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On arithmetic continuity

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abstract: In this article we introduce the concept of arithmetic continuity and arithmetic compactness and prove some intresting results related to these notions.

Key Words: Continuity; sequences; summability; compactness.

Contents

1. Introduction

A sequence $x = (x_k)$ defined on $\mathbb{N}$ and $n \in \mathbb{N}$ the notation $\sum_{k|n} x_k$ means the finite sum of all the numbers x_k as k ranges over the integers that divide n including 1 and n. In general for integers k and n we write $k|n$ to mean 'k divides n' or 'n is a multiple of k'. We use the symbol $< m, n >$ to denote the greatest common divisor of two integers m and n.

W.H.Ruckle [11], introduced the notions arithmetically summable and arithmetically convergent as follows.

(i) A sequence $x = (x_k)$ defined on $\mathbb{N}$ is called *arithmetically summable* if for each $\varepsilon > 0$ there is an integer n such that for every integer m we have $\left|\sum_{k|m} x_k - \sum_{k|<m,n>} x_k\right| < \varepsilon$.

(ii) A sequence $y = (y_k)$ is called *arithmetically convergent* if for each $\varepsilon > 0$ there is an integer n such that for every integer m we have $\left|y_m - y_{(<m,n>)}\right| < \varepsilon$.

From above two definitions it is clear that a sequence $x = (x_k)$ is arithmetically summable if and only if the sequence $y = (y_k)$ defined by $y_n = \sum_{k|n} x_k$ is arithmetically convergent, but the sequence $y = (y_k)$ is not convergent in the ordinary sense and is in fact periodic.

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A sequence $x = (x_k)$ is called periodic if there is a number n such that $x_{k+n} = x_k$ for all $k \in \mathbb{N}$, the smallest such integer n is called the period of the sequence x. Denote by **P** the linear space of all periodic sequences and denote by **QP** the closure of **P** in the space ℓ_∞ of all bounded sequences. For details about the spaces **P** and **QP** we refer to [2,3,8,9,13]. For details of arithmetically summable and arithmetical functions we refer to [7,12,14].

A subset E of $\mathbb{R}$ is compact if any open covering of E has a finite subcovering, where $\mathbb{R}$ is the set of real numbers. This is equivalent to the statement that any sequence $x = (x_n)$ of points in E has a convergent subsequence whose limit is in E. A real function f is continuous if and only if $(f(x_n))$ is a convergent sequence whenever (x_n) is. Regardless of limit, this is equivalent to the statement that $(f(x_n))$ is Cauchy whenever (x_n) is. Using the idea of continuity of a real function and the idea of compactness in terms of sequences, we introduce the concept of *arithmetic continuity* and *arithmetic compactness* and establish some interesting results related to these notions. For details on continuity of real valued functions we refer to [1,4,5,6,10]

2. Arithmetic Continuity

Definition 2.1. *A function f defined on a subset E of $\mathbb{R}$ is said to be arithmetic continuous if it transforms arithmetic convergent sequences to arithmetic convergent sequences. In other words, the sequence (x_n) is arithmetic convergent implies the sequence $(f(x_n))$ is arithmetic convergent.*

Theorem 2.2. *Sum of two arithmetic continuous functions is again arithmetic continuous.*

Proof: Let f and g be arithmetic continuous functions on a subset E of $\mathbb{R}$. To prove that the function $f+g$ is arithmetic continuous function on E. Let $\varepsilon > 0$ and (x_n) be any arithmetic convergent sequence on E. By the definition of arithmetic continuity the sequences $(f(x_n))$ and $(g(x_n))$ are arithmetic convergent sequences. Since $(f(x_n))$ and $(g(x_n))$ are arithmetic convergent sequences therefore for $\varepsilon > 0$ and a positive integer m

$$|f(x_n) - f(x_{<n,m>})| < \frac{\varepsilon}{2} \text{ for each } n \text{ and } |g(x_n) - g(x_{<n,m>})| < \frac{\varepsilon}{2} \text{ for each } n.$$

Now

$$\begin{aligned} |(f+g)(x_n) - (f+g)(x_{<n,m>})| &= |f(x_n) + g(x_n) - f(x_{<n,m>}) - g(x_{<n,m>})| \\ &\leq |f(x_n) - f(x_{<n,m>})| + |g(x_n) - g(x_{<n,m>})| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for each } n. \end{aligned}$$

This proves the theorem. $\square$

Theorem 2.3. *The difference of two arithmetic continuous functions is again arithmetic continuous.*

Proof: The proof is similar to the above theorem. □

Theorem 2.4. *If f is an arithmetic continuous function then $|f|$ is arithmetic continuous.*

Proof: Let f be an arithmetic continuous function on a subset E of $\mathbb{R}$. Let (x_n) be any arithmetic convergent sequence in E. Then by definition of arithmetic continuity the sequence $(f(x_n))$ is arithmetic convergent.
Therefore for $\varepsilon > 0$ there exists a positive integer m such that

$$|f(x_n) - f(x_{<n,m>})| < \varepsilon \text{ for each } n.$$

Now

$$\begin{aligned} ||f|(x_n) - |f|(x_{<n,m>})| &= ||f(x_n)| - |f(x_{<n,m>})|| \\ &\leq |f(x_n) - f(x_{<n,m>})| < \varepsilon \text{ for each } n. \end{aligned}$$

Hence the result. □

Theorem 2.5. *The composition of two arithmetic continuous functions is again arithmetic continuous.*

Proof: Let f and g be two arithmetic continuous functions on a subset E of $\mathbb{R}$. To prove that the function $f \circ g(x_n) = f(g(x_n))$ is arithmetic continuous function. Let (x_n) be any arithmetic convergent sequence. Since g is arithmetic continuous,so the sequence $(g(x_n))$ is also arithmetic convergent. Furthermoer, it is given that f is arithmetic continuous, hence it transforms arithmetic convergent sequence $(g(x_n))$ to arithmetic convergent sequence $(f(g(x_n)))$.
Hence the result follows. □

Theorem 2.6. *If f is uniformly continuous on a subset E of $\mathbb{R}$ then it is arithmetic continuous.*

Proof: Let f be uniformly continuous and (x_n) be any arithmetic convergent sequence in E. Since f is uniformly continuous in E, for a given $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in E$ with $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$.
Again the sequence (x_n) is arithmetic convergent, hence for the same $\delta > 0$ there exists a positive integer m such that

$$\begin{aligned} |x_n - x_{<n,m>}| < \delta \text{ for each } n &\Rightarrow |f(x_n) - f(x_{<n,m>})| < \varepsilon \text{ for each } n \\ &\Rightarrow \text{the sequence } (f(x_n)) \text{ is arithmetic convergent.} \\ &\Rightarrow \text{the function} f \text{is arithmetic continuous.} \end{aligned}$$

This completes the proof. □

3. Arithmetic convergence of sequence of functions

Definition 3.1. *A sequence of functions (f_n) defined on a subset E of $\mathbb{R}$ is said to be arithmetic convergent if for any $\varepsilon > 0$ and $\forall\ x \in E$ there exists a positive integer m such that*

$$|f_n(x) - f_{<n,m>}(x)| < \varepsilon\ \forall x \in E \quad \textit{and} \quad n \in \mathbb{N}.$$

Theorem 3.2. *If (f_n) be a sequence of arithmetic functions defined on a subset E of $\mathbb{R}$ and x_0 is a point in E such that*

$$\lim_{x \to x_0} f_n(x) = y_n,\ \ n = 1, 2, 3 \dots .$$

then (y_n) is arithmetic convergent.

Proof: Since the sequence (f_n) is arithmetic convergent therefore for $\varepsilon > 0$ and a positive integer m

$$|f_n(x) - f_{<n,m>}(x)| < \varepsilon\ \forall x \in E \quad \text{and} \quad n \in \mathbb{N}.$$

Keeping n, m fixed and letting $x \to x_o$,

$$|y_n - y_{<n,m>}| < \varepsilon\ \forall\ n.$$

Hence the sequence (y_n) is arithmetic convergent. □

Theorem 3.3. *If (f_n) is a sequence of arithmetic continuous functions and (f_n) is uniformly convergent to a function f on a subset E of $\mathbb{R}$, then f is arithmetic continuous.*

Proof: Let $\varepsilon > 0$ and (x_n) be any arithmetic convergent sequence on a subset E of $\mathbb{R}$. Since $f_n \to f$ uniformly, we have for a positive integer n_1 such that

$$|f_n(x) - f(x)| < \varepsilon\ \forall n \geq n_1 \text{ and } \forall x \in E. \tag{3.1}$$

In particular for $n = n_1$

$$|f_{n_1}(x) - f(x)| < \varepsilon\ \forall x \in E. \tag{3.2}$$

Furthermore (f_n) is given to be a sequence of arithmetic continuous functions, therefore

$$|f_{n_1}(x_n) - f_{n_1}(x_{<n,m>})| < \varepsilon. \tag{3.3}$$

Also from (3.1) we get

$$|f_{n_1}(x_{<n,m>}) - f(x_{<n,m>})| < \varepsilon. \tag{3.4}$$

Therefore using (3.2),(3.3),(3.4) we get

$$\begin{aligned}|f(x_n)-f(x_{<n,m>})| &= |f(x_n)-f_{n_1}(x_n)+f_{n_1}(x_n)-f_{n_1}(x_{<n,m>}) \\ &\quad +f_{n_1}(x_{<n,m>})-f(x_{<n,m>})| \\ &\leq |f(x_n)-f_{n_1}(x_n)|+|f_{n_1}(x_n)-f_{n_1}(x_{<n,m>})| \\ &\quad +|f_{n_1}(x_{<n,m>})-f(x_{<n,m>})| \\ &< \varepsilon+\varepsilon+\varepsilon=3\varepsilon\end{aligned}$$

Hence f is arithmetic continuous, which concludes the proof. □

Theorem 3.4. *The set of all arithmetic continuous functions defined on a subset E of $\mathbb{R}$ is a closed subset of all continuous function on E, i.e. $\overline{acf(E)}$=$acf(E)$, where $acf(E)$ denotes the set of all arithmetic continuous functions defined on E and $\overline{acf(E)}$ denotes the set of all limit points of $acf(E)$.*

Proof: Let f be any element of $\overline{acf(E)}$. Then there exists a sequence of points in $acf(E)$ such that $\lim f_n = f$. Now let (x_n) be any arithmetic convergent sequence in E. Since (f_n) converges to f, there exists a positive integer n_1 such that

$$|f(x)-f_n(x)|<\varepsilon \;\forall n\geq n_1 \text{ and } \forall x\in E. \tag{3.5}$$

Also since f_{n_1} is arithmetic continuous on E, there exists a positive integer m such that

$$|f_{n_1}(x_n)-f_{n_1}(x_{<n,m>})|<\varepsilon. \tag{3.6}$$

Again from(3.5),

$$|f_{n_1}(x_n)-f(x_n)|<\varepsilon \;, n=1,2,3\ldots. \tag{3.7}$$

Also

$$|f_{n_1}(x_{<n,m>})-f(x_{<n,m>})|<\varepsilon. \tag{3.8}$$

Using (3.6),(3.7),(3.8) we get

$$\begin{aligned}|f(x_n)-f(x_{<n,m>})| &= |f(x_n)-f_{n_1}(x_n)+f_{n_1}(x_n)-f_{n_1}(x_{<n,m>}) \\ &\quad +f_{n_1}(x_{<n,m>})-f(x_{<n,m>})| \\ &\leq |f(x_n)-f_{n_1}(x_n)|+|f_{n_1}(x_n)-f_{n_1}(x_{<n,m>})| \\ &\quad +|f_{n_1}(x_{<n,m>})-f(x_{<n,m>})| \\ &< \varepsilon+\varepsilon+\varepsilon=3\varepsilon.\end{aligned}$$

Hence f is arithmetic continuous on E. This completes the proof of the theorem. □

Corollary 3.5. *The set of all arithmetic continuous functions defined on a subset E of $\mathbb{R}$ is a complete subspace of the space of all continuous functions.*

4. Arithmetic Compactness

Definition 4.1. *A subset E of $\mathbb{R}$ is called aritmetic compact if every sequence of points in E has arithmetic convergent subsequence.*

Theorem 4.2. *An arithmetic continuous image of an arithmetic compact subset of $\mathbb{R}$ is arithmetic compact.*

Proof: Let f be an arithmetic continuous function on a subset E of $\mathbb{R}$ and E is arithmetic compact. Let (y_n) be a sequence of points in $f(E)$. Then we can write $y_n = f(x_n)$ where $(x_n) \in E$ for each $n \in \mathbb{N}$. Since E is arithmetic compact, there exists an arithmetic convergent subsequence (x_{n_k}) of (x_n). Again it is given that f is arithmetic continuous, this implies that $f(x_{n_k})$ is arithmetic convergent subsequence of $f(x_n)$. Hence $f(E)$ is arithmetic compact. $\square$

Corollary 4.3. *An arithmetic continuous image of a compact subset of $\mathbb{R}$ is arithmetic compact.*

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