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Limit behavior of a heat loss problem on oscillating thin layer

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ABSTRACT: The aim of this work is to study the limit behavior of weak solutions of a thermal problem (where the heat loss is considered), of a containing structure, an oscillating thin layer of thickness, periodicity and heat loss parameter depending of ε. The epiconvergence method is considered to find the limit problems with interface conditions.

Key Words: Limit behavior, heat loss, epiconvergence method, limit problems.

Contents

1. Introduction

Consider a body which occupies a bounded three dimensional domain, $\Omega \subset \mathbb{R}^3$, with a Lipschitz boundary $\partial\Omega$, composed on a thin layer B_ε, with oscillating border $\Sigma_\varepsilon^\pm$, of average interface Σ, where $B_\varepsilon = \left\{x \in \Omega : |x_3| \le \varepsilon\varphi(\frac{x'}{\varepsilon})\right\}$ and $x = (x', x_3)$, $\Omega_\varepsilon = \Omega \backslash B_\varepsilon$, ε be a positive small enough parameter and φ be a $]0,1[\times]0,1[$-periodic function (see figure 1). Let consider a thermal problem on the body occupying the domain Ω, where a very hight heat loss on the layer B_ε is considered. the last body mentioned is subject to an outside temperature f, $f : \Omega \to \mathbb{R}$, and cooled at the

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boundary $\partial\Omega$. The problem is modeled with the following equations

$$\begin{cases} -\Delta_{\mathrm{p}}\, u^{\varepsilon} = f \quad \text{in } \Omega_{\varepsilon}, \\ -\Delta_{\mathrm{p}}\, u^{\varepsilon} + \dfrac{1}{\varepsilon^{\alpha}} g(x, u^{\varepsilon}) = f \quad \text{in } B_{\varepsilon}, \\ [u^{\varepsilon}] = 0 \quad \text{on } \Sigma_{\varepsilon}^{\pm}, \\ [|\nabla u^{\varepsilon}|^{p-2} \dfrac{\partial u^{\varepsilon}}{\partial n}] = 0 \quad \text{on } \Sigma_{\varepsilon}^{\pm}, \\ u^{\varepsilon} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}^{\varepsilon}$}$$

the heat loss and its parameter are expressed by $g(x, u^{\varepsilon})$ and $\frac{1}{\varepsilon^{\alpha}}$ where the unknown u^{ε} be the temperature, n be the outward normal to $\partial\Omega$, $p > 1, \alpha \geq 0$, $f \in L^{\infty}(\Omega)$, we denote by $[v]$ the jump of the function v, Δ_p is p-Laplace operator defined on sobolev space $W_0^{1,p}(\Omega)$ to its dual space $W^{-1,p'}(\Omega)$, g is a positive function defined by

$$\begin{array}{rcl} g : \Omega \times \mathbb{R} & \longrightarrow & \mathbb{R}^{+} \\ (x, s) & \longmapsto & g(x, s) \end{array}, \text{ and let us } \begin{array}{rcl} G : \Omega \times \mathbb{R} & \longrightarrow & \mathbb{R}^{+} \\ (x, s) & \longmapsto & \displaystyle\int_0^s g(x, t) dt, \end{array}$$

we remark that $G(x, 0) = 0$ in Ω.

In setting of absence of heat loss, the asymptotic analysis of linear and nonlinear thermal problems with heat conductivity, are widely treated by many authors, for example in a structure with plate case, we can refer the reader to Ait Moussa et al. in [1], Brillard et al. in [4,6] and Sanchez-Palencia et al. in [8], in other hand for a structure with an oscillating layer, Messaho et al. have studied the problem with very hight conductivity in the layer in [7]. In the physics view point, for a thermal problem, the heat loss is present. hence the interest of our choice of this work. Now, the aim of this paper, is to study the problem of existence and limit behaviour of a weak solutions for elliptic problem expressed by ($\mathcal{P}^{\varepsilon}$). The limit behaviour consist, via the epiconvergence method (see for instance [2,5]), to find a minimization limit problem linked to the minimization problem linked to ($\mathcal{P}^{\varepsilon}$).

Where ε being a positive parameter intended to tend towards zero and φ_{ε} is a bounded real function, $]0, \varepsilon[^2$-periodic.

This paper is organized in the following way. In section 2, we give some notations and assumptions on the function g, functional spaces for our study and results for the epiconvergence notion that will be used throughout this paper. We study problem (3.1) and its limit asymptotic in section 3.

2. Preliminaries and assumptions

2.1. Notations and assumptions

In this subsection, we will give the notations and some assumptions on g, that will be used throughout this paper:

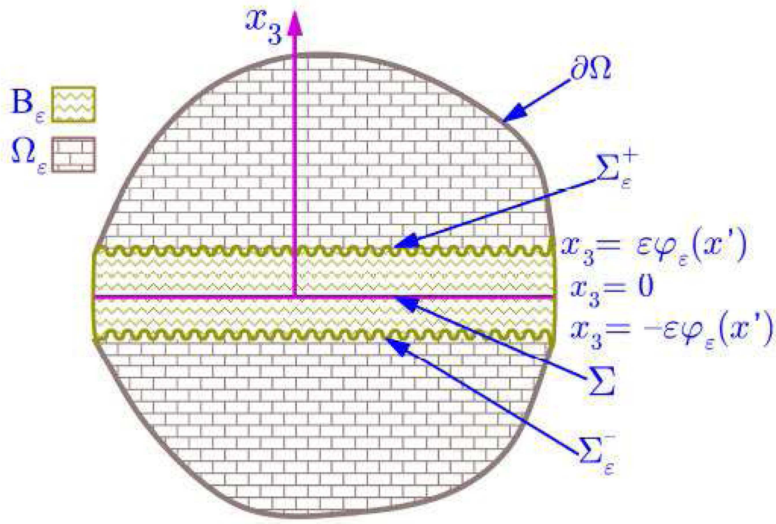

Figure 1: Domain Ω.

$x = (x', x_3)$ where $x' = (x_1, x_2)$, $Y =]0,1[\times]0,1[$, $\varphi\colon \mathbb{R}^2 \to [a_1, a_2]$ where φ is Y-periodic and $a_2 \geq a_1 > 0$, $\varphi_\varepsilon(x') = \varphi(\frac{x'}{\varepsilon})$, $m(\varphi) = \int_Y \varphi(x')dx'$,	$\eta(t) = \lim_{\varepsilon\to 0} \varepsilon^{1-t}$ with $t \geq 0$. C will denote any constant with respect to ε.

Now, we assume that g satisfying to the following conditions

- $g(x, .)$ is a convex function for $x \in \Omega$, (H1)
- $\exists\gamma > 0, \forall t \in \mathbb{R}\ \ |t|^{p-1} \leq g(x,t) \leq \gamma.(1 + |t|^{p-1}),$ for $x \in \Omega$, (H2)
- $\begin{cases} \text{there exists a continuous } \omega : \mathbb{R}^+ \to \mathbb{R}^+ \\ \text{which is increasing and vanishing at the origin, such that} \\ \forall t \in \mathbb{R},\ \forall x, y \in \Omega,\ |g(x,t) - g(y,t)| \leq \omega(|x-y|)(1 + |t|^{p-1}). \end{cases}$ (H3)

2.2. Functional setting

First, let us introduce the Banach space $\mathcal{W} = W_0^{1,p}(\Omega)$. Let

$$\mathcal{W}_0(\Sigma) = \Big\{u \in \mathcal{W} : u_{|_\Sigma} = 0\Big\}.$$

We can show easily that $\mathcal{W}_0(\Sigma)$ is a Banach space endowed with the norm of the sobolev space $W_0^{1,p}(\Omega)$.

$$\mathbb{G}^\alpha = \begin{cases} \mathcal{W} & \text{if } \alpha \leq 1, \\ \mathcal{W}_0 & \text{if } \alpha > 1. \end{cases}$$

$$\mathbb{D}^\alpha = \begin{cases} \mathcal{D}(\Omega) & \text{if } \alpha \leq 1, \\ \left\{ u \in \mathcal{D}(\Omega) : u_{|\Sigma} = 0 \right\} & \text{if } \alpha > 1. \end{cases}$$

It is known that $\overline{\mathbb{D}^\alpha} = \mathbb{G}^\alpha$.

2.3. Epiconvergence notion

In this subsection, we will give a notion of operator's sequence convergence, named epiconvergence, who is a special case of the $\Gamma-$convergence introduced by De Giorgi (1979) [5]. It is well suited to the asymptotic analysis of sequences of minimization problems.

Definition 2.1 ([2, Definition 1.9])**.** *Let $(\mathbb{X}, \tau)$ be a metric space and $(F^\varepsilon)_\varepsilon$ and F be functionals defined on $\mathbb{X}$ and with value in $\mathbb{R} \cup \{+\infty\}$. F^ε epi-converges to F in $(\mathbb{X}, \tau)$, noted $\tau - \mathop{\mathbf{epilim}}_{\varepsilon \to 0} F^\varepsilon = F$, if the following assertions are satisfied*

- *For all $x \in \mathbb{X}$, there exists x_ε^0, $x_\varepsilon^0 \xrightarrow{\tau} x$ such that $\limsup_{\varepsilon \to 0} F^\varepsilon(x_\varepsilon^0) \leq F(x)$.*
- *For all $x \in \mathbb{X}$ and all x_ε with $x_\varepsilon \xrightarrow{\tau} x$, $\liminf_{\varepsilon \to 0} F^\varepsilon(x_\varepsilon) \geq F(x)$.*

Note the following stability result of the epi-convergence.

Proposition 2.1 ([2, p. 40])**.** *Suppose that F^ε epi-converges to F in $(\mathbb{X}, \tau)$ and that $\Phi \colon \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$, is τ-continuous. Then $F^\varepsilon + \Phi$ epi-converges to $F + \Phi$ in $(\mathbb{X}, \tau)$.*

Theorem 2.2 ([2, theorem 1.10])**.** *Suppose that*

1. *F^ε admits a minimizer on $\mathbb{X}$,*
2. *The sequence $(\overline{u}^\varepsilon)$ is τ-relatively compact,*
3. *The sequence F^ε epi-converges to F in this topology τ.*

Then every cluster point $\overline{u}$ of the sequence $(\overline{u}^\varepsilon)$ minimizes F on $\mathbb{X}$ and

$$\lim_{\varepsilon' \to 0} F^{\varepsilon'}(\overline{u}^{\varepsilon'}) = F(\overline{u}),$$

if $(\overline{u}^{\varepsilon'})_{\varepsilon'}$ denotes the subsequence of $(\overline{u}^\varepsilon)_\varepsilon$ which converges to $\overline{u}$.

Lemma 2.1 ([7, Annex]). *Let $\varphi \in L^{\infty}(\Sigma)$, a Y-periodic, $Y =]0,1[\times]0,1[$. Let*

$$\varphi_\varepsilon(x) = \varphi(\frac{x}{\varepsilon}), \text{ for a small enough } \varepsilon > 0.$$

So that

$$\begin{aligned} \varphi_\varepsilon &\to m(\varphi) \quad \text{in } L^s(\Sigma) \text{ for } 1 \le s < \infty, \\ \varphi_\varepsilon &\rightharpoonup^* m(\varphi) \quad \text{in } L^\infty(\Sigma). \end{aligned}$$

3. Main results

3.1. Study of the problem $(\mathcal{P}^\varepsilon)$

Note that the problem $(\mathcal{P}^\varepsilon)$ is equivalent to the minimization problem

$$\inf_{v \in \mathcal{W}} \Big\{ \frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} G(x,v) - \int_\Omega f.v \Big\}. \tag{3.1}$$

So, the problems $(\mathcal{P}^\varepsilon)$ and (3.1) are the same solutions in $\mathcal{W}$. Next, we will interest to the study of the problem minimization (3.1) and the existence of its weak solutions is given in the following proposition

Proposition 3.1. *Under the assumptions (H1), (H2) and for every $f \in L^\infty(\Omega)$, the problem (3.1) admits an unique solution u^ε in $\mathcal{W}$.*

The proof of this proposition is based on classical convexity arguments see for example [3].

In the sequel, our question is to ask how establishing the limit behaviour of the solution u^ε of problem (3.1). In order to establish this behavoir, we use the epiconvergence method (see definition 2.1), to starting this, in one hand, we need to determinate the space and its topology in what applying the epiconvergence method. So, let us introduce the following operator sequence $(m^\varepsilon)_{\varepsilon>0}$, like that in [7], defined on $\mathcal{W}$ valued in $L^p(\Sigma)$ by

$$m^\varepsilon(u)(x') = \frac{1}{2\varepsilon\varphi_\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} u(x',x_3)dx_3, \ \forall u \in \mathcal{W}. \tag{3.2}$$

We give in lemma, as follows, the estimations on u^ε and $m^\varepsilon(u^\varepsilon)$

Lemma 3.1. *For every $f \in L^\infty(\Omega)$, the families $(u^\varepsilon)_{\varepsilon>0}$ and $(m^\varepsilon(u^\varepsilon))_{\varepsilon>0}$ satisfy to*

$$\frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} |u^\varepsilon|^p \le C, \tag{3.3}$$

$$\frac{1}{\varepsilon^{\alpha-1}}\int_\Sigma |m^\varepsilon(u^\varepsilon)|^p \, dx' \le C. \tag{3.4}$$

Moreover u^ε is bounded in $\mathcal{W}$.

Proof: Since u^ε is the solution of the problem (3.1), we have

$$\int_\Omega |\nabla u^\varepsilon|^{p-2}\,\nabla u^\varepsilon \nabla v + \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} g(x,u^\varepsilon)v = \int_\Omega fv, \quad \forall v \in \mathcal{W}.$$

In particular, by taking $v = u^\varepsilon$, we obtain

$$\|\nabla u^\varepsilon\|^p_{L^p(\Omega)} + \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} g(x,u^\varepsilon)u^\varepsilon = \int_\Omega fu^\varepsilon.$$

According to the inequalities of Hölder and Young, one has

$$\begin{aligned}\|\nabla u^\varepsilon\|^p_{L^p(\Omega)} + \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} g(x,u^\varepsilon)u^\varepsilon &\le C\,\|\nabla u^\varepsilon\|_{L^p(\Omega)} \\ &\le C + \frac{1}{p}\,\|\nabla u^\varepsilon\|^p_{L^p(\Omega)}.\end{aligned}$$

So that

$$\|\nabla u^\varepsilon\|^p_{L^p(\Omega)} + \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} g(x,u^\varepsilon)u^\varepsilon \le C. \tag{3.5}$$

From the assumption (H2), we have

$$\|u^\varepsilon\|^p_{L^p(B_\varepsilon)} \le \int_{B_\varepsilon} g(x,u^\varepsilon)u^\varepsilon,$$

so, (3.5) becomes

$$\|\nabla u^\varepsilon\|^p_{L^p(\Omega)} + \frac{1}{\varepsilon^\alpha}\,\|u^\varepsilon\|^p_{L^p(B_\varepsilon)} \le C. \tag{3.6}$$

Therefore, we will have the assertion (3.3) and we deduce that (u^ε) is bounded in $\mathcal{W}$.

We have

$$\int_\Sigma |m^\varepsilon(u^\varepsilon)|^p\,dx' = \int_\Sigma (\frac{1}{2\varepsilon\varphi_\varepsilon})^p \left|\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} u^\varepsilon dx_3\right|^p dx', \tag{3.7}$$

since $0 < a_1 \le \varphi_\varepsilon \le a_2$, and according to the inequality of Hölder, (3.7) becomes

$$\begin{aligned}\int_\Sigma |m^\varepsilon(u^\varepsilon)|^p\,dx' &\le \int_\Sigma \frac{1}{2\varepsilon\varphi_\varepsilon}\Big(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |u^\varepsilon|^p\,dx_3\Big)dx' \\ &\le \frac{1}{2\varepsilon a_1}\int_\Sigma \Big(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |u^\varepsilon|^p\,dx_3\Big)dx' \\ &\le \frac{1}{2\varepsilon a_1}\int_{B_\varepsilon} |u^\varepsilon|^p\,dx.\end{aligned} \tag{3.8}$$

According to (3.3), we obtain

$$\int_{\Sigma} |m^{\varepsilon}(u^{\varepsilon})|^{p}\, dx' \leq C\varepsilon^{\alpha-1}.$$

□

Now, in the following proposition, we will discuss, according to the reals values of α, to obtain some information of the solution $(u^{\varepsilon})_{\varepsilon}$ of problem (3.1) when ε close to zero.

Proposition 3.2. *The solution of the problem (3.1), $(u^{\varepsilon})_{\varepsilon}$, possess a subsequence denoted also $(u^{\varepsilon})_{\varepsilon}$ weakly convergent toward an element u^{*} in $\mathcal{W}$ satisfying*

$$u^{*}{}_{|\Sigma} \in L^{p}(\Sigma), \tag{3.9}$$

$$\forall \alpha > 1,\; u^{*}{}_{|\Sigma} = 0. \tag{3.10}$$

Proof: According to lemma 3.1, the sequence u^{ε} is bounded in $\mathcal{W}$, it follows that there exists an element $u^{*} \in \mathcal{W}$ and a subsequence of u^{ε}, still denoted by u^{ε} such that $u^{\varepsilon} \rightharpoonup u^{*}$ in $\mathcal{W}$. So

$$u^{\varepsilon}{}_{|\Sigma} \rightharpoonup u^{*}{}_{|\Sigma} \text{ in } L^{p}(\Sigma),$$

thanks to [7, Lemma 4.2], we have

$$\begin{aligned}\int_{\Sigma} \left|m^{\varepsilon}u^{\varepsilon} - u^{\varepsilon}{}_{|\Sigma}\right|^{p} &\leq C\varepsilon^{p-1}\int_{B_{\varepsilon}} |\nabla u^{\varepsilon}|^{p} \\ &\leq C\varepsilon^{p-1}\int_{\Omega} |\nabla u^{\varepsilon}|^{p},\end{aligned}$$

so

$$\int_{\Sigma} \left|m^{\varepsilon}u^{\varepsilon} - u^{\varepsilon}{}_{|\Sigma}\right|^{p} \leq C\varepsilon^{p-1}. \tag{3.11}$$

For $\alpha = 1$, according to the evaluation (3.4), the sequence $m^{\varepsilon}(u^{\varepsilon})$ is bounded in $L^{p}(\Sigma)$, so possess a subsequence, still denoted by $m^{\varepsilon}(u^{\varepsilon})$ weakly convergent in $L^{p}(\Sigma)$, as $u^{\varepsilon}_{|\Sigma} \rightharpoonup u^{*}{}_{|\Sigma}$ in $L^{p}(\Sigma)$ and from (3.11), we conclude that $m^{\varepsilon}(u^{\varepsilon}) \rightharpoonup u^{*}{}_{|\Sigma}$ in $L^{p}(\Sigma)$.

For $\alpha > 1$, from (3.4) we have $m^{\varepsilon}(u^{\varepsilon}) \rightharpoonup 0$ in $L^{p}(\Sigma)$. We show easily that, with the same way like the case $\alpha = 1$, that $u^{*}{}_{|\Sigma} = 0$. Hence the proof of proposition 3.2 is complete. □

The limit behavior of the problem (3.1), will be derived with the epiconvergence method, (see definition 2.1).

3.2. Limit behavior of the problem (3.1)

In this subsection, we will interest, according to the values of α, to find the limit problem of the problem (3.1). Consider the following energy functional given by

$$\mathcal{F}^{\varepsilon}(u) = \frac{1}{p}\int_{\Omega} |\nabla u|^p + \frac{1}{\varepsilon^{\alpha}}\int_{B_{\varepsilon}} G(x,u), \quad \forall u \in \mathcal{W}, \tag{3.1}$$

$$\Phi(u) = -\int_{\Omega} fu, \quad \forall u \in \mathcal{W}. \tag{3.2}$$

We denote by τ_f the weak topology on $\mathcal{W}$.

Theorem 3.1. *According to the values of α, there exists a functional F^{α} defined on $\mathcal{W}$ with value in $\mathbb{R} \cup \{+\infty\}$ such that $\tau_f - \underset{\varepsilon\to 0}{\textbf{epilim}}\ \mathcal{F}^{\varepsilon} = \mathcal{F}_{\alpha}$ in $\mathcal{W}$, where the functional $\mathcal{F}_{\alpha}$ is given by*

1. *If $0 \le \alpha < 1$:*
$$\mathcal{F}_{\alpha}(u) = \frac{1}{p}\int_{\Omega} |\nabla u|^p, \quad \forall u \in \mathcal{W}.$$
2. *If $\alpha = 1$:*
$$\mathcal{F}_{\alpha}(u) = \frac{1}{p}\int_{\Omega} |\nabla u|^p + 2m(\varphi)\int_{\Sigma} G(x', u_{|\Sigma}), \quad \forall u \in \mathcal{W}.$$
Where $G(x', u_{|\Sigma}) = G((x',0), u_{|\Sigma})$.
3. *If $\alpha > 1$:*
$$\mathcal{F}_{\alpha}(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_{\Omega} |\nabla u|^p & \text{if } u \in \mathbb{G}^{\alpha}, \\ +\infty & \text{if } u \in \mathcal{W}\setminus\mathbb{G}^{\alpha}. \end{cases}$$

First, to start the proof of this theorem, we need the following lemma

Lemma 3.2. *Let $u, u^{\varepsilon} \in \mathcal{W}$ such that*

$$u^{\varepsilon} \rightharpoonup u \text{ weakly in } \mathcal{W}, \text{when } \varepsilon \text{ close to } 0. \tag{3.3}$$

Under the assumptions (H1) and (H2), we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{B_{\varepsilon}} g(x, u^{\varepsilon}{}_{|\Sigma})(u^{\varepsilon} - u^{\varepsilon}{}_{|\Sigma}) = 0$$

Proof: **[Proof of Lemma 3.2.]** It is clear, from (H2), that there exists a constant $C > 0$ such that

$$\int_{B_{\varepsilon}} \left|g(x, u^{\varepsilon}{}_{|\Sigma})\right|^{p'} dx \le C\varepsilon\Big(1 + \int_{\Sigma} \left|u^{\varepsilon}{}_{|\Sigma}\right|^{p} dx\Big).$$

Since ${u^\varepsilon}_{|\Sigma} \rightharpoonup u_{|\Sigma}$ in $L^p(\Sigma)$, consequently $({u^\varepsilon}_{|\Sigma})$ is a bounded sequence in $L^p(\Sigma)$, and we have

$$\int_{B_\varepsilon} \left|g(x, {u^\varepsilon}_{|\Sigma})\right|^{p'} dx \leq C\varepsilon. \tag{3.4}$$

We have

$$\begin{aligned}
\int_{B_\varepsilon} \left|u^\varepsilon - {u^\varepsilon}_{|\Sigma}\right|^p dx &= \int_{B_\varepsilon} \left|\int_0^{x_3} \frac{\partial u^\varepsilon}{\partial x_3} dx_3\right|^p dx \\
&\leq \int_{B_\varepsilon} \left|\int_{-\varepsilon\varphi_\varepsilon(x')}^{\varepsilon\varphi_\varepsilon(x')} \frac{\partial u^\varepsilon}{\partial x_3} dx_3\right|^p dx \\
&\leq \int_\Sigma \int_{-\varepsilon\varphi_\varepsilon(x')}^{\varepsilon\varphi_\varepsilon(x')} (2\varepsilon\varphi_\varepsilon(x'))^{p-1} \left(\int_{-\varepsilon\varphi_\varepsilon(x')}^{\varepsilon\varphi_\varepsilon(x')} \left|\frac{\partial u^\varepsilon}{\partial x_3}\right|^p dx_3 \right) dx_3 dx' \\
&\leq C\varepsilon^p \int_{B_\varepsilon} |\nabla u^\varepsilon|^p dx \\
&\leq C\varepsilon^p \int_\Omega |\nabla u^\varepsilon|^p dx,
\end{aligned}$$

since (u^ε) is a bounded sequence in $\mathcal{W}$, thus

$$\int_{B_\varepsilon} \left|u^\varepsilon - {u^\varepsilon}_{|\Sigma}\right|^p \leq C\varepsilon^p dx. \tag{3.5}$$

From the Hölder inequality, we have

$$\frac{1}{\varepsilon} \int_{B_\varepsilon} g(x, {u^\varepsilon}_{|\Sigma})(u^\varepsilon - {u^\varepsilon}_{|\Sigma}) \leq \frac{1}{\varepsilon} \left\|g(x, {u^\varepsilon}_{|\Sigma})\right\|_{L^{p'}(B_\varepsilon)} \left\|u^\varepsilon - {u^\varepsilon}_{|\Sigma}\right\|_{L^{p'}(B_\varepsilon)}, \tag{3.6}$$

according to (3.4) and (3.5), (3.8) becomes

$$\frac{1}{\varepsilon} \int_{B_\varepsilon} g(x, {u^\varepsilon}_{|\Sigma})(u^\varepsilon - {u^\varepsilon}_{|\Sigma}) \leq C\varepsilon^{\frac{1}{p'}}, \tag{3.7}$$

so we obtain

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{B_\varepsilon} g(x, {u^\varepsilon}_{|\Sigma})(u^\varepsilon - {u^\varepsilon}_{|\Sigma}) = 0. \tag{3.8}$$

□

Proof: [**Proof of theorem 3.1**] (a) we are going to determine the upper **epilimit**: Let $u \in \mathbb{G}^\alpha \subset \mathcal{W}$, by the density argument, there exists a sequence (u^n) in $\mathbb{D}^\alpha$ such that

$$\begin{aligned}
&u^n \to u \text{ in } \mathbb{G}^\alpha, \text{ when } n \to +\infty, \\
&\text{so that } u^n \to u \text{ in } \mathcal{W}.
\end{aligned}$$

Let θ be a smooth function satisfying

$$\theta(s) = 1 \text{ if } |s| \leq 1,\ \theta(s) = 0 \text{ if } |s| \geq 2 \ \text{ and } \ \left|\theta'(s)\right| \leq 2 \ \forall s \in \mathbb{R},$$

and for every $x \in \Omega$, we set

$$\theta_\varepsilon(x) = \theta(\frac{x_3}{\varepsilon\varphi_\varepsilon});$$

we define

$$u^{\varepsilon,n} = \theta_\varepsilon(x) u^n{}_{|\Sigma} + (1 - \theta_\varepsilon(x)) u^n,$$

it is easy to see that $u^{\varepsilon,n} \in \mathbb{G}^\alpha$ and $u^{\varepsilon,n} \to u^n$ in $\mathbb{G}^\alpha$, when $\varepsilon \to 0$. Since

$$\mathcal{F}^\varepsilon(u^{\varepsilon,n}) = \frac{1}{p}\int_\Omega |\nabla u^{\varepsilon,n}|^p\, dx + \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} G(x, u^{\varepsilon,n}) dx,$$

so that

$$\begin{aligned} \mathcal{F}^\varepsilon(u^{\varepsilon,n}) &= \frac{1}{p}\int_\Omega |\nabla u^{\varepsilon,n}|^p\, dx + \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} G(x, u^{\varepsilon,n}) dx \\ &= \frac{1}{p}\int_{|x_3|>2\varepsilon\varphi_\varepsilon} |\nabla u^n|^p\, dx + \frac{1}{p}\int_{\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon} |\nabla u^{\varepsilon,n}|^p\, dx + \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} G(x, u^{\varepsilon,n}) dx. \end{aligned} \tag{3.9}$$

Since φ_ε is bounded, we verify easily that

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{p}\int_{\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon} |\nabla u^{\varepsilon,n}|^p\, dx\right\} = 0. \tag{3.10}$$

We have

$$\int_{B_\varepsilon} G(x, u^{\varepsilon,n}) dx = \int_{B_\varepsilon} G(x, u^n{}_{|\Sigma}) dx$$

according to the assumption (H3), there exists a constance $C > 0$ such that

$$\int_{B_\varepsilon} G(x, u^n{}_{|\Sigma}) dx \leq \int_{B_\varepsilon} G((x', 0), u^n{}_{|\Sigma}) dx + C\int_{B_\varepsilon} \omega(\varepsilon\varphi_\varepsilon(x'))(1 + \left|u^n{}_{|\Sigma}\right|^p) dx,$$

since $\varphi_\varepsilon(.) \leq a_2$ on Σ, $\varepsilon\varphi_\varepsilon$ converges to 0, ω is continuous and increasing (see the assumption (H3)), we have

$$\begin{aligned} \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} G(x, u^n{}_{|\Sigma}) dx &\leq \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} G((x', 0), u^n{}_{|\Sigma}) dx + \frac{C\omega(a_2\varepsilon)}{\varepsilon^\alpha}\int_{B_\varepsilon} (1 + \left|u^n{}_{|\Sigma}\right|^p) dx \\ &\leq 2\varepsilon^{1-\alpha}\int_\Sigma \varphi_\varepsilon G((x', 0), u^n{}_{|\Sigma}) dx' \\ &\quad + 2C\omega(a_2\varepsilon)\varepsilon^{1-\alpha}\int_\Sigma \varphi_\varepsilon(1 + \left|u^n{}_{|\Sigma}\right|^p) dx', \end{aligned}$$

1. If $\alpha \leq 1$: Since when $\varepsilon \to 0$, $\varphi_{\varepsilon} \rightharpoonup^{*} m(\varphi)$ in $L^{\infty}(\Sigma)$, $\omega(a_2\varepsilon) \to 0$ and $\varepsilon^{1-\alpha} \to \eta(\alpha)$, it follows that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\alpha}} \int_{B_{\varepsilon}} G(x, u^{n}{}_{|\Sigma})dx \leq 2m(\varphi)\eta(\alpha) \int_{\Sigma} G((x',0), u^{n}{}_{|\Sigma})dx',$$

so by passage to the limit ($\delta \to 0$), we obtain

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\alpha}} \int_{B_{\varepsilon}} G(x, u^{n}{}_{|\Sigma})dx \leq 2m(\varphi)\eta(\alpha) \int_{\Sigma} G((x',0), u^{n}{}_{|\Sigma})dx',$$

by passage to the upper limit, we have

$$\begin{aligned}\limsup_{\varepsilon\to 0} \mathcal{F}^{\varepsilon}(u^{\varepsilon,n}) &= \limsup_{\varepsilon\to 0} \Big(\frac{1}{p}\int_{|x_3|>2\varepsilon\varphi_{\varepsilon}} |\nabla u^{n}|^{p}\,dx + \frac{1}{\varepsilon^{\alpha}}\int_{B_{\varepsilon}} G(x, u^{n}{}_{|\Sigma})dx\Big) \\ &= \frac{1}{p}\int_{\Omega} |\nabla u^{n}|^{p}\,dx + 2m(\varphi)\eta(\alpha)\int_{\Sigma} G((x',0), u^{n}{}_{|\Sigma})dx'.\end{aligned}$$

Recall that, if $\alpha < 1$ that imply $\eta(\alpha) = 0$, consequently

$$\begin{aligned}\limsup_{\varepsilon\to 0} \mathcal{F}^{\varepsilon}(u^{\varepsilon,n}) &= \limsup_{\varepsilon\to 0} \Big(\frac{1}{p}\int_{|x_3|>2\varepsilon\varphi_{\varepsilon}} |\nabla u^{n}|^{p}\,dx + \frac{1}{\varepsilon^{\alpha}}\int_{B_{\varepsilon}} G(x, u^{n}{}_{|\Sigma})dx\Big) \\ &= \frac{1}{p}\int_{\Omega} |\nabla u^{n}|^{p}\,dx.\end{aligned}$$

2. If $\alpha > 1$:
   By passage to the upper limit, with fact $u^{n}{}_{|\Sigma} = 0$, we have

$$\begin{aligned}\limsup_{\varepsilon\to 0} \mathcal{F}^{\varepsilon}(u^{\varepsilon,n}) &= \limsup_{\varepsilon\to 0} \Big(\frac{1}{p}\int_{|x_3|>2\varepsilon\varphi_{\varepsilon}} |\nabla u^{n}|^{p}\,dx\Big) \\ &= \frac{1}{p}\int_{\Omega} |\nabla u^{n}|^{p}\,dx.\end{aligned}$$

Since $u^{n} \to u$ in $\mathbb{G}^{\alpha}$, when $n \to +\infty$. According to the classical result, diagonalization's lemma [2, Lemma 1.15], there exists a function $n(\varepsilon) : \mathbb{R}^{+} \to \mathbb{N}$ increasing to $+\infty$ when $\varepsilon \to 0$, such that $u^{\varepsilon,n(\varepsilon)} \to u$ in $\mathbb{G}^{\alpha}$ when $\varepsilon \to 0$. when n approaches $+\infty$, we will have

1. If $\alpha \neq 1$:

$$\begin{aligned}\limsup_{\varepsilon\to 0} \mathcal{F}^{\varepsilon}(u^{\varepsilon,n(\varepsilon)}) &\leq \limsup_{n\to+\infty}\limsup_{\varepsilon\to 0} \mathcal{F}^{\varepsilon}(u^{\varepsilon,n}) \\ &\leq \frac{1}{p}\int_{\Omega} |\nabla u|^{p}\,dx.\end{aligned}$$

2. If $\alpha = 1$:

$$\begin{aligned}\limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^{\varepsilon,n(\varepsilon)}) &\le \limsup_{n\to+\infty}\limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^{\varepsilon,n}) \\ &\le \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + 2m(\varphi)\eta(\alpha)\int_\Sigma G((x',0),u_{|\Sigma})dx' \\ &\le \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + 2m(\varphi)\int_\Sigma G(x',u_{|\Sigma})dx'.\end{aligned}$$

If $\alpha > 1$, let us $u \in \mathcal{W}\setminus\mathbb{G}^\alpha$, it is clear that, for every $u^\varepsilon \in \mathcal{W}$, $u^\varepsilon \rightharpoonup u$ in $\mathcal{W}$, we have

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) \le +\infty.$$

(b) We are going to determine the lower epi-limit. Let $u \in \mathbb{G}^\alpha$ and (u^ε) be a sequence in $\mathcal{W}$ such that $u^\varepsilon \rightharpoonup u$ in $\mathcal{W}$, so that

$$\nabla u^\varepsilon \rightharpoonup \nabla u \quad \text{in } L^p(\Omega)^3. \tag{3.11}$$

1. If $\alpha \neq 1$: Since

$$\mathcal{F}^\varepsilon(u^\varepsilon) \ge \frac{1}{p}\int_\Omega |\nabla u^\varepsilon|^p\,dx.$$

According to (3.11) and by passage to the lower limit, we obtain

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) \ge \frac{1}{p}\int_\Omega |\nabla u|^p\,dx.$$

2. If $\alpha = 1$: If $\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) = +\infty$, there is nothing to prove, because

$$\frac{1}{p}\int_\Omega |\nabla u|^p\,dx + 2m(\varphi)\int_\Sigma G(x',u_{|\Sigma})dx' \le +\infty.$$

Otherwise, $\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) < +\infty$, there exists a subsequence of $\mathcal{F}^\varepsilon(u^\varepsilon)$ still denoted by $\mathcal{F}^\varepsilon(u^\varepsilon)$ and a constant $C > 0$, such that $\mathcal{F}^\varepsilon(u^\varepsilon) \le C$, which implies that

$$\frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} G(x,u^\varepsilon)dx \le C, \tag{3.12}$$

and according to (H2), we have

$$\frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} |u^\varepsilon|^p\,dx \le C. \tag{3.13}$$

Recall that

$$\mathcal{F}^\varepsilon(u^\varepsilon) = \frac{1}{p}\int_\Omega |\nabla u^\varepsilon|^p + \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} G(x,u^\varepsilon)$$

From the assumption (H1), $G(x,.)$ is a convex function for any $x \in \Omega$, so using the sub-differential inequality of

$$v \to \int_{B_\varepsilon} G(x,v)dx,$$

we have

$$\int_{B_\varepsilon} G(x,u^\varepsilon)dx \geq \int_{B_\varepsilon} G(x,{u^\varepsilon}_{|\Sigma})dx + \int_{B_\varepsilon} g(x,{u^\varepsilon}_{|\Sigma})(u^\varepsilon - {u^\varepsilon}_{|\Sigma})dx$$

thus

$$\mathcal{F}^\varepsilon(u^\varepsilon) \geq \frac{1}{p}\int_\Omega |\nabla u^\varepsilon|^p + \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} G(x,{u^\varepsilon}_{|\Sigma})dx + \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} g(x,{u^\varepsilon}_{|\Sigma})(u^\varepsilon - {u^\varepsilon}_{|\Sigma})dx$$

thanks to lemma 3.2 and $\alpha = 1$, we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} g(x,{u^\varepsilon}_{|\Sigma})(u^\varepsilon - {u^\varepsilon}_{|\Sigma})dx = 0 \tag{3.14}$$

so there exists a constant $C > 0$ such that

$$\begin{aligned}\int_{B_\varepsilon} G(x,{u^\varepsilon}_{|\Sigma})dx &\leq \int_{B_\varepsilon} G(x,u^\varepsilon)dx - \int_{B_\varepsilon} g(x,{u^\varepsilon}_{|\Sigma})(u^\varepsilon - {u^\varepsilon}_{|\Sigma})dx \\ &\leq \rho_\varepsilon + \int_{B_\varepsilon} G(x,u^\varepsilon)dx\end{aligned}$$

where $\rho_\varepsilon = -\int_{B_\varepsilon} g(x,{u^\varepsilon}_{|\Sigma})(u^\varepsilon - {u^\varepsilon}_{|\Sigma})dx$, according to (3.12), we obtain

$$\int_{B_\varepsilon} G(x,{u^\varepsilon}_{|\Sigma})dx \leq \rho_\varepsilon + C\varepsilon^\alpha.$$

From (3.4), let us define the following sequence

$$v_\varepsilon = m^\varepsilon(G(x,{u^\varepsilon}_{|\Sigma}))$$

we can show easily that

$$\int_\Sigma v_\varepsilon dx' \leq \frac{C}{\varepsilon}\int_{B_\varepsilon} G(x,{u^\varepsilon}_{|\Sigma})dx$$

so we have

$$\int_\Sigma |v_\varepsilon|\, dx' \leq C(\frac{\rho_\varepsilon}{\varepsilon} + \varepsilon^{\alpha-1}).$$

Let us show that

$$\int_\Sigma \left|v_\varepsilon - G((x',0),{u^\varepsilon}_{|\Sigma})\right| dx' = 0.$$

To simplify the writing, we put $G(x', u^{\varepsilon}{}_{|\Sigma}) = G((x', 0), u^{\varepsilon}{}_{|\Sigma})$.
We have

$$\begin{aligned}\int_{\Sigma} \left|v_{\varepsilon} - G(x', u^{\varepsilon}{}_{|\Sigma})\right| dx' &= \int_{\Sigma} \left|v_{\varepsilon} - G(x', u^{\varepsilon}{}_{|\Sigma})\right| dx' \\ &= \int_{\Sigma} \left| \frac{1}{2\varepsilon\varphi_{\varepsilon}(x')} \int_{-\varepsilon\varphi_{\varepsilon}(x')}^{\varepsilon\varphi_{\varepsilon}(x')} \Big(G(x, u^{\varepsilon}{}_{|\Sigma}) - G(x', u^{\varepsilon}{}_{|\Sigma}) \Big) \right| dx' \\ &\leq \frac{C}{\varepsilon} \int_{\Sigma} \int_{-\varepsilon\varphi_{\varepsilon}(x')}^{\varepsilon\varphi_{\varepsilon}(x')} \left| G(x, u^{\varepsilon}{}_{|\Sigma}) - G(x', u^{\varepsilon}{}_{|\Sigma}) \right| dx'\end{aligned}$$

according to (H3), we obtain

$$\begin{aligned}\int_{\Sigma} \left|v_{\varepsilon} - G(x', u^{\varepsilon}{}_{|\Sigma})\right| dx' &\leq \frac{C}{\varepsilon} \int_{\Sigma} \int_{-\varepsilon\varphi_{\varepsilon}(x')}^{\varepsilon\varphi_{\varepsilon}(x')} w(\varepsilon\varphi_{\varepsilon}(x'))(1 + \left|u^{\varepsilon}{}_{|\Sigma}\right|^{p}) dx' \\ &\leq Cw(a_2\varepsilon) \int_{\Sigma} (1 + \left|u^{\varepsilon}{}_{|\Sigma}\right|^{p}) dx' \\ &\leq Cw(a_2\varepsilon)\end{aligned}$$

from (H3) and by passing to limit when ε close to 0, hence

$$\lim_{\varepsilon \to 0} \int_{\Sigma} \left|v_{\varepsilon} - G(x', u^{\varepsilon}{}_{|\Sigma})\right| dx' = 0$$

let us $x' \mapsto \delta_{\varepsilon}(x')$ close to 0 in $L^1(\Sigma)$ such that

$$v_{\varepsilon} = G(x', u^{\varepsilon}{}_{|\Sigma}) + \delta_{\varepsilon}$$

we have

$$\begin{aligned}\mathcal{F}^{\varepsilon}(u^{\varepsilon}) &\geq \frac{1}{p} \int_{\Omega} |\nabla u^{\varepsilon}|^{p} + \frac{1}{\varepsilon^{\alpha}} \int_{B_{\varepsilon}} G(x, u^{\varepsilon}{}_{|\Sigma}) dx + \frac{1}{\varepsilon^{\alpha}} \int_{B_{\varepsilon}} g(x, u^{\varepsilon}{}_{|\Sigma})(u^{\varepsilon} - u^{\varepsilon}{}_{|\Sigma}) dx \\ &\geq \frac{1}{p} \int_{\Omega} |\nabla u^{\varepsilon}|^{p} + 2\varepsilon^{1-\alpha} \int_{\Sigma} \varphi_{\varepsilon}(x') v_{\varepsilon} dx' + \frac{1}{\varepsilon^{\alpha}} \int_{B_{\varepsilon}} g(x, u^{\varepsilon}{}_{|\Sigma})(u^{\varepsilon} - u^{\varepsilon}{}_{|\Sigma}) dx \\ &\geq \frac{1}{p} \int_{\Omega} |\nabla u^{\varepsilon}|^{p} + 2\varepsilon^{1-\alpha} \int_{\Sigma} \varphi_{\varepsilon}(x') G(x', u^{\varepsilon}{}_{|\Sigma}) dx' + \int_{\Sigma} \varphi_{\varepsilon}(x') \delta_{\varepsilon} dx' \\ &+ \frac{1}{\varepsilon^{\alpha}} \int_{B_{\varepsilon}} g(x, u^{\varepsilon}{}_{|\Sigma})(u^{\varepsilon} - u^{\varepsilon}{}_{|\Sigma}) dx.\end{aligned}$$

From the sub-gradient inequality of $G(x', .)$, we obtain

$$\begin{aligned}\mathcal{F}^{\varepsilon}(u^{\varepsilon}) \geq \frac{1}{p} \int_{\Omega} |\nabla u^{\varepsilon}|^{p} &+ 2\varepsilon^{1-\alpha} \int_{\Sigma} \varphi_{\varepsilon}(x') G(x', u_{|\Sigma}) dx' + \varepsilon^{1-\alpha} \int_{\Sigma} \varphi_{\varepsilon}(x') \delta_{\varepsilon} dx' \\ &+ \varepsilon^{1-\alpha} \int_{\Sigma} \varphi_{\varepsilon}(x') g(x', u_{|\Sigma})(u^{\varepsilon}{}_{|\Sigma} - u_{|\Sigma}) dx \\ &+ \frac{1}{\varepsilon^{\alpha}} \int_{B_{\varepsilon}} g(x, u^{\varepsilon}{}_{|\Sigma})(u^{\varepsilon} - u^{\varepsilon}{}_{|\Sigma}) dx.\end{aligned}$$

Thanks to lemma 2.1, so $\varphi_\varepsilon \to m(\varphi)$ in $L^{p'}(\Sigma)$ and $\varphi_\varepsilon \rightharpoonup^* m(\varphi)$ in $L^\infty(\Sigma)$, according to (3.14), $\delta_\varepsilon \to 0$ in $L^1(\Sigma)$ and by passing to the lower limit, we obtain

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) \geq \frac{1}{p}\int_\Omega |\nabla u|^p + 2m(\varphi)\,\eta(\alpha)\int_\Sigma G(x', u_{|\Sigma}).$$

Since $\alpha = 1$, thus

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) \geq \frac{1}{p}\int_\Omega |\nabla u|^p + 2m(\varphi)\int_\Sigma G(x', u_{|\Sigma}).$$

If $\alpha > 1$:
Let $u \in \mathcal{W} \setminus \mathbb{G}^\alpha$ and $u^\varepsilon \in \mathcal{W}$, such that $u^\varepsilon \rightharpoonup u$ in $\mathcal{W}$.
Assume that

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) < +\infty.$$

So there exists a constant $C > 0$ and a subsequence of $\mathcal{F}^\varepsilon(u^\varepsilon)$, still denoted by $\mathcal{F}^\varepsilon(u^\varepsilon)$, such that

$$\mathcal{F}^\varepsilon(u^\varepsilon) < C. \tag{3.15}$$

And according to (H2), so $u\varepsilon$ verify the assertions of lemma 3.1 and we can apply proposition 3.2, thus $u_{|\Sigma} = 0$, so that $u \in \mathbb{G}^\alpha$ what contradicts the fact that $u \notin \mathbb{G}^\alpha$, so that

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) = +\infty.$$

Hence the proof of theorem 3.1 is complete. □

In the sequel, one is interested to limit problem determination partner to the problem (3.1), when ε approaches zero. Thanks to the epi-convergence results, (see theorem 2.2, Proposition 2.1) and Theorem 3.1, and according to τ_f-continuity of Φ in $\mathcal{W}$, one has $\mathcal{F}^\varepsilon + \Phi\ \tau_f -$ **epilimit** toward $\mathcal{F}_\alpha + \Phi$ in $\mathcal{W}$.

Proposition 3.3. *For every $f \in L^\infty(\Omega)$ and according to the parameter values of α, there exists $u^* \in \mathcal{W}$ satisfying*

$$u^\varepsilon \rightharpoonup u^* \text{ in } \mathcal{W},$$
$$F_\alpha(u^*) + \Phi(u^*) = \inf_{v\in\mathbb{G}^\alpha}\Big\{F_\alpha(v) + \Phi(v)\Big\}.$$

Proof: Thanks to lemma 3.1, the family (u^ε) is bounded in $\mathcal{W}$, therefore it possess a τ_f- cluster point u^* in $\mathcal{W}$. And thanks to a classical epi-convergence result (see theorem 2.2), one has u^* is a solution of the problem Find

$$\inf_{v\in\mathcal{W}}\Big\{\mathcal{F}_\alpha(v) + \Phi(v)\Big\}. \tag{3.16}$$

Since for $\alpha > 1$, $\mathcal{F}_\alpha$ equals $+\infty$ on $\mathcal{W} \setminus \mathbb{G}^\alpha$, (3.16) becomes

$$\inf_{v \in \mathbb{G}^\alpha} \Big\{ \mathcal{F}_\alpha(v) + \Phi(v) \Big\}. \tag{3.17}$$

According to the uniqueness of solutions of the problem (3.16), so that u^ε admits an unique τ_f-cluster point u^*, and therefore $u^\varepsilon \rightharpoonup u^*$ in $\mathcal{W}$. □

Conclusion 3.1. In this work, it shown that the limit behavior of a constituted structure of two mediums without heat loss by an oscillating non linear thin layer of thickness ε, which the heat loss parameter depends on the negative powers of ε, is describes by a problem with interface Σ, (Σ the middle interface of the thin layer). Following the powers of ε, to the interface Σ, we have, on the interface Σ, the heat continuity, a bidimensional problem appears with a constant heat loss parameter or the heat is null.

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