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The Common-Neighbourhood of a Graph

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ABSTRACT: The most widely used and well-known vulnerability measures of a connected graph are based on the neighbourhood concept. It takes into account neighbour-integrity, edge-integrity and accessibility number. In this work we define and examine common-neighbourhood of a connected graph as a new global connectivity measure. Our measure examines the neighbourhoods of all pairs of vertices of any connected graph. We show that, for connected graphs G_1 and G_2 of the same order, if the dominating number of G_1 is bigger than the dominating number of G_2, then the common-neighbourhood of G_1 is less than the common-neighbourhood of G_2. We give some theorems and obtain some results on common-neighbourhood of a graph. We consider all the graphs in this paper as connected, undirected and without loops.

Key Words: Vertex-neighbourhood, connectivity, stability, common- neighbourhood.

Contents

1. Introduction

The stability and reliability of a network are of prime importance to network designers. The vulnerability value of a communication network shows the resistance of the network after the disruption of some centres or connection lines until communication within the network breaks down. As the network begins to loose connection lines or centres, there would eventually be a loss of its effectiveness. If the communication network is modelled as a simple, undirected, connected and unweighted graph G, deterministic measures tend to provide a worst-case analysis of some aspects of the overall disconnection process.
Let G be a graph with the vertex set $V = V(G)$and edge set $E = E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. The reliability of a graph can be measured by various parameters. The best known reliability measure of a graph is connectivity, denoted by $k(G)$, defined as the minimum number of vertices whose deletion results in a disconnected or trivial graph. This parameter has been extensively studied.

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The *open neighbourhood* of $v \in V$ is $N(v) = \{u : u \in V | uv \in E\}$ and the *closed neighbourhood* of v is $N[v] = \{v\} \bigcup N(v)$. For a set $S \subseteq V$, its open neighbourhood $N(S) = \bigcup_{v \in S} N(v)$ and its closed neighbourhood $N[S] = N(S) \cup S$.
Connectivity, integrity, toughness, neighbour-integrity are all worst-case measures and as such do not always reflect what happens throughout the graph [1-5]. For example, a tree and a graph obtained by appending an pendant-vertex to a complete graph both have connectivity 1. Nevertheless, for large number of vertices the latter graph is far more reliable than the former. Recent interest in the vulnerability and reliability of networks has given rise to a host of other measures, some of which are more global in nature. These parameters, such as average connectivity, average degree and average distance of a graph, have been found to be more useful in some circumstances then the corresponding local measures, [6-19]. For example, the average distance between vertices in graph was introduced as a tool in architecture and later turned out to be more valuable than the diameter when analyzing transportation networks.
While the ordinary connectivity is the minimum number of removed vertices which separates at least one connected pair of vertices, the average connectivity is a measure for the expected number of vertices that have to be removed to separate a randomly chosen pair of vertices.

Definition 1.1. *Let $G = (V, E)$ be a simple graph of order n and let u and v be two distinct vertices of G. Two vertices u and v in a graph G are said to be **k-neighbour**, if there are k distinct vertices which are neighbours of both u and v. The k-neighbour of $u$and v vertices of G is denoted by $N(u, v)$. This new definition is given by us.*

In this paper we investigate the **common-neighbourhood**, a new measure for reliability and stability of a graph. The common-neighbourhood gives the expected number of vertices to constitute neighbourhood between a randomly chosen pair of vertices. Although other global measures of reliability, such as the toughness and integrity of a graph, are NP hard, the common-neighbourhood can be computed in polynomial time, this makes it much more attractive for applications.

Definition 1.2. *If the order of G is n, then the common-neighbourhood of G is denoted by $\bar{N}(G)$, is defined to be*

$$\bar{N}(G) = \frac{\sum\limits_{u,v \in V(G)} |N(u,v)|}{n-1} \quad for\, n \geq 3$$

where $\sum\limits_{u,v \in V(G)} |N(u, v)|$ is equal to the number of paths of length 2 occurring in the graph G. For any vertex v there exist exactly $\binom{\deg(v)}{2}$ such paths, i.e. paths of *the form $u_1\, v\, u_2$ with the vertex v in the middle. In order to determine the value $\bar{N}(G)$ one only needs $O(n^2)$ steps since*

$$\bar{N}\,(G) = \frac{\sum_{v \in V(G)} \binom{\deg(v)}{2}}{n-1}.$$

As examples, we consider the two graphs in Figure 1, but the second is a more reliable network then the first. This is reflected in the common-neighbourhood since $\bar{N}\,(G_1) = \frac{3}{4}$ and $\bar{N}\,(G_2) = \frac{15}{4}$.

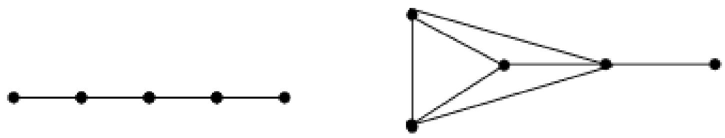

Figure 1: $P_5 = G_1$ G_2

Apart from having the same connectivity, the two graphs in Figure 1 have the same number vertices but a different number of edges. The difference in common-neighbourhood is a result of the increased number of edges.
In Figure 2 we show two graphs with the same numbers of vertices and edges, but $\bar{N}\,(G_3) = \frac{6}{4}$ while $\bar{N}\,(G_4) = \frac{7}{4}$.

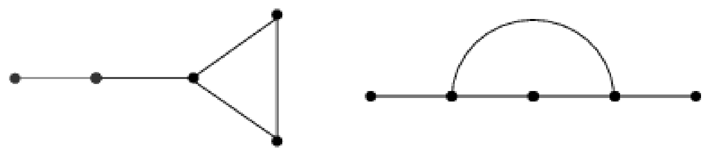

Figure 2: G_1 G_2

2. Basic Results on Common-Neighbourhood of Some Graphs

In the following results for the common-neighbourhood of a variety of families of graphs can be seen clearly.

1) $\bar{N}(P_n) = \frac{n-2}{n-1} = 1 - \frac{1}{n-1}$

2) $\bar{N}(C_n) = \frac{n}{n-1} = 1 + \frac{1}{n-1}$

3) $\bar{N}(K_n) = \frac{n(n-2)}{2}$

4) $\bar{N}(K_{1,n-1}) = \frac{n-2}{2}$

5) $\bar{N}(K_{m,n}) = \dfrac{\binom{m}{2} n + \binom{n}{2} m}{m+n-1}$

6) $\bar{N}(W_{1,n-1}) = \dfrac{\binom{n-1}{2} + 3(n-1)}{n-1}$

For a connected graph G, let the nodes of G be labelled as $v_1, v_2, ..., v_n$ and the adjacency matrix $A = A(G) = [a_{i,j}]$ of G is the binary matrix of order n

$$a_{ij} = \begin{cases} 1, \text{ if } v_i \text{ is adjacent with } v_j \\ 0, \text{ otherwise} \end{cases}$$

For a connected graph G, we define the distance $d(u, v)$ between two vertices u and v as the minimum of the lengths of the $u - v$ paths in G. Under the distance function, the set $V(G)$ is a metric space. The eccentricity $e(v)$ of a vertex v of connected graph G is the number $\max\limits_{v,u \in V(G)} d(v, u)$. The radius $rad(G)$ is defined as $\min\limits_{u \in V(G)} e(v)$ while the diameter $diam(G)$ is $\max\limits_{v \in V(G)} e(v)$. It follows that $diam(G) = \max\limits_{v,u \in V(G)} d(v, u)$.

Definition 2.1. [20, 21] *An independent set of vertices of a graph G is a set whose elements are pairwise nonadjacent. The independence number $\beta(G)$ of G is the maximum cardinality among all independent sets of vertices of G.*

Definition 2.2. [20, 21] *A vertex is said to cover other vertices in a graph G if it is incident to these vertices in G. A cover in G is a set of vertices that covers all edges of G. The minimum cardinality of a cover in a graph G, denoted by $\alpha(G)$, is called the covering number of G.*

Therefore $\alpha(G) + \beta(G) = n$

Definition 2.3. [20, 21] *A vertex dominating set for a graph G is a set S of vertices such that every vertex of G belongs to S or is adjacent to a vertex of S. The minimum cardinality of a vertex dominating set in a graph G is called the vertex dominating number of G and is denoted by $\sigma(G)$. For every graph G, $\sigma(G) \leq \beta(G)$.*

Lemma 2.1. *Let u and v be two vertices of a connected graph G. If the $d(u, v) > 2$ then $N(u, v) = 0$, whereas if $d(u, v) = 2$ then $1 \leq N(u, v) \leq n - 2$.*

Lemma 2.2. *For any connected graph with n vertices,* $n-2 \leq \sum_{u,v \in V(G)} N(u,v) \leq \frac{n(n-1)(n-2)}{2}$ *where* $n > 2$.

Theorem 2.1. *Let G be a graph of order* $n \geq 3$.

a) $\bar{N}(G) = 0$ *if and only if G is a null graph.*

b) $\bar{N}(G) > 0$ *if and only if G is a connected graph at least of order 3.*

Proof is clear.

Lemma 2.3. *Let G be a connected graph of order n. The common-neighbourhood of G is minimum if and only if* $G = P_n$ *and maximum if and only if* $G = K_n$*. It can be easily seen from Lemma 2.1.*

Theorem 2.2. *For a connected graph G, the common-neighbourhood of G is* $\frac{1}{2} \leq \bar{N}(G) \leq \frac{n(n-2)}{2}$.

Proof: From the Lemma 2.1. and Theorem 2.1. G must be connected graph and at least the path P_3. Then $\bar{N}(P_3) = \frac{1}{2}$. G can be the complete graph order of n, K_n at most. In complete graph K_n, $N(u,v) = n-2$ for each $u, v \in V(K_n)$. It is obtained from the definition of common-neighbourhood and Lemma 2.2

$$\bar{N}(K_n) = \frac{\binom{n}{2}(n-2)}{n-1} = \frac{n(n-2)}{2}$$

Consequently, for any connected graph G, its common-neighbourhood is $\frac{1}{2} \leq \bar{N}(G) \leq \frac{n(n-2)}{2}$ □

Theorem 2.3. *Let G be connected graph with n vertices having* $K_{1,n-1}$ *as a spanning subgraph then*

$$\bar{N}(G) \geq \frac{(n-2)}{2}.$$

Proof: The right side of the inequality can be seen from common-neighbourhood of $K_{1,n-1}$. In $K_{1,n-1}$, the number of (u,v) pairs which have the property $d(u,v) = 2$ is $\binom{n-1}{2}$ and for each pair of vertices. And from the definition of common-neighbourhood it can be obtained easily. □

Theorem 2.4. $\bar{N}(G) < \bar{N}(G+e)$.

Proof: From the definition of $(G+e)$, to add an e edge between any vertices which are disjoint increase k-neighbour value for at least any (v_i, v_j) vertex pair of G. Since we are working on graphs which do not contain loops, we cannot add an

edge between adjacent vertices. This increases $N(v_i, v_j)$ value.
In the definition common-neighbourhood, if $\sum\limits_{u,v\in V(G)} |N(u,\, v)|$ increases, then the $\bar{N}(G)$ increases also. Hence, $\bar{N}(G) < \bar{N}(G+e)$. □

Theorem 2.5. *Let G be a graph of order n and P_n be a path graph. If $diam(G) < diam(P_n)$, then*

$$\bar{N}(G) \geq \bar{N}(P_n).$$

Proof: Let G and P_n be two graphs whose orders are the same and $diam(G) < diam(P_n)$, from Lemma 2.3 it is obvious that P_n has the minimum value of common-neighbourhood. Hence, this shows that the k-neighbourhoods in G are greater than in P_n. By the definition of common-neighbourhood, if the number of neighbourhoods in G_1 is higher, then the number of $N(u, v)$ in G will be high. Then $\bar{N}(G) > \bar{N}(P_n)$. □

Theorem 2.6. *Let G be connected graph except tree. Then, $\bar{N}(G) < \frac{\sum\limits_{i=1}^{n} \deg(v_i)}{2(n-1)}$, for all v_i vertices.*

Proof: For the graph G max $\left(\sum\limits_{i=1}^{n} \deg(v_i)\right) = 2m$, where m denotes the number of edges of G. m gets its maximum value $\frac{n(n-1)}{2}$ only in complete graph.

$$\sum N(u, v) = (n-2)\frac{n(n-1)}{2} = \frac{(n-2)}{2}\sum_{i=1}^{n} \deg\,(v_i).$$

Thus,

$$\bar{N}(G) = \frac{\left(\frac{n-2}{2}\right)\sum\limits_{i=1}^{n} \deg\,(v_i)}{2\,(n-1)} = \frac{(n-2)\sum\limits_{i=1}^{n} \deg\,(v_i)}{2\,(n-1)}.$$

This value is the maximum value in K_n complete graph. If we remove any $e_i = (u_i, v_i)$ edge from K_n , the neighbourhood values of the vertices u_i and v_i decrease 1. Consequently, for any connected graph G, the value of common-neighbourhood

$$\bar{N}(G) > \frac{\sum\limits_{i=1}^{n} \deg\,(v_i)}{2\,(n-1)}.$$

□

3. Common-Neighbourhood and Other Parameters

Certainly other parameters provide bounds on the common-neighbourhood of a graph. In this section we give some theorems relating to common-neighbourhood and graph parameters.

Definition 3.1. *A graph in which all vertices are of equal degree is called a regular graph. For any non-regular graph G, $\Delta(G)$ denotes maximum vertex degree and $\delta(G)$ denotes minimum vertex degree of the graph G.*

Theorem 3.1. *Let G be a tree and for $n > 3$, $\bar{N}(G) \geq \frac{\beta(G)}{n-1}$.*

Proof: From the *Lemma 2.1*, If $d(u,v) = 2$, then $N(u,v) \geq 1$. For all (u,v) pairs of a maximum independent set of G, $d(u,v) \geq 2$. From *Theorem 2.1* if u and v is adjacent then $N(u,v) = 0$.

Then, in the definition $\bar{N}(G) = \frac{\sum\limits_{u,v\in V(G)} N(u,v)}{n-1}$, $\sum\limits_{u,v\in V(G)} N(u,v) \geq \beta(G)$

We divide the both sides of this inequality by $n-1$, we obtain $\frac{\sum\limits_{u,v\in V(G)} N(u,v)}{n-1} \geq \frac{\beta(G)}{n-1}$. □

Theorem 3.2. *Let G_1 and G_2 be graphs each having n vertices. If $\sigma(G_1) < \sigma(G_2)$, then $\bar{N}(G_1) > \bar{N}(G_2)$.*

Proof: Since the minimum dominating number denotes that the neighbourhood between vertex pairs of that graph is more.
If $\sigma(G_1) < \sigma(G_2)$ then for any vertex pair in G_1 k-neighbour value is bigger than in G_2.
Consequently, $\sum\limits_{u,v\in G_1} |N(u,v)| > \sum\limits_{x,y\in G_2} |N(x,y)|$ If we divide the both sides of the inequality by $n-1$, we obtain the following inequality.

$$\frac{\sum\limits_{u,v\in G_1} |N(u,v)|}{n-1} > \frac{\sum\limits_{x,y\in G_2} |N(x,y)|}{n-1}$$

and by the definition of common-neighbourhood, $\bar{N}(G_1) > \bar{N}(G_2)$. □

The results of the common-neighbourhood of the above graphs relating with α, β, Δ and *diam* are given in the following.

1) $\bar{N}(P_n) = \frac{diam(P_n)-1}{n-1}$

2) $\bar{N}(P_n) \leq \frac{\beta+diam(P_n)}{n-1}$

3) $\bar{N}(C_n) = \frac{\alpha+\beta}{n-1}$

4) $\bar{N}(C_n) = \begin{cases} \frac{\Delta diam(C_n)+1}{n-1}, & \text{if } n \text{ is odd} \\ \frac{\Delta diam(C_n)}{n-1}, & \text{if } n \text{ is even} \end{cases}$

5) $\bar{N}(K_{1,n}) = \begin{cases} \frac{\left\lceil \frac{\beta}{3} \right\rceil \Delta}{n-1}, & \text{if } n \text{ is odd} \\ \frac{\left\lceil \frac{\beta}{2} \right\rceil \Delta - 3}{n-1}, & \text{if } n \text{ is even} \end{cases}$

Theorem 3.3. *Let G be $K_{1,n-1}$ graph.* $\bar{N}\,(G) = \frac{\Delta(G)-\delta(G)}{2}$.

Proof: If we put this value in the common-neighbourhood definition for $K_{1,n-1}$, we obtain the following equality. $\bar{N}(K_{1,\ n-1}) = \frac{n-2}{2} = \frac{n-1-1}{2} = \frac{\Delta(G)-\delta(G)}{2}$.
And this value is the minimum value that common-neighbourhood can get. To be far away from $K_{1,n-1}$ when we add edges to $K_{1,n-1}$, the value of $\Delta(G)$ is still $n-1$ however $\delta(G)$ increases. From *Theorem 2.4*, $\bar{N}\,(G)$ increases and finally $\bar{N}\,(G) = \frac{\Delta(G)-\delta(G)}{2}$ is obtained. □

4. Algorithm for the Common-Neighbourhood Number of a Graph

In this section, we offer an algorithm for the common-neighbourhood number of a graph. The complexity of this algorithm is $O(n^2)$. Data of this algorithm are adjacency matrix and the order of the graph.

A [i, j]: The adjacency matrix of the graph.
CN: Common-Neighbourhood Number of the graph
n: the order of the graph G

```
sumneigh=0
For i=1 to n do
   degv=0
   For j=1 to n do
      degv=degv+A [i, j]
   fact1=1
   fact2=1
   For j=1 to degv do
      fact1=fact1*j
   For j=1 to (degv-2) do
      fact2=fact2*j
   sumneigh=sumneigh + (fact1)/(2-fact2)
Repeat
CN = sumneigh / (n-1)
END.
```

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