath})|\big]dx\,dt \\
&\le \epsilon(n,\beta,\jmath,\imath,s,m)
\end{aligned}$$

and so, thanks to (4.35)

$$\begin{aligned}
&\int_{Q_T} \big[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a_n(x,t,T_\ell(u_n),\nabla T_\ell(v_\jmath)\chi^s_\jmath)\big] \\
&\qquad\times\big[\nabla T_\ell(u_n) - \nabla T_\ell(v_\jmath)\chi^s_\jmath\big]dx\,dt \le \epsilon(n,\beta,\jmath,\imath,s,m)
\end{aligned}$$

Now observe that

$$\begin{aligned}
&\int_{Q_T} \big[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a_n(x,t,T_\ell(u_n),\nabla T_\ell(u)\chi^s)\big] \\
&\times\big[\nabla T_\ell(u_n) - \nabla T_\ell(u)\chi^s\big]\rho_m(u_n)dx\,dt \\
&= \int_{Q_T} \big[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a_n(x,t,T_\ell(u_n),\nabla T_\ell(v_\jmath)\chi^s_\jmath)\big] \\
&\times\big[\nabla T_\ell(u_n) - \nabla T_\ell(v_\jmath)\chi^s_\jmath\big]\rho_m(u_n)dx\,dt \\
&+ \int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(v_\jmath)\chi^s_\jmath)\big[\nabla T_\ell(u_n) - \nabla T_\ell(v_\jmath)\chi^s_\jmath)\big]\rho_m(u_n)dx\,dt \\
&- \int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u)\chi^s)\big[\nabla T_\ell(u_n) - \nabla T_\ell(u)\chi^s\big]\rho_m(u_n)dx\,dt \\
&+ \int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\big[\nabla T_\ell(v_\jmath)\chi^s_\jmath - \nabla T_\ell(u)\chi^s\big]\rho_m(u_n)dx\,dt.
\end{aligned}$$

Passing to the limit in n and $\jmath$ in the last three terms on the right-hand side of the last equality, we get

$$\begin{aligned}
&\int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(v_\jmath)\chi^s_\jmath)\big[\nabla T_\ell(u_n) - \nabla T_\ell(v_\jmath)\chi^s_\jmath\big]\rho_m(u_n)dx\,dt \\
&- \int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u)\chi^s)\big[\nabla T_\ell(u_n) - \nabla T_\ell(u)\chi^s\big]\rho_m(u_n)dx\,dt = \epsilon(n,\jmath)
\end{aligned}$$

and

$$\int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\big[\nabla T_\ell(v_\jmath)\chi^s_\jmath - \nabla T_\ell(u)\chi^s\big]\rho_m(u_n)dx\,dt = \epsilon(n,\jmath). \tag{4.70}$$

This implies that

$$\begin{aligned}
&\int_{Q_T} \big[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a_n(x,t,T_\ell(u_n),\nabla T_\ell(u)\chi^s)\big]\\
&\times\big[\nabla T_\ell(u_n) - \nabla T_\ell(u)\chi^s\big]\rho_m(u_n)dx\,dt\\
&= \int_{Q_T} \big[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a_n(x,t,T_\ell(u_n),\nabla T_\ell(v_\jmath)\chi^s_\jmath)\big]\\
&\times\big[\nabla T_\ell(u_n) - \nabla T_\ell(v_\jmath)\chi^s_\jmath\big]\rho_m(u_n)dx\,dt + \epsilon(n,\jmath).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\int_{Q_T} \big[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a_n(x,t,T_\ell(u_n),\nabla T_\ell(u)\chi^s)\big]\\
&\qquad\times\big[\nabla T_\ell(u_n) - \nabla T_\ell(u)\chi^s\big]dx\,dt\\
&= \int_{Q_T} \big[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a_n(x,t,T_\ell(u_n),\nabla T_\ell(u)\chi^s)\big]\\
&\qquad\times\big[\nabla T_\ell(u_n) - \nabla T_\ell(u)\chi^s\big]\rho_m(u_n)dx\,dt\\
&+\int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\big[\nabla T_\ell(u_n) - \nabla T_\ell(u)\chi^s\big](1-\rho_m(u_n))dx\,dt\\
&-\int_{Q_T} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u)\chi^s)\big[\nabla T_\ell(u_n) - \nabla T_\ell(u)\chi^s\big](1-\rho_m(u_n))dx\,dt.
\end{aligned} \tag{4.71}$$

Since $\rho_m(u_n) = 1$ in $\{|u_n| \leq m\}$ and $\{|u_n| \leq \ell\} \subset \{|u_n| \leq m\}$ for m large enough, we deduce from (4.71) that

$$\begin{aligned}
&\int_{Q_T} \big[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a_n(x,t,T_\ell(u_n),\nabla T_\ell(u)\chi^s)\big]\\
&\qquad\times\big[\nabla T_\ell(u_n) - \nabla T_\ell(u)\chi^s\big]dx\,dt\\
&= \int_{Q_T} \big[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a_n(x,t,T_\ell(u_n),\nabla T_\ell(u)\chi^s)\big]\\
&\qquad\times\big[\nabla T_\ell(u_n) - \nabla T_\ell(u)\chi^s\big]\rho_m(u_n)dx\,dt\\
&+\int_{\{|u_n|>\ell\}} a_n(x,t,T_\ell(u_n),\nabla T_\ell(u)\chi^s)\nabla T_\ell(u)\chi^s(1-\rho_m(u_n))dx\,dt.
\end{aligned}$$

It is easy to see that the last terms of the last equality tend to zero as $n \to +\infty$, which implies that

$$\begin{aligned}
&\int_{Q_T} \big[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a_n(x,t,T_\ell(u_n),\nabla T_\ell(u)\chi^s)\big]\\
&\qquad\times\big[\nabla T_\ell(u_n) - \nabla T_\ell(u)\chi^s\big]dx\,dt\\
&= \int_{Q_T} \big[a(x,t,T_\ell(u),\nabla T_\ell(u)) - a(x,t,T_\ell(u),\nabla T_\ell(u)\chi^s)\big]\\
&\times\big[\nabla T_\ell(u) - \nabla T_\ell(u)\chi^s\big]\rho_m(u_n)dx\,dt + \epsilon(n,\jmath)\\
&\leq \epsilon(n,\jmath,\beta,m,s).
\end{aligned} \tag{4.72}$$

To pass to the limit in (4.72) as $n, \jmath, m, s$ tend to infinity, we obtain

$$\lim_{s\to+\infty}\lim_{n\to+\infty}\int_{Q_T}\big[a_n(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a_n(x,t,T_\ell(u_n),\nabla T_\ell(u)\chi^s)\big] \times\big[\nabla T_\ell(u_n) - \nabla T_\ell(u)\chi^s\big]dx\,dt = 0. \tag{4.73}$$

This implies by the lemma 4.6, the desired statement and hence the proof of Proposition 4.5 is achieved. □

Remark 4.8. *Observe that for every $\sigma > 0$,*

$$meas\{(x,t)\in Q_T : |\nabla u_n - \nabla u| > \sigma\} \le meas\{(x,t)\in Q_T : |\nabla u_n| > \ell\}$$
$$+meas\{(x,t)\in Q_T : |\nabla u| > \ell\} + meas\{(x,t)\in Q_T : |\nabla T_\ell(u_n) - \nabla T_\ell(u)| > \sigma\},$$

then as a consequence of (4.36), it follows that ∇u_n converges to ∇u in measure and therefore, always reasoning for a subsequence,

$$\nabla u_n \to \nabla u \qquad a.\ e.\ in\ Q_T. \tag{4.74}$$

Step 4: Equi-integrability of the nonlinearitie $g_n(x,t,u_n,\nabla u_n)$.

We shall now prove that $g_n(x,t,u_n,\nabla u_n) \to g(x,t,u,\nabla u)$ strongly in $L^1(Q_T)$ by using Vitali's theorem. Since $g_n(x,t,u_n,\nabla u_n) \to g(x,t,u,\nabla u)$ a.e. in Q_T, thanks to (4.22) and (4.74), it suffices to prove that $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable in Q_T. Let $E \subset Q_T$ be a measurable subset of Q_T. We have for any $m > 0$:

$$\begin{aligned}
&\int_E |g_n(x,t,u_n,\nabla u_n)|dx\,dt = \int_{E\cap\{u_n\le m\}} |g_n(x,t,u_n,\nabla u_n)|dx\,dt\\
&+\int_{E\cap\{u_n>m\}} |g_n(x,t,u_n,\nabla u_n)|dx\,dt\\
&\le \tfrac{d(m)}{\alpha}\int_E a_n(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dx\,dt + d(m)\int_E c_2(x,t)dx\,dt\\
&+\int_E |f_n|dx\,dt + \int_{\{u_{0n}>m\}} |u_{0n}|dx\,dt,
\end{aligned}$$

where we have used (3.4) and (4.13). Therefore, it is easy to see that there exists $\delta > 0$ such that

$$|\,E| < \delta \Rightarrow \int_E |g_n(x,t,u_n,\nabla u_n)|dx\,dt \le \epsilon, \quad \forall n\in\mathbb{N}$$

which shows that $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable in Q_T as required.

Step 5:

In this step we prove that u satisfies (3.9).

Lemma 4.9. *The limit u of the approximate solution u_n of (4.6) satisfies*

$$\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}} a(x,t,u,\nabla u)dx\,dt = 0. \tag{4.75}$$

Proof: Observe that for any fixed $m \ge 0$ one has

$$\begin{aligned}
&\int_{\{m\le|u_n|\le m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\
&=\int_{Q_T} a_n(x,t,u_n,\nabla u_n)\big(\nabla T_{m+1}(u_n)-\nabla T_m(u_n)\big)\,dx\,dt\\
&=\int_{Q_T} a_n(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n)\,dx\,dt\\
&-\int_{Q_T} a_n(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)\,dx\,dt
\end{aligned} \tag{4.76}$$

According to (4.43) (with $z_n = T_m(u_n)$ or $z_n = T_{m+1}(u_n)$, one is at liberty to pass to the limit as n tends to $+\infty$ for fixed $m \ge 0$ and to obtain

$$\begin{aligned}
&\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n dx\,dt\\
&=\int_{Q_T} a(x,t,T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u)\,dx\,dt\\
&-\int_{Q_T} a(x,t,T_m(u),\nabla T_m(u)\nabla T_m(u)\,dx\,dt\\
&=\int_{\{m\le|u|\le m+1\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt.
\end{aligned} \tag{4.77}$$

Taking the limit as m tends to $+\infty$ in (4.77) and using the estimate (4.31) it possible to conclude that (4.76) holds true and the proof of Lemma 4.9 is complete. □

Step 6:

In this step, u is shown to satisfies (3.11) . Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let K be a positive real number such that $suppS' \subset [-K,K]$. Pointwise multiplication of the approximate equation (4.6) by $S'(u_n)$ leads to

$$\begin{aligned}
&\frac{\partial S(u_n)}{\partial t} - \operatorname{div}\big(S'(u_n)a_n(x,t,u_n,\nabla u_n)\big) + S''(u_n)a_n(x,t,u_n,\nabla u_n)\nabla u_n\\
&-\operatorname{div}\big(S'(u_n)\Phi_n(u_n)\big) + S''(u_n)\Phi_n(u_n)\nabla u_n + g_n(x,t,u_n,\nabla u_n)S'(u_n) = fS'(u_n).
\end{aligned} \tag{4.78}$$

It what follows we pass to the limit as n tends to $+\infty$ in each term of (4.78).

- Since S' is bounded, and $S(u_n)$ converges to $S(u)$ a.e. in Q_T and in $L^\infty(Q_T)$ weak $*$. Then $\frac{\partial S(u_n)}{\partial t}$ converges to $\frac{\partial S(u)}{\partial t}$ in $\mathcal{D}'(Q_T)$ as n tends to $+\infty$

▶ Since $suppS \subset [-K, K]$, we have

$$S'(u_n)a_n(x,t,u_n,\nabla u_n) = S'(u_n)a_n(x,t,T_K(u_n),\nabla T_K(u_n)) \text{ a. e. in } Q_T.$$

The pointwise convergence of u_n to u as n tends to $+\infty$, the bounded character of $S\prime\prime$, (4.22) and (4.37) of Proposition 4.5 imply that

$$S'(u_n)a_n(x,t,T_K(u_n),\nabla T_K(u_n)) \rightharpoonup S'(u)a(x,t,T_K(u),\nabla T_K(u)) \text{ weakly in}$$

$(L_{\overline{M}}(Q_T))^N$, for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ as n tends to $+\infty$, because $S(u) = 0$ for $|u| \geq K$ a. e. in Q_T. And the term $S'(u)a(x,t,T_K(u),\nabla T_K(u))$=$S'(u)a(x,t,u,\nabla u)$ a. e. in Q_T.

▶ Since $suppS' \subset [-K, K]$, we have

$$S''(u_n)a_n(x,t,u_n),\nabla u_n)\nabla u_n = S''(u_n)a_n(x,t,T_K(u_n),\nabla T_K(u_n))\nabla T_K(u_n)$$

a. e. in Q_T. The pointwise convergence of $S''(u_n)$ to $S''(u)$ as n tends to $+\infty$, the bounded character of S'', (4.22), (4.37) and (4.37) imply that

$$S'(u_n)a_n(x,t,u_n,\nabla u_n)\nabla u_n \rightharpoonup S'(u)a(x,t,T_K(u),\nabla T_K(u))\nabla T_K(u) \text{ weakly in}$$

$L^1(Q_T)$, as n tends to $+\infty$. And

$$S''(u)a(x,t,T_K(u),\nabla T_K(u))\nabla T_K(u) = S''(u)a(x,t,u,\nabla)\nabla u \text{ a. e. in } Q_T.$$

▶ Since $suppS' \subset [-K, K]$, we have

$$S'(u_n)\Phi_n(u_n) = S'(u_n)\Phi_n(T_K(u_n)) \text{ a. e. in } Q_T.$$

As a consequence of (3.6), (4.3) and (4.22), it follows that:

$$S'(u_n)\Phi_n(u_n) \to S'(u)\Phi(T_K(u)) \text{ strongly in } (E_M(Q_T))^N,$$

as n tends to $+\infty$. The term $S'(u)\Phi(T_K(u))$ is denoted by $S'(u)\Phi(u)$.

▶ Since $S \in W^{1,\infty}(\mathbb{R})$ with $suppS' \subset [-K, K]$, we have

$$S''(u_n)\Phi_n(u_n)\nabla u_n = \Phi_n(T_K(u_n))\nabla S''(u_n) \text{ a. e. in } Q_T,$$

we have, $\nabla S''(u_n)$ converges to $\nabla S''(u)$ weakly in $(L_M(Q_T))^N$ as n tends to $+\infty$, while $\Phi_n(T_K(u_n))$ is uniformly bounded with respect to n and converges a. e. in Q_T to $\Phi(T_K(u))$ as n tends to $+\infty$. Therefore

$$S''(u_n)\Phi_n(u_n)\nabla u_n \rightharpoonup \Phi(T_K(u))\nabla S''(u) \text{ weakly in } L_M(Q_T).$$

▶ Due to (4.5) and (4.22), we have $f_nS'(u_n)$ converges to $fS'(u)$ strongly in $L^1(Q_T)$, as n tends to $+\infty$.

- ▶ Due to (4.22) and the fact that $g_n(x,t,u_n,\nabla u_n) \to g(x,t,u,\nabla u)$ strongly in $L^1(Q_T)$, we have $g_n S'(u_n)$ converges to $gS'(u)$ strongly in $L^1(Q_T)$, as n tends to $+\infty$.

As a consequence of the above convergence result, we are in a position to pass to the limit as n tends to $+\infty$ in equation (4.78) and to conclude that u satisfies (3.11). Remark that, S' has a compact support, we have $S(u_n)$ is bounded in $L^\infty(Q_T)$. by (4.78) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial S(u_n)}{\partial t}$ is bounded in $L^1(Q_T)+W^{-1,x}L_M(Q_T)$. a consequence, an Aubin's type Lemma (see e.g., [30], Corollary 4) (see also [23]) implies that $S(u_n)(t=0)$ lies in a compact set of $C^0([0,T];L^1(\Omega))$. It follows that, $S(u_n)(t=0)$ converges to $S(u)(t=0)$ strongly in $L^1(\Omega)$. Due to (4.4), we conclude that $S(u_n)(t=0) = S(u_n(x,0))$ converges to $S(u)(t=0)$ strongly in $L^1(\Omega)$. Then we conclude that $S(u)(t=0) = S(u_0)$ in Ω.

As a conclusion of step 1 to step 6, the proof of theorem 4.1 is complete. □

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Youssef EL HADFI
Laboratory LAMA, Department of Mathematics,
Faculty of Sciences-Fez, P.O. Box 1796 Atlas Fez,
Morocco
E-mail address: yelhadfi@gmail.com

and

Abdelmoujib BENKIRANE
Laboratory LAMA, Department of Mathematics,
Faculty of Sciences-Fez, P.O. Box 1796 Atlas Fez,
Morocco
E-mail address: abd.benkirane@gmail.com

and

Mostafa EL MOUMNI
Laboratory LAMA, Department of Mathematics,
Faculty of Sciences-Fez, P.O. Box 1796 Atlas Fez,
Morocco
E-mail address: mostafaelmoumni@gmail.com