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New Trends In Laplace Type Integral Transforms With Applications

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ABSTRACT: In this paper, the authors provided a discussion on one and two dimensional Laplace transforms and generalized Stieltjes transform and their applications in evaluating special series and integrals. Finally, we implemented the joint Laplace – Fourier transforms to construct exact solution for a variant of the KdV equation. Illustrative examples are also provided.

Key Words: Laplace transform; Generalized Stieltjes transform; Kelvin's function; Airy differential equation; Korteweg – de Vries equation.

Contents

Introduction

In this work, we consider some methods consisting Laplace, Stieljes, Fourier transforms to evaluate some certain integrals and series and also to find the solution of some integral equations. The authors have already studied several methods to evaluate series, integrals and solve fractional differential equations, specially the popular Laplace transform method, [1], [2], [3], [4], [5], [6], [7], [8] and this work is a completion for their previous researches.

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1. One dimensional Laplace transform

Definition 1.1. *Laplace transform of function $f(t)$ is as follows*

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt := F(s).$$

If $L\{f(t)\} = F(s)$, then $L^{-1}\{F(s)\}$ is given by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds,$$

where $F(s)$ is analytic in the region $\mathrm{Re}(s) > c$.

Theorem 1.2 (Buschman)**.** *If the functions $f(t), g(t), k(t)$ be analytical and real on $(0,\infty)$ such that $g(0) = 0$ and $g(\infty) = \infty$. Then*

$$L\{k(t)f[g(t)]\} = \int_0^{+\infty} R(s,u)F(u)du.$$

in which

$$Q(s,p) = \int_0^{+\infty} e^{-pu}R(s,u)du = e^{-sh(p)}k[h(p)]h'(p), \quad h = g^{-1},$$

Proof: See [14]. □

Definition 1.3. *Kratzel function was first introduced by Kratzel and then studied by Puri [18]*

$$Z_\rho^\nu(z) = \int_0^\infty t^{\nu-1}\exp(-t^\rho - \frac{z}{t})dt, \quad \nu \in C, \rho \in R,$$

for example we have

$$Z_1^\nu(\frac{x^2}{4}) = \int_0^\infty t^{\nu-1}\exp(-t - \frac{x^2}{4\,t})dt = \frac{2^{\nu+1}}{x^\nu}K_{-\nu}(x),$$

which is in the form of an integral representation of modified Bessel function and can be expressed in terms of Airy function as below. Let$\nu = -\frac{1}{3}, x = \frac{2}{3}\eta^{\frac{3}{2}}$

$$Z_1^{-\frac{1}{3}}(\frac{\eta^3}{9}) = \int_0^\infty t^{-\frac{4}{3}}\exp(-t - \frac{\eta^3}{9\,t})dt = 2\sqrt[6]{3}\sqrt{\frac{\eta}{3}}K_{\frac{1}{3}}(\frac{2}{3}\eta^{\frac{3}{2}}) = 2\pi\sqrt[6]{3}Ai(\eta).$$

Example 1.4. *Prove that the following relationship holds true*

$$\int_0^\infty Z_\rho^\nu(z)dz = \frac{1}{\rho}\Gamma(\frac{\nu+1}{\rho}).$$

Proof. *Taking Laplace transform of the Kratzel function we have*

$$L\{Z_\rho^\nu(z);s\} = \int_0^\infty e^{-sz}\left(\int_0^\infty t^{\nu-1}\exp(-t^\rho - \frac{z}{t})dt\right)dz,$$

now changing the order of integrals

$$L\{Z_\rho^\nu(z);s\} = \frac{1}{s}\int_0^\infty \frac{t^\nu e^{-t^\rho}}{t+(\frac{1}{s})}dt = \int_0^\infty t^\nu e^{-t^\rho}\frac{dt}{1+st},$$

in suffices to let $s = 0$*in the last integral and making a change of variable* $t^\rho = w$ *to get*

$$\int_0^\infty Z_\rho^\nu(z)dz = \frac{1}{\rho}\Gamma(\frac{\nu+1}{\rho}).$$

Lemma 1.5. *The following identities hold true*

$$1 - \int_0^{+\infty} z^\mu K_\nu(z)dz = 2^{\mu-1}\Gamma(\tfrac{\mu+\nu+1}{2})\Gamma(\tfrac{\mu-\nu+1}{2})$$
$$2 - \int_0^{+\infty} \ln z K_0(z)dz = -\tfrac{\pi}{2}(\ln 2 + \gamma)$$

Where $\gamma = .577215664901532860606512090082....$ *is Euler – Maschroni constant*

Proof: 1 – We use integral representation for modified Bessel function of order v, to get

$$\int_0^{+\infty} z^\mu K_\nu(z)dz = \int_0^{+\infty} z^\mu\left\{\frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^{+\infty}\exp(-t-\frac{z^2}{4t})\frac{dt}{t^{\nu+1}}\right\}dz.$$

Changing the order of integration to get

$$\int_0^{+\infty} z^\mu K_\nu(z)dz = \left(\frac{1}{2}\right)^{\nu+1}\int_0^{+\infty} e^{-t}\left(\int_0^{+\infty} z^{\nu+\mu}e^{-\frac{z^2}{4t}}dz\right)\frac{dt}{t^{\nu+1}}.$$

In the inner integral, introducing the change of variable $\frac{z^2}{4\,t} = w$

$$\int_0^{+\infty} z^\mu K_\nu(z)dz = 2^{\mu-1}\{\int_0^{+\infty} w^{\frac{\mu+\nu-1}{2}}e^{-w}dw\}\{\int_0^{+\infty} t^{\frac{\mu-\nu-1}{2}}e^{-t}dt\}.$$

After calculating each integral by definition of Laplace transform, one has

$$\int_0^{+\infty} z^\mu K_\nu(z)dz = 2^{\mu-1}\Gamma(\frac{\mu+\nu+1}{2})\Gamma(\frac{\mu-\nu+1}{2}).$$

2 – In the above relation, let us differentiate with respect to y , after simplifying , one gets

$$\int_0^{+\infty} z^\mu \ln z\, K_\nu(z)dz = \ln 2\; 2^{\mu-1}\Gamma(\tfrac{\mu+\nu+1}{2})\Gamma(\tfrac{\mu-\nu+1}{2}) + 2^{\mu-2}\Gamma'(\tfrac{\mu+\nu+1}{2})\Gamma(\tfrac{\mu-\nu+1}{2}) +$$
$$+ 2^{\mu-2}\Gamma(\tfrac{\mu+\nu+1}{2})\Gamma'(\tfrac{\mu-\nu+1}{2}).$$

By setting $\mu = \nu = 0$ we have

$$\int_0^{+\infty} \ln z \, K_0(z) dz = \frac{\ln 2}{2}(\Gamma(\frac{1}{2}))^2 + \frac{1}{2}\Gamma(\frac{1}{2})\Gamma'(\frac{1}{2}) = \frac{\pi}{2}(\ln 2 + (-2\ln 2 - \gamma))$$
$$= -\frac{\pi}{2}(\ln 2 + \gamma).$$

□

Example 1.6. *Find the inverse Laplace transform of the function* $\int_a^b \frac{d\beta}{(s+\lambda)^\beta}$.
Solution. We have

$$\int_a^b \frac{d\beta}{(s+\lambda)^\beta} = \frac{1}{\ln(s+\lambda)}\left[(s+\lambda)^{-a} - (s+\lambda)^{-b}\right].$$

Then by using inverse of Laplace transform we get

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{(s+\lambda)^{-a} - (s+\lambda)^{-b}}{\ln(s+\lambda)} e^{st} ds,$$

now making a change of variable $s + \lambda = p$ *the following result will be obtained*

$$f(t) = e^{-\lambda t}\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{p^{-a} - p^{-b}}{\ln p} e^{pt} dp.$$

At this point, we evaluate the complex integral by virtue of Titchmarsh theorem [3]

$$f(t) = e^{-\lambda t}\frac{1}{\pi}\int_0^\infty Im\left(\frac{(re^{-i\pi})^{-a} - (re^{-i\pi})^{-b}}{\ln(-r)}\right) e^{-tr} dr$$
$$= \frac{e^{-\lambda t}}{\pi}\int_0^\infty Im\left(\frac{r^{-a}e^{ia\pi} - r^{-b}e^{ib\pi}}{\ln|r| + i\pi}\right) e^{-tr} dr,$$

multiplying the numerator and denominator by the conjugate of the denominator , we ge

$$f(t) = \frac{e^{-\lambda t}}{\pi}\int_0^\infty \frac{\ln r(r^b\cos a\pi - r^a\cos b\pi) - \pi(r^b\sin a\pi - r^a\sin b\pi)}{r^{a+b}(\ln^2 r + \pi^2)} e^{-tr} dr.$$

Lemma 1.7. *The following relationship holds true*

$$L\{f(t^3)\} = \frac{1}{3\pi}\int_0^\infty \left(\frac{s}{u}\right)^{\frac{1}{2}} K_{\frac{1}{3}}\left[2\left(\frac{s}{3u^{\frac{1}{3}}}\right)^{\frac{3}{2}}\right] F(u) du.$$

Proof. By using Buschman's theorem (see [14]) for $k(t) = 1, g(t) = t^3$ *and therefore* $h(p) = \sqrt[3]{p}$ *we have*

$$R(s,u) = L^{-1}\{\frac{e^{-sp^{\frac{1}{3}}}}{3p^{\frac{2}{3}}}\},$$

on the other hand by using table of Laplace transform we can see that

$$L^{-1}\{\frac{e^{-3p^{\frac{1}{3}}}}{p^{\frac{2}{3}}}\} = \frac{1}{\pi}\sqrt{\frac{3}{u}}K_{\frac{1}{3}}(\frac{2}{\sqrt{u}}),$$

Using the fact that $L^{-1}\{F(as)\} = \frac{1}{a}f(\frac{t}{a})$*then for* $a = (\frac{s}{3})^3$*we obtain*

$$L^{-1}\{\frac{e^{-sp^{\frac{1}{3}}}}{3p^{\frac{2}{3}}}\} = \frac{1}{3\pi}(\frac{s}{u})^{\frac{1}{2}}K_{\frac{1}{3}}[2(\frac{s}{3u^{\frac{1}{3}}})^{\frac{3}{2}}],$$

and therefore

$$L\{f(t^3)\} = \frac{1}{3\pi}\int_0^{\infty}\left(\frac{s}{u}\right)^{\frac{1}{2}}K_{\frac{1}{3}}\left[2\left(\frac{s}{3u^{\frac{1}{3}}}\right)^{\frac{3}{2}}\right]F(u)du.$$

Corollary 1.8. *The following relationship holds true*

$$L\{e^{-t^6}\} = \frac{1}{6\sqrt{\pi}}\int_0^{\infty}\left(\frac{s}{u}\right)^{\frac{1}{2}}K_{\frac{1}{3}}\left[2\left(\frac{s}{3u^{\frac{1}{3}}}\right)^{\frac{3}{2}}\right]e^{\frac{u^2}{4}}erfc(\frac{u}{2})du.$$

Solution. Let us consider the function $f(t) = e^{-t^2}$ *so we have*

$$L\{e^{-t^2}\} = \int_0^{+\infty}e^{-st-t^2}dt = e^{\frac{s^2}{4}}\int_0^{+\infty}e^{-(t+\frac{s}{2})^2}dt.$$

Making a change of variable $t + \frac{s}{2} = w$*the error function will be appeared as below*

$$L\{e^{-t^2}\} = e^{\frac{s^2}{4}}\int_{\frac{s}{2}}^{+\infty}e^{-w^2}dw = \frac{\sqrt{\pi}}{2}e^{\frac{s^2}{4}}erfc(\frac{s}{2}).$$

Now using the previous lemma we can write

$$L\{e^{-t^6}\} = \frac{1}{6\sqrt{\pi}}\int_0^{\infty}\left(\frac{s}{u}\right)^{\frac{1}{2}}K_{\frac{1}{3}}\left[2\left(\frac{s}{3u^{\frac{1}{3}}}\right)^{\frac{3}{2}}\right]e^{\frac{u^2}{4}}erfc(\frac{u}{2})du.$$

2. Inverse Laplace transform of some functions by using conformal mapping

Definition 2.1. *The function* $f : C \to C$ *defined as*

$$f(z) = a(z + \frac{b^2}{z}),$$

is named Joukowsky map. This function is a conformal mapping (analytic and angle preserving). Jokouwsky map is extensively used in aerodynamics and physics *which transforms a circle into an ellipse, because if $z = x + iy$be a point on a circle of radius $r$then*

$$f(z) = u + iv = a((x + iy) + \frac{b^2}{(x + iy)}) = \frac{ax(r^2 + b^2)}{r^2} + i\frac{ay(r^2 - b^2)}{r^2},$$

and consequently

$$\frac{u^2}{(r^2 + b^2)^2} + \frac{v^2}{(r^2 - b^2)^2} = \frac{a^2}{r^2},$$

which is the equation of the desired ellipse. One can deduce from the above relationship that if the radius of the circle tends to $b$then the ellipse will tend to a line segment on $x$axis between the points $x = 2ab$and $x = -2ab$.

Theorem 2.2. *The following relationship holds true*

$$L^{-1}\left\{\frac{e^{-a\sqrt{s^2+2\lambda s+\mu^2}}}{Q(s)\sqrt{s^2 + 2\lambda s + \mu^2}}; t\right\}$$
$$= \begin{cases} -e^{-\lambda t}(\sqrt{\frac{t-a}{t+a}})^{\nu} I_{\nu}((\lambda^2 - \mu^2)\sqrt{t^2 - a^2}); & \lambda^2 - \mu^2 > 0 \\ -e^{-\lambda t}(\sqrt{\frac{t-a}{t+a}})^{\nu} J_{\nu}((\lambda^2 - \mu^2)\sqrt{t^2 - a^2}); & \lambda^2 - \mu^2 < 0 \end{cases}.$$

Where, $Q(s) = (\sqrt{s^2 + 2\lambda s + \mu^2} + s + \lambda)^{-\nu}$.

Proof: By using inverse of Laplace transform we have

$$\begin{aligned} L^{-1}\{F(s); t\} &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\frac{e^{-a\sqrt{s^2+2\lambda s+\mu^2}}}{Q(s)\sqrt{s^2 + 2\lambda s + \mu^2}}ds \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\frac{e^{-a\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}}}{Q(s)\sqrt{(s + \lambda)^2 - (\lambda^2 - \mu^2)}}ds. \end{aligned}$$

It is clear that the function $F(s)$ has branch points at $s = -\lambda \pm \sqrt{\lambda^2 - \mu^2}$ therefore according to the figure 2 consider the branch cuts. Case 1: Assume that $\lambda^2 - \mu^2 > 0$ then we integrate the function $e^{st}F(s)$on the path Øč$_1$ and then let it tend to infinity.

By integrating along the path indicated in the figure 1 and using residue theorem we have

$$\begin{aligned} &\frac{1}{2\pi i}\oint_{\Gamma_1} \frac{e^{-a\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}+st}}{Q(s)\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}}ds \\ &= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{e^{-a\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}+st}}{Q(s)\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}}ds + \frac{1}{2\pi i}\int_{C_R} \frac{e^{-a\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}+st}}{Q(s)\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}}ds \\ &+ \frac{1}{2\pi i}\int_{AB} \frac{e^{-a\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}+st}}{Q(s)\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}}ds + \frac{1}{2\pi i}\int_{\Omega} \frac{e^{-a\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}+st}}{Q(s)\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}}ds \\ &+ \frac{1}{2\pi i}\int_{DC} \frac{e^{-a\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}+st}}{Q(s)\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}}ds + \frac{1}{2\pi i}\int_{C'_R} \frac{e^{-a\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}+st}}{Q(s)\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}}ds = 0. \end{aligned}$$

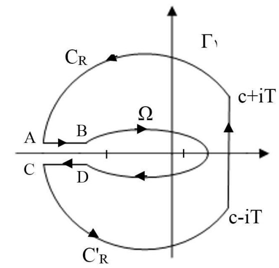

Figure 1:

One can show that summation of integrals along AB and DC is zero. Then

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{e^{-a\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}+st}}{Q(s)\sqrt{(s+\lambda)^2+(\lambda^2-\mu^2)}}ds=-\frac{1}{2\pi i}\int_{C_R}\frac{e^{-a\sqrt{(Re^{i\theta}+\lambda)^2-(\lambda^2-\mu^2)}+Re^{i\theta}t}}{Q(Re^{i\theta})\sqrt{(Re^{i\theta}+\lambda)^2+(\lambda^2-\mu^2)}}Rie^{i\theta}d\theta$$
$$-\frac{1}{2\pi i}\int_{C'_R}\frac{e^{-a\sqrt{(Re^{i\theta}+\lambda)^2-(\lambda^2-\mu^2)}+Re^{i\theta}t}}{Q(Re^{i\theta})\sqrt{(Re^{i\theta}+\lambda)^2-(\lambda^2-\mu^2)}}Rie^{i\theta}d\theta-\frac{1}{2\pi i}\int_{\Omega}\frac{e^{-a\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}+st}}{Q(s)\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}}ds.$$

On the other hand if $R\to\infty$one can show that integrals along the arcs C_R and C'_Ralso tend to zero. Hence we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-a\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}+st}}{Q(s)\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}}ds=-\frac{1}{2\pi i}\int_{\Omega}\frac{e^{-a\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}+st}}{Q(s)\sqrt{(s+\lambda)^2-(\lambda^2-\mu^2)}}ds,$$

now we make a change of variables $s+\lambda=p$in the right hand side integral to get

$$I=-\frac{e^{-\lambda t}}{2\pi i}\int_{\Omega_1}\frac{e^{-a\sqrt{p^2-(\lambda^2-\mu^2)}+pt}}{R(p)\sqrt{p^2-(\lambda^2-\mu^2)}}dp,$$

where $R(p)=(\sqrt{p^2+(\mu^2-\lambda^2)}+p)^{\nu}$and Ω_1 is obtained by shifting the ellipse Ω in direction of horizontal axis by λ(if $\lambda>0$shifting to the left and if $\lambda<0$ shifting to the right). Therefore it suffices to evaluate the integral$I$. This integral can be rewritten as follows

$$I=-\frac{e^{-\lambda t}}{2\pi i}\int_{\Omega_1}\frac{e^{-a(\sqrt{p^2-(\lambda^2-\mu^2)}+p)}}{R(p)\sqrt{p^2-(\lambda^2-\mu^2)}}e^{(t+a)p}dp.$$

Now let us make a change of variables $w=\sqrt{p^2-(\lambda^2-\mu^2)}+p$ in the above integral, this is in fact the inverse of the Jokouwsky map $p=\frac{1}{2}(w+\frac{(\lambda^2-\mu^2)}{w})$ so we can transform the ellipse $\Omega_1$to a circle of radius r. Therefore we have

$$I=-\frac{e^{-\lambda t}}{2\pi i}\int_{|w|=r}e^{-aw+\frac{1}{2}(t+a)(w+\frac{(\lambda^2-\mu^2)}{w})}\frac{dw}{w^{\nu+1}},$$

now we make a change of variables $z = \sqrt{\frac{t-a}{t+a}}\,w$ to get

$$I = -\frac{e^{-\lambda t}}{2\pi i}\left(\sqrt{\frac{t-a}{t+a}}\right)^{\nu}\int_{|z|=\sqrt{\frac{t-a}{t+a}}r} e^{\frac{1}{2}\sqrt{t^2-a^2}\left(z+\frac{(\lambda^2-\mu^2)}{z}\right)}\frac{dz}{z^{\nu+1}},$$

according to the definition of Jokouwsky map if the ellipse Ω_1 tends to a line segment then r tends to $\lambda^2-\mu^2$ therefore one can rewrite the above equation as below

$$\begin{aligned} I &= -\frac{e^{-\lambda t}}{2\pi}\left(\sqrt{\frac{t-a}{t+a}}\right)^{\nu}\int_{-\pi}^{\pi} e^{\frac{1}{2}(\lambda^2-\mu^2)\sqrt{t^2-a^2}(e^{i\theta}+e^{-i\theta})-i\nu\theta}d\theta \\ &= -\frac{e^{-\lambda t}}{2\pi}\left(\sqrt{\frac{t-a}{t+a}}\right)^{\nu}\int_{-\pi}^{\pi} e^{(\lambda^2-\mu^2)\sqrt{t^2-a^2}\cos\theta}\cos\nu\theta d\theta. \end{aligned}$$

On the other hand, using the fact that $I_\nu(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\alpha}\cos\nu\alpha\, d\alpha$ (see [15]) then one gets

$$L^{-1}\left\{\frac{e^{-a\sqrt{s^2+2\lambda s+\mu^2}}}{Q(s)\sqrt{s^2+2\lambda s+\mu^2}}; s\to t\right\} = -e^{-\lambda t}\left(\sqrt{\frac{t-a}{t+a}}\right)^{\nu} I_\nu((\lambda^2-\mu^2)\sqrt{t^2-a^2}).$$

Case 2: Assume that $\lambda^2-\mu^2<0$ then the branch points of the function $F(s)$ are $s=-\lambda\pm i\sqrt{\mu^2-\lambda^2}$ therefore we integrate the function $e^{st}F(s)$ along the path Γ_2 then we let $R$to tend to infinity. Similarly to case 1 we obtain the following result

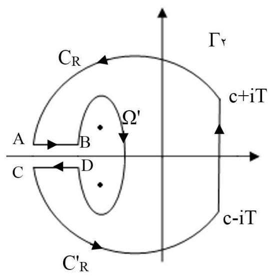

Figure 2:

$$L^{-1}\left\{\frac{e^{-a\sqrt{s^2+2\lambda s+\mu^2}}}{Q(s)\sqrt{s^2+2\lambda s+\mu^2}}; s\to t\right\} = -e^{-\lambda t}\left(\sqrt{\frac{t-a}{t+a}}\right)^{\nu} J_\nu((\mu^2-\lambda^2)\sqrt{t^2-a^2}).$$

□

Lemma 2.3. *The following relationship holds true*

$$L^{-1}\left\{\frac{F(\sqrt{s^2+2\mu s+\lambda^2})}{Q(s)\sqrt{s^2+2\mu s+\lambda^2}};t\right\}$$
$$=\begin{cases} -e^{-\lambda t}\int_0^\infty f(u)\sqrt{\frac{t-u}{t+u}})^\nu I_\nu((\mu^2-\lambda^2)\sqrt{t^2-u^2})du, & \lambda^2-\mu^2<0 \\ -e^{-\lambda t}\int_0^\infty f(u)\sqrt{\frac{t-u}{t+u}})^\nu J_\nu((\mu^2-\lambda^2)\sqrt{t^2-u^2})du, & \lambda^2-\mu^2>0 \end{cases}.$$

Proof: Let us assume that $g(t)=\int_0^t f(u)du$ then we know that $G(p)=L\{g(t);p\}$ $=\frac{F(p)}{p}$ in which $F(p)=L\{f(t);p\}$ so by substituting $p=\sqrt{s^2+2\mu s+\lambda^2}$ we can write

$$\frac{F(\sqrt{s^2+2\mu s+\lambda^2})}{Q(s)\sqrt{s^2+2\mu s+\lambda^2}}=\int_0^\infty \frac{e^{-t\sqrt{s^2+2\mu s+\lambda^2}}}{Q(s)}\left(\int_0^t f(u)du\right)dt,$$

changing the order of integrals we have

$$\frac{F(\sqrt{s^2+2\mu s+\lambda^2})}{Q(s)\sqrt{s^2+2\mu s+\lambda^2}}=\int_0^\infty f(u)(\int_u^\infty \frac{e^{-t\sqrt{s^2+2\mu s+\lambda^2}}}{Q(s)}dt)du$$
$$=\int_0^\infty f(u)\frac{e^{-u\sqrt{s^2+2\mu s+\lambda^2}}}{Q(s)\sqrt{s^2+2\mu s+\lambda^2}}du,$$

now we take inverse of Laplace transform

$$h(t)=L^{-1}\{\frac{F(\sqrt{s^2+2\mu s+\lambda^2})}{Q(s)\sqrt{s^2+2\mu s+\lambda^2}};t\}$$
$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\{\int_0^\infty f(u)\frac{e^{-u\sqrt{s^2+2\mu s+\lambda^2}}}{Q(s)\sqrt{s^2+2\mu s+\lambda^2}}du\}ds.$$

Once again change the order of integrals to get

$$h(t)=\int_0^\infty f(u)\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-u\sqrt{s^2+2\mu s+\lambda^2}}}{Q(s)\sqrt{s^2+2\mu s+\lambda^2}}e^{st}ds\right)du,$$

therefore by using the solution to the previous theorem we obtain

$$h(t)=\begin{cases} -e^{-\lambda t}\int_0^\infty f(u)\sqrt{\frac{t-u}{t+u}})^\nu I_\nu((\mu^2-\lambda^2)\sqrt{t^2-u^2})du, & \lambda^2-\mu^2<0 \\ -e^{-\lambda t}\int_0^\infty f(u)\sqrt{\frac{t-u}{t+u}})^\nu J_\nu((\mu^2-\lambda^2)\sqrt{t^2-u^2})du, & \lambda^2-\mu^2>0 \end{cases}.$$

$\square$

3. Evaluation of certain series

Lemma 3.1. *The following relationships hold true*

$$\sum_{n=1}^{\infty} F(n) = \int_0^{\infty} \frac{e^{-\frac{x}{2}} f(x)}{2\sinh\frac{x}{2}}\, dx = \int_0^{\infty} \frac{f(x)}{e^x - 1}\, dx, \tag{3.1}$$

$$\sum_{n=1}^{\infty} (-1)^n F(n) = \int_0^{\infty} \frac{e^{-\frac{x}{2}} f(x)}{2\cosh\frac{x}{2}}\, dx = \int_0^{\infty} \frac{f(x)}{e^x + 1}\, dx \tag{3.2}$$

in which $F(s) = L\{f(t); t \to s\}$.

Proof: By using the definition of Laplace transform

$$F(s) = \int_0^{\infty} e^{-sx} f(x) dx,$$

the values of the function $F(s)$for natural arguments will be

$$F(n) = \int_0^{\infty} e^{-nx} f(x) dx,$$

at this point, we sum the above relationship over all positive integers to get

$$\sum_{n=1}^{\infty} F(n) = \int_0^{\infty} \sum_{n=1}^{\infty} e^{-nx} f(x) dx = \int_0^{\infty} \frac{f(x)}{e^x - 1} dx = \int_0^{\infty} \frac{e^{-\frac{x}{2}} f(x)}{2\sinh\frac{x}{2}}\, dx \quad ,$$

and also

$$\sum_{n=1}^{\infty} (-1)^{n-1} F(n) = \int_0^{\infty} \sum_{n=1}^{\infty} (-1)^{n-1} e^{-nx} f(x) dx = \int_0^{\infty} \frac{f(x)}{e^x + 1} dx = \int_0^{\infty} \frac{e^{-\frac{x}{2}} f(x)}{2\cosh\frac{x}{2}} dx.$$

□

Example 3.2. *For evaluating the series* $\sum_{n=1}^{\infty} \frac{8}{(2n-1)^2+4}$ *we use Laplace transform of the function*

$$f(t) = 2e^{\frac{t}{2}} \sin t.$$

We know that $F(s) = \frac{2}{(s-\frac{1}{2})^2+1}$ *is the Laplace transform of the function* $f(t)$*. Therefore substituting in* (**??**) *we get the following result*

$$\sum_{n=1}^{\infty} \frac{2}{(n-\frac{1}{2})^2 + 1} = \int_0^{\infty} \frac{\sin t}{\sinh\frac{t}{2}} dt.$$

By integrating of the following function over the closed path indicated in figure 3 we have

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{e^{iz}}{\sinh\frac{z}{2}} dz = 0$$

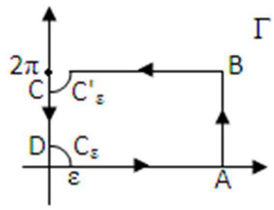

Figure 3:

It means that

$$\int_{\varepsilon}^{R}\frac{e^{ix}}{\sinh\frac{x}{2}}dx+\int_{0}^{2\pi}\frac{e^{i(R+iy)}}{\sinh\frac{(R+iy)}{2}}idy+\int_{R}^{\varepsilon}\frac{e^{i(x+2\pi i)}}{\sinh\frac{(x+2\pi i)}{2}}dx+\int_{0}^{-\frac{\pi}{2}}\frac{e^{i(2\pi i+\varepsilon e^{i\theta})}}{\sinh\frac{(2\pi i+\varepsilon e^{i\theta})}{2}}i\varepsilon e^{i\theta}d\theta$$
$$-\int_{\varepsilon}^{2\pi-\varepsilon}\frac{e^{i(iy)}}{\sinh\frac{(iy)}{2}}idy+\int_{\frac{\pi}{2}}^{0}\frac{e^{i(\varepsilon e^{i\theta})}}{\sinh\frac{(\varepsilon e^{i\theta})}{2}}i\varepsilon e^{i\theta}d\theta=0,$$

which by manipulating can be rewritten as below

$$(1+e^{-2\pi})\int_{0}^{+\infty}\frac{e^{ix}}{\sinh\frac{x}{2}}dx=i\pi\frac{1-e^{-2\pi}}{1+e^{-2\pi}}-\frac{1}{1+e^{-2\pi}}\int_{0}^{2\pi}\frac{e^{-y}}{\sin\frac{y}{2}}dy,$$

taking the imaginary part of the above relationship one gets

$$\sum_{n=1}^{\infty}\frac{8}{(2n-1)^2+4}=\int_{0}^{+\infty}\frac{\sin w}{\sinh\frac{w}{2}}dw=\pi\frac{1-e^{-2\pi}}{1+e^{-2\pi}}=\pi\tanh\pi.$$

Example 3.3. *Consider the following alternative series* $\sum_{n=1}^{\infty}\frac{(-1)^n 4(2n-1)}{(2n-1)^2+4}$.

For evaluating the above series, we use again Laplace transform of the function

$$f(t)=2e^{\frac{t}{2}}\cos t.$$

It is clear that $F(s)=\frac{2s-1}{(s-\frac{1}{2})^2+1}$ *is the Laplace transform of the function* $f(t)$. *Therefore substituting in* (**??**) *we get the following result*

$$\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)}{(n-\frac{1}{2})^2+1}=\int_{0}^{\infty}\frac{\cos t}{cosh\frac{t}{2}}dt,$$

By using the table of integrals or as previous example by complex integration around rectangle, we get the following

$$\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)}{(n-\frac{1}{2})^2+1}=\int_{0}^{\infty}\frac{\cos t}{cosh\frac{t}{2}}\,dt=\frac{\pi}{\cosh 0.5\pi}$$

4. Evaluation of certain series containing Legendre polynomials

Definition 4.1. *Legendre polynomials are the solutions of the following ordinary differential equation called Legendre differential equation*

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right]+n(n+1)P_n(x)=0.$$

Lemma 4.2. *The Legendre polynomials are orthogonal, it means that*

$$\int_{-1}^{1}P_n(x)P_m(x)dx=\frac{2}{2n+1}\delta_{mn},$$

or in other words

$$\|P_n(x)\|=\sqrt{\frac{2}{2n+1}}.$$

Proof: See [19]. □

Theorem 4.3. *The following relationship holds true*

$$\sum_{n=0}^{\infty}\frac{(-1)^nL_n(t)L_n(\xi)}{n+\frac{1}{2}}=e^{\frac{t+\xi}{2}}\int_0^{+\infty}J_0(\frac{tu}{2})J_0(\frac{\xi}{2u})\frac{du}{u^2+1}.$$

Proof. It is well known that the generating function of Legendre polynomials is as below

$$\sum_{n=0}^{\infty}t^nP_n(x)=\frac{1}{\sqrt{1-2tx+t^2}},$$

substituting $t=1-\frac{1}{p}$in the above relationship we obtain

$$\sum_{n=0}^{\infty}(1-\frac{1}{p})^nP_n(x)=\frac{1}{\sqrt{1-2x(1-\frac{1}{p})+(1-\frac{1}{p})^2}}=\frac{p}{\sqrt{p^2-2xp(p-1)+(p-1)^2}},$$

one can rewrite the above relationship as below

$$\sum_{n=0}^{\infty}\frac{1}{p}(1-\frac{1}{p})^nP_n(x)=\frac{1}{\sqrt{2(1-x)}}\cdot\frac{1}{\sqrt{(p-\frac{1}{2})^2+\frac{1+x}{4(1-x)}}},$$

on the other hand we know that $L\{L_n(t);p\}=\frac{1}{p}(1-\frac{1}{p})^n$ and

$$L\{J_0(\frac{1}{2}\sqrt{\frac{1+x}{1-x}}t);p\}=\frac{e^{-\frac{t}{2}}}{\sqrt{(p-\frac{1}{2})^2+\frac{1+x}{4(1-x)}}},$$

therefore, if we take inverse Laplace transform of both sides of the above equation, we get

$$\sum_{n=0}^{\infty} L_n(t)P_n(x) = \frac{e^{\frac{t}{2}}}{\sqrt{2(1-x)}} J_0\left(\frac{t}{2}\sqrt{\frac{1+x}{1-x}}\right).$$

Hence we can write

$$\left(\sum_{n=0}^{\infty} L_n(t)P_n(x)\right)\left(\sum_{m=0}^{\infty} L_m(\xi)P_m(-x)\right) = \frac{e^{\frac{t+\xi}{2}}}{2\sqrt{1-x^2}} J_0\left(\frac{t}{2}\sqrt{\frac{1+x}{1-x}}\right) J_0\left(\frac{\xi}{2}\sqrt{\frac{1-x}{1+x}}\right),$$

in other words

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (-1)^m (L_n(t)L_m(\xi))P_n(x)P_m(x) = \frac{e^{\frac{t+\xi}{2}}}{2\sqrt{1-x^2}} J_0\left(\frac{t}{2}\sqrt{\frac{1+x}{1-x}}\right) J_0\left(\frac{\xi}{2}\sqrt{\frac{1-x}{1+x}}\right).$$

If we integrate both sides of the above equation with respect to the variable x from -1 to 1, because of the orthogonality of the Legendre functions, we get the following result

$$\sum_{n=0}^{\infty} \frac{(-1)^n L_n(t)L_n(\xi)}{n+\frac{1}{2}} = \frac{1}{2} e^{\frac{t+\xi}{2}} \int_{-1}^{1} J_0\left(\frac{t}{2}\sqrt{\frac{1+x}{1-x}}\right) J_0\left(\frac{\xi}{2}\sqrt{\frac{1-x}{1+x}}\right) \frac{dx}{\sqrt{1-x^2}}.$$

Making a change of variable $u = \sqrt{\frac{1+x}{1-x}}$, yields

$$\sum_{n=0}^{\infty} \frac{(-1)^n L_n(t)L_n(\xi)}{n+\frac{1}{2}} = e^{\frac{t+\xi}{2}} \int_{0}^{+\infty} J_0\left(\frac{tu}{2}\right) J_0\left(\frac{\xi}{2u}\right) \frac{du}{u^2+1}.$$

Special case: Let $t = \xi = 0$in the previous lemma then regarding $L_n(0) = J_0(0) = 1$ we will have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+\frac{1}{2}} = \int_{0}^{+\infty} \frac{du}{u^2+1} = \frac{\pi}{2}.$$

5. Two dimensional Laplace transform

Definition 5.1. *Two dimensional Laplace transform of the function $f(x,y)$ is defined as*

$$F(p,q) = \int_{0}^{+\infty}\int_{0}^{+\infty} e^{-px-qy} f(x,y)\,dxdy, \tag{5.1}$$

while its inverse is given by

$$f(x,y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty}\int_{c'-i\infty}^{c'+i\infty} F(p,q)e^{px+qy}\,dpdq, \tag{5.2}$$

where $F(p,q)$ is analytic in the regions $Rep > c, Req > c'$ ([9], [10], [11] , [16]).

Lemma 5.2. *The following relationship holds true*

$$L_2^{p,q}\{\frac{ber(\sqrt[4]{xy})}{\sqrt{xy}}\} = \frac{\pi}{\sqrt{pq}} J_0(\frac{1}{8\sqrt{pq}}).$$

Proof: By series expansion of Kelvin function of order zero we have

$$ber(x) = 1 - \frac{(\frac{x}{2})^4}{(2!)^2} + \frac{(\frac{x}{2})^8}{(4!)^2} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{4k}}{(2k!)^2},$$

hence

$$\frac{ber(\sqrt[4]{xy})}{\sqrt{xy}} = \sum_{k=0}^{\infty} \frac{(-1)^k (xy)^{k-\frac{1}{2}}}{2^{4k}(2k!)^2},$$

now we take Laplace transform of the above relationship with respect to the variables x, y

$$L_2^{p,q}\{\frac{ber(\sqrt[4]{xy})}{\sqrt{xy}}\} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{4k}(2k!)^2} \frac{\Gamma(k+\frac{1}{2})\Gamma(k+\frac{1}{2})}{(pq)^{k+\frac{1}{2}}}.$$

Using the elementary relation $\Gamma(m+\frac{1}{2}) = \frac{(2m)!\sqrt{\pi}}{2^{2m}m!}$ we obtain

$$L_2^{p,q}\{\frac{ber(\sqrt[4]{xy})}{\sqrt{xy}}\} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{8k}} \frac{\pi}{(k!)^2(pq)^{k+\frac{1}{2}}}.$$

on the other hand we have also the following expansion for Bessel's function

$$J_0(x) = 1 - \frac{(\frac{x}{2})^2}{1^2} + \frac{(\frac{x}{2})^4}{1^2 \times 2^2} - \frac{(\frac{x}{2})^6}{1^2 \times 2^2 \times 3^2} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k}}{(k!)^2}.$$

Hence one has

$$\frac{\pi}{\sqrt{pq}} J_0(\frac{1}{8\sqrt{pq}}) = \pi \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{\sqrt{pq}})^{2k+1}}{2^{8k}(k!)^2} = \sum_{k=0}^{\infty} \frac{(-1)^k \pi}{2^{8k}(k!)^2(pq)^{k+\frac{1}{2}}},$$

and finally we get the following result

$$L_2^{p,q}\{\frac{ber(\sqrt[4]{xy})}{\sqrt{xy}}\} = \frac{\pi}{\sqrt{pq}} J_0(\frac{1}{8\sqrt{pq}}).$$

□

6. Stieltjes transform

Definition 6.1. *The generalized Stieltjes transform is defined as follows*

$$F(y) = S_\rho\{f(t); t\ toy\} = \int_0^\infty \frac{f(t)}{(t+y)^\rho} dt, |\text{argy}| < \pi$$

and its inverse is as below [21]

$$S_\rho^{-1}\{F(y); y \to t\} = -\frac{1}{2\pi i} t^\rho \int_C (1+y)^{\rho-1} F'(ty)dy;\ \rho > 0$$

In the special case of ρ=1, the above relationship leads to the ordinary Stieltjes transform

$$S\{f(t); t \to y\} = \int_0^\infty \frac{f(t)}{t+y}. \tag{6.1}$$

Example 6.2. *Find the inverse Stieltjes transform of the function $\frac{1}{\sqrt{s}(s+a)}$ for $a \in R$.*

Solution. Considering the inverse Stietjes transform of convolution of functions we have

$$S^{-1}\left\{\frac{1}{\sqrt{s}(s+a)}; s \to t\right\} = S^{-1}\left\{\frac{1}{\sqrt{s}}\right\} \otimes S^{-1}\left\{\frac{1}{s+a}\right\}$$

on the other hand Stieltjes transform is the second iterate of the Laplace transform hence

$$S^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\pi\sqrt{t}},\ S^{-1}\left\{\frac{1}{s+a}\right\} = \delta(t-a)$$

therefore the final result will be

$$S^{-1}\left\{\frac{1}{\sqrt{s}(s+a)}; s \to t\right\} = \delta(t-a)\int_0^\infty \frac{1}{\pi\sqrt{u}(u-t)}du + \frac{1}{\pi\sqrt{t}}\int_0^\infty \frac{\delta(u-a)}{(u-t)}du$$

making a change of variable $u = w^2$

$$S^{-1}\left(\frac{1}{\sqrt{s}(s-t)}; s \to t\right) = \frac{1}{\pi\sqrt{t}(a-t)}$$

Lemma 6.3. *Assume that $S\{f(x); x \to s\} = F(s)$ then the following relationship holds true*

$$f(x) = \frac{1}{2\pi i}\{F(xe^{-i\pi}) - F(xe^{i\pi})\} \tag{6.2}$$

Proof: By definition of inverse Stieltjes transform we have

$$S^{-1}\{F(s); s \to x\} = -\frac{s}{2\pi i}\int_C F'(sw)dw$$

making a change of variable $F(sw) = \eta$ we will have

$$S^{-1}\{F(s); s \to x\} = -\frac{1}{2\pi i}\int_C d\eta \tag{6.3}$$

in which C is a closed simple path avoiding the origin and a branch cut on the negative x-axis. Therefore we obtain

$$S^{-1}\{F(s); s \to x\} = \frac{1}{2\pi i}\{F(xe^{-i\pi}) - F(xe^{i\pi})\}$$

In special case, for $q(s) = \sqrt{s}$ one has

$$L^{-1}\{F(\sqrt{s}); s \to t\} = \frac{1}{2t\sqrt{\pi t}} \int_0^\infty \tau f(\tau) \exp\left(-\frac{\tau^3}{4t}\right) d\tau$$

$\square$

Corollary 6.4. *The following relationship holds true*

$$S\{\sin(a\sqrt{t})J_0(b\sqrt{t})\} = \pi e^{-a\sqrt{s}} I_0(b\sqrt{s})$$

Proof: Taking inverse Stieltjes transform we have
which could be rewritten as below

$$S^{-1}\{F(s)\} = \frac{1}{2i}\left\{e^{ia\sqrt{t}} I_0(ib\sqrt{t}) - e^{-ia\sqrt{t}} I_0(ib\sqrt{t})\right\} = \sin(a\sqrt{t})J_0(b\sqrt{t})$$

$\square$

Theorem 6.5 (Schouten-Vanderpol for Stieltjes transform). *Consider the function $f(t)$ and its Stieltjes transform $F(s)$ which are analytic over the region* $\mathrm{Re}s > s_0$. *If $q(s)$ is another analytic function over* $\mathrm{Re}s > s_0$. *Then the inverse Stieltjes transform of the function $F(q(s))$ will be obtained as below*

$$S^{-1}\{F(q(s))\} = \frac{1}{2\pi i}\{F(q(te^{-i\pi})) - F(q(te^{i\pi}))\}$$

Proof: Assume that $G(s) = F(q(s))$ then from the previous lemma we have

$$g(t) = S^{-1}\{G(s)\} = \frac{1}{2\pi i}\{G(te^{-i\pi}) - G(te^{i\pi})\}$$

it means that

$$S^{-1}\{F(q(s))\} = \frac{1}{2\pi i}\{F(q(te^{-i\pi})) - F(q(te^{i\pi}))\}$$

$\square$

Example 6.6. *Solve the following Stieltjes type singular integral equation*

$$\int_0^\infty \frac{\phi(t)}{t+s} dt = G(\eta(s))$$

Solution. The above integral equation is indeed the definition of Stieltjes transform, therefore by using the inverse of Stieltjes transform and Schouten-Vanderpol theorem we can write

$$\phi(t)=\frac{1}{2\pi i}\{G(\eta(te^{-i\pi}))-G(\eta(te^{i\pi}))\}$$

Special case: Let us consider the following integral equation

$$\int_0^\infty \frac{\phi(t)}{t+s}dt=\frac{1}{s\sqrt{s}}$$

therefore, we have

$$\phi(t)=\frac{1}{2\pi i}\left\{\frac{1}{(te^{-i\pi})^{\frac{3}{2}}}-\frac{1}{(te^{i\pi})^{\frac{3}{2}}}\right\}=\frac{1}{\pi t\sqrt{t}}$$

Definition 6.7. *Elliptic integrals were first investigated by Giulio Fagnano and Leonhard Euler. Generally any function f which could be expressed as below is called an elliptic function*

$$f(x)=\int_c^x R(t,P(t))dt,$$

in which R is a rational function of its two arguments, P is a polynomial of degree 3 or 4 with no repeated roots, and c is a constant.

In general, integrals in this form cannot be expressed in terms of elementary functions. Exceptions to this general rule are when P has repeated roots, or when $R(x,y)$ contains no odd powers of y. However, with the appropriate reduction formula, every elliptic integral can be brought into a form that involves integrals over rational functions and the three Legendre canonical forms (i.e. the elliptic integrals of the first, second and third kind).But complete elliptic functions of the first and second kind can be written as follows

$$\begin{array}{l} K(k^2)=\int_0^{\frac{\pi}{2}}(1-k^2\sin^2\theta)^{-\frac{1}{2}}d\theta=\frac{\pi}{2}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right),\\ E(k^2)=\int_0^{\frac{\pi}{2}}(1-k^2\sin\theta)^{\frac{1}{2}}d\theta=\frac{\pi}{2}{}_2F_1\left(\frac{1}{2},-\frac{1}{2};1;k^2\right).\end{array}$$

Example 6.8. *The following relationship holds true*

$$S_\rho\left\{\frac{1}{1+t}\right\}={}_2F_1\left(1,\rho;\rho+1,\left(1-\frac{1}{s}\right)\right)$$

Solution. Considering the definition of the generalized Stieltjes transform and changing the variables $t+s=w$ we have

$$S_\rho\left\{\frac{1}{1+t}\right\}=\int_0^{+\infty}\frac{dt}{(t+s)^\rho(1+t)}=\int_s^{+\infty}\frac{w^{-\rho}}{(w+1-s)}dw$$

which can be rewritten as follows

$$S_\rho\left\{\frac{1}{1+t}\right\} = \int_s^{+\infty} w^{-(\rho+1)}\left\{1-\frac{1-s}{w}+\frac{(1-s)^2}{w^2}-\dots\right\}$$
$$= \int_s^{+\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(1-s)^n}{w^{n+\rho+1}}dw$$

We can rewrite the above relationship by using hyper-geometric functions

$$S_\rho\left\{\frac{1}{1+t}\right\} = \frac{1}{s^\rho}\sum_{n=0}^{\infty}\frac{1}{n+\rho}\left(1-\frac{1}{s}\right)^n = {}_2F_1\left(1,\rho;\rho+1,\left(1-\frac{1}{s}\right)\right)$$

7. Airy functions

George Biddell Airy (1801–1892) was particularly involved in optics for this reason, he was also interested in the calculation of light intensity in the neighborhood of a caustic (see [12,13]). For this purpose, he introduced the function defined by the integral

$$W(m) = \int_0^{+\infty}\cos\left[\frac{\pi}{2}(w^3-mw)\right]dw,$$

which is the solution of the following differential equation

$$W'' + \frac{\pi^2}{12}mW = 0.$$

In 1928 Jeffreys introduced the notation used nowadays

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it(x+\frac{t^2}{3})}dt = \frac{1}{\pi}\int_0^{+\infty}\cos\left(\frac{t^3}{3}+xt\right)dt, \tag{7.1}$$

which is the solution of the the following homogeneous ODE called Airy ODE

$$y'' - xy = 0. \tag{7.2}$$

7.1. Solution to non homogenous linear KdV via Joint Laplace – Fourier transforms.

The KdV equations are attracting many researchers, and a great deal of works has already been done in some of these equations. In this section, we will implement the joint Laplace – Fourier transforms to construct exact solution for a variant of the KdV equation.

Problem 1. *Solving the following non homogenous linear Kdv.*

$$u_t + \alpha u + \beta u_x + \gamma u_{xxx} = Ai(x),$$
$$u(x,0) = f(x).$$

Solution. By taking joint Laplace – Fourier transform of equation and using boundary condition, we get the following transformed equation

$$\hat{\hat{U}}(w,s)=\frac{F(w)}{s-(i\gamma w^3-iw\beta-\alpha)}+\frac{G(w)}{s\{s-(i\gamma w^3-iw\beta-\alpha)\}}.$$

For the sake of simplicity, let us assume that$\tau=i\gamma w^3-iw\beta-\alpha$*, and using inverse Laplace transform of transformed equation to obtain*

$$\hat{U}(w,t)=L^{-1}\{\frac{F(w)}{s-\tau};s->t\}+L^{-1}\{\frac{G(w)}{s(s-\tau)};s->t\}=F(w)e^{\tau t}+G(w)\int_0^t e^{\tau u}du.$$

At this point, inverting Fourier transform to get

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-ixw)\{F(w)e^{\tau t}+G(w)\int_0^t e^{\tau u}du\}dw.$$

By setting back $\tau=i\gamma w^3-iw\beta-\alpha$*, and changing the order of integration, to get*

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-ixw)\{F(w)e^{(i\gamma w^3-iw\beta-\alpha)t}+G(w)\int_0^t e^{(i\gamma w^3-iw\beta-\alpha)u}du\}dw,$$

or

$$\begin{aligned}u(x,t)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-ixw)\{F(w)e^{(i\gamma w^3-iw\beta-\alpha)t}\\&+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-ixw)G(w)\{\int_0^t e^{(i\gamma w^3-iw\beta-\alpha)u}du\}dw,\end{aligned}$$

equivalently

$$\begin{aligned}u(x,t)&=\frac{e^{-\alpha t}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-i(\beta+x)w}F(w)e^{i(-\gamma t)w^3}\\&+\int_0^t e^{-\alpha u}\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-i(\beta+x)w}G(w)e^{i(-\gamma u)w^3}dw\}du.\end{aligned}$$

The inner integrals can be evaluated by convolution for Fourier transform as below

$$u(x,t)=e^{-\alpha t}\{f(x+\beta)*Ai(\frac{x+\beta}{\sqrt[3]{-3\gamma t}})\}+\int_0^t e^{-\alpha u}Ai(x+\beta)*Ai(\frac{x+\beta}{\sqrt[3]{-3\gamma u}})\,du.$$

Note. Where * denotes convolution for Fourier transform.

8. Conclusion

The paper is devoted to study Laplace, Stieltjes integral transforms and their applications in evaluating integrals and series. The authors also discussed Laguerre series as well. The one dimensional Laplace and Fourier Transforms provide powerful method for analyzing linear systems. The main purpose of this work is to develop methods for evaluating some special integrals, series and solution to a variant of non- homogenous KdV equation.

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References

1. A.Aghili, H.Zeinali, Advances in Laplace type integral transforms with applications, Indian Journal of Science and Technology, Vol. 7(6), 877–890, June 2014..
2. A.Aghili, H.Zeinali, Solution to time fractional wave equation in the presence of friction via integral transform, Communications on applied nonlinear analysis, Vol.21(3), 67-88, June 2014.
3. A.Aghili, H.Zeinali, Solution to Fractional Schrodinger and Airy Differential Equations via Integral Transforms, British journal of mathematics and computer science, Vol. 4(18), 2630-2664, July 2014.
4. A.Aghili, H.Zeinali, Two dimensional Laplace transform for non- homogenous forth order partial differential equations, Journal of Mathematical Research and Applications (JMRA), Vol.1, 48-54, December 2013.
5. A.Aghili, H.Zeinali, Integral transform method for solving Volterra singular integral equations and non homogenous time Fractional PDEs, Gen.Math.Notes, Vol.14(1), 6-20, January 2013.
6. A.Aghili, H.Zeinali, New identities for Laplace type Integral transforms with applications, International Journal of Mathematical archive (IJMA), Vol.4(10), 1-14, October 2013.
7. A.Aghili, H.Zeinali, Integral transform methods for solving fractional PDEs and evaluation of certain integrals and series. Intern journal of physics and mathematical sciences, Vol.2(4), 27-40, October-December 2012.n Mathematical Sciences
8. A.Aghili, H.Zeinali, Fourier, Laplace integral transform with applications. Mathematical Sciences Research Journal,Vol.17(9), 245-262, December 2013.
9. A.Aghili,B.Salkhordeh Moghaddam, Laplace transform pairs of n- dimensions and a wave equation. Intern. Math. Journal, Vol. 5, 2004,no. 4,377-382.
10. A.Aghili, B.Salkhordeh Moghaddam, Multi-dimensional Laplace transform and systems of partial differential equations. Intern. Math. Journal, Vol. 1, 2006, no. 21-24. 6.
11. A.Aghili, B.Salkhordeh Moghaddam.,Laplace transform pairs of N- dimensions and second order linear differential equations with constant coefficients, Annales Mathematicae et informaticae,35(2008) pp.3-10.
12. G.B.Airy, On the intensity of light in the neighbourhood of a caustic, Trans. Camb. Phil. Soc. 6,1838, pp.379–402.
13. G.B.Airy, Supplement to a paper, On the intensity of light in the neighbourhood of a caustic, Trans. Camb. Phil. Soc. 8,1849, pp.595–599.
14. A.Apelblat, Laplace transforms and their applications, Nova science publishers, Inc, Newyork, 2012.
15. W.W.Bell, Special functions for scientists and engineers, D.Van Nostrand company LTD, Canada, 1968.
16. R.S.Dahiya, J.Saberi – Nadjafi, theorems on n - dimensional Laplace Transforms and their applications. 15th annual conference of applied mathematics, Univ. of Central Oklahoma.Electronic Journal of Diferential Equations, Conference 02,1999, pp.61-74.
17. B.Davies, Integral transforms and their applications, Springer, USA, 2001.
18. P.M.Jordan, P.Puri, Thermal stresses in a spherical shell under three thermo-elastic models, J. Therm. Stresses, 24, 2001, pp47-70.
19. G.M.Phillips, Interpolation and approximation by polynomials, Springer, 2003.

20. I.Podlubny, Fractional Differential Equations, Academic Press, San Diego, CA,1999.

21. J.H.Schwarz, The generalized Stieljes transform and its inverse, Journal of mathematical physics 46, 2005.

22. O.Vallee, M.Soares, Airy Functions And Applications To Physics, Imperial College Press, London, 2010.

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