Bol. Soc. Paran. Mat. (3s.) **v. 35** 1 (2017): 9–22.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v35i1.23568

On symmetric biadditive mappings of semiprime rings

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abstract: Let R be a ring with centre $Z(R)$. A mapping $D(.,.) : R\times\ R \longrightarrow R$ is said to be symmetric if $D(x, y) = D(y, x)$ for all $x, y \in R$. A mapping $f : R \longrightarrow R$ defined by $f(x) = D(x, x)$ for all $x \in R$, is called trace of D. It is obvious that in the case $D(.,.) : R\times\ R \longrightarrow R$ is a symmetric mapping, which is also biadditive (i.e. additive in both arguments), the trace f of D satisfies the relation $f(x + y) = f(x) + f(y) + 2D(x, y)$, for all $x, y \in R$. In this paper we prove that a nonzero left ideal L of a 2-torsion free semiprime ring R is central if it satisfies any one of the following properties: (i) $f(xy) \mp [x, y] \in Z(R)$, (ii) $f(xy) \mp [y, x] \in Z(R)$, (iii) $f(xy) \mp xy \in Z(R)$, (iv) $f(xy) \mp yx \in Z(R)$, (v) $f([x, y]) \mp [x, y] \in Z(R)$, (vi) $f([x, y]) \mp [y, x] \in Z(R)$, (vii) $f([x, y]) \mp xy \in Z(R)$, (viii) $f([x, y]) \mp yx \in Z(R)$, (ix) $f(xy) \mp f(x) \mp [x, y] \in Z(R)$, (x) $f(xy) \mp f(y) \mp [x, y] \in Z(R)$, (xi) $f([x, y]) \mp f(x) \mp [x, y] \in Z(R)$, (xii) $f([x, y]) \mp f(y) \mp [x, y] \in Z(R)$, (xiii) $f([x, y]) \mp f(xy) \mp [x, y] \in Z(R)$, (xiv) $f([x, y]) \mp f(xy) \mp [y, x] \in Z(R)$, (xv) $f(x)f(y) \mp [x, y] \in Z(R)$, (xvi) $f(x)f(y) \mp [y, x] \in Z(R)$, (xvii) $f(x)f(y) \mp xy \in Z(R)$, (xviii) $f(x)f(y) \mp yx \in Z(R)$, (xix) $f(x) \circ f(y) \mp [x, y] \in Z(R)$, (xx) $f(x) \circ f(y) \mp xy \in Z(R)$, (xxi) $f(x) \circ f(y) \mp yx \in Z(R)$, (xxii) $f(x)f(y) \mp x \circ y \in Z(R)$, (xxiii) $[x, y] - f(xy) + f(yx) \in Z(R)$, for all $x, y \in L$, where f stands for the trace of a symmetric biadditive mapping $D(.,.) : R\times\ R \longrightarrow R$.

Key Words: Semiprime rings, Left ideals, Symmetric biadditive mappings.

Contents

1. Introduction

Throughout the paper R will denote an associative ring with centre $Z(R)$. A ring R is said to be prime (resp. semiprime) if $aRb = 0$ implies that either $a = 0$ or $b = 0$ (resp. $aRa = 0$ implies that $a = 0$). We shall write for each pair of elements $x, y \in R$ the commutator $[x, y] = xy - yx$ and skew commutator $x \circ y = xy + yx$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A derivation d is inner if there exists an element $a \in R$ such that $d(x) = [a, x]$ for all $x \in R$. A mapping $D(.,.) : R\times\ R \longrightarrow R$ is said to be symmetric if $D(x, y) = D(y, x)$ for all $x, y \in R$. A mapping $f : R \longrightarrow R$ defined by $f(x) = D(x, x)$ for all $x \in R$, is called trace

2000 *Mathematics Subject Classification:* 16W25, 16R50, 16N60.
Submitted June 21, 2014. Published October 17, 2015

of D. It is obvious that in the case $D(.,.) : R \times R \longrightarrow R$ is a symmetric mapping, which is also biadditive (i.e. additive in both arguments), the trace f of D satisfies the relation $f(x+y) = f(x) + f(y) + 2D(x,y)$, for all $x, y \in R$. A biadditive mapping $D(.,.) : R \times R \longrightarrow R$ is said to be a biderivation on R if $D(xy,z) = D(x,z)y + xD(y,z)$ and $D(x,yz) = D(x,y)z + yD(x,z)$ for all $x, y \in R$.

Gy. Maksa [6] introduced the concept of a symmetric biderivation. It was shown in [7] that symmetric biderivations are related to general solution of some functional equations. Some results on symmetric biderivation in prime and semiprime rings can be found in [8] and [9]. There has been ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations. Recently many authors viz. [1], [2], [3] and [4] have obtained commutativity of prime and semiprime rings with derivations satisfying certain polynomial identities. In this paper we prove that a nonzero left ideal of a semiprime ring admitting a biadditive map is central if it satisfies some polynomial identities.

2. Preliminary result

We make extensive use of basic commutator identities $[xy,z] = [x,z]y + x[y,z]$ and $[x,yz] = [x,y]z + y[x,z]$. Moreover, we shall require the following lemma:

Lemma 2.1. *[5, Lemma 1.1.5] If R is a semiprime ring, then the center of a nonzero one sided ideal is contained in the center of R. As an immediate consequence, any commutative one sided ideal is contained in the center of R.*

3. Main Results

Theorem 3.1. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f(xy) \mp [x,y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: Suppose

$$f(xy) - [x,y] \in Z(R) \text{ for all } x, y, \in L. \tag{3.1}$$

Replacing y by $y+z$ in (3.1), we get

$$f(xy) + f(xz) + 2D(xy,xz) - [x,y] - [x,z] \in Z(R) \text{ for all } x, y, z \in L. \tag{3.2}$$

Since R is 2-torsion free, (3.2) yields that

$$D(xy,xz) \in Z(R) \text{ for all } x, y, z \in L. \tag{3.3}$$

Substituting y for z in (3.3), we get

$$f(xy) = D(xy,xy) \in Z(R) \text{ for all } x, y \in L. \tag{3.4}$$

In view of (3.1), (3.4) yields that

$$[x, y] \in Z(R) \text{ for all } x, y \in L. \tag{3.5}$$

Then

$$[[x, y], r] = 0 \text{ for all } x, y \in L, r \in R. \tag{3.6}$$

Replace x by xy in (3.6), to get

$$[[x, y]y, r] = 0 \text{ for all } x, y \in L, r \in R. \tag{3.7}$$

This implies that

$$[x, y][y, r] = 0 \text{ for all } x, y \in L, r \in R. \tag{3.8}$$

Replacing r by rx in (3.8), we get

$$[x, y]r[y, x] = 0 \text{ for all } x, y \in L, r \in R. \tag{3.9}$$

This implies that

$$[x, y]R[x, y] = 0 \text{ for all } x, y \in L. \tag{3.10}$$

Since R is semiprime, we get $[x, y] = 0$ for all $x, y \in L$, Hence $L \subseteq Z(R)$ by Lemma 2.1.

The proof is same for the case $f(xy) + [x, y] \in Z(R)$ for all $x, y \in L$. □

Theorem 3.2. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f(xy) \mp [y, x] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: The proof runs on the same parallel lines as of Theorem 3.1. □

Theorem 3.3. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f(xy) \mp xy \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: By hypothesis

$$f(xy) - xy \in Z(R) \text{ for all } x, y \in L. \tag{3.11}$$

Replacing y by $y + z$, we get

$$f(xy) + f(xz) + 2D(xy, xz) - xy - xz \in Z(R) \text{ for all } x, y, z \in L. \tag{3.12}$$

Comparing (3.11) and (3.12) we obtain

$$2D(xy, xz) \in Z(R) \text{ for all } x, y, z \in L. \tag{3.13}$$

Since R is 2-torsion free, we have

$$D(xy, xz) \in Z(R) \text{ for all } x, y, z \in L. \tag{3.14}$$

Substituting y for z in (3.14), we get

$$f(xy) = D(xy, xy) \in Z(R) \text{ for all } x, y \in L. \tag{3.15}$$

Using (3.11), we have $xy \in Z(R)$ for all $x, y \in L$. This imples that $[x, y] \in Z(R)$ for all $x, y \in L$. Arguing in the similar manner as in Theorem 3.1, we get the result . □

Theorem 3.4. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f(xy) \mp yx \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: The proof runs on the same parallel lines as of Theorem 3.3. □

Theorem 3.5. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f([x, y]) \mp [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: Let

$$f([x, y]) - [x, y] \in Z(R) \text{ for all } x, y \in L. \tag{3.16}$$

Replacing y by $y + z$, we have $f([x, y] + [x, z]) - [x, y] - [x, z] \in Z(R)$ i.e. $f([x, y]) + f([x, z]) + 2D([x, y], [x, z]) - [x, y] - [x, z] \in Z(R)$ for all $x, y, z \in L$. Using (3.16), we get

$$2D([x, y], [x, z]) \in Z(R) \text{ for all } x, y, z \in L. \tag{3.17}$$

Substituting y for z in (3.17) and using the fact that R is 2-torsion free, we find

$$f([x, y]) \in Z(R) \text{ for all } x, y \in L. \tag{3.18}$$

Using (3.16) and (3.18), we obtain $[x, y] \in Z(R)$ for all $x, y \in L$. Arguing in the similar manner as in Theorem 3.1, we get the result .

Similarly one can prove the result if $f([x, y]) + [x, y] \in Z(R)$ for all $x, y \in L$. □

Using similar arguments as we have done in the proof of Theorem 3.5, we can prove the following:

Theorem 3.6. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f([x, y]) \mp [y, x] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Theorem 3.7. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f([x, y]) \mp xy \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: Let

$$f([x, y]) - xy \in Z(R) \text{ for all } x, y \in L. \tag{3.19}$$

Replacing y by $y+z$ in (3.19), we have $f([x, y]+[x, z])-xy-xz \in Z(R)$ for all x, $y, z \in L$. This implies that

$$f([x, y]) + f([x, z]) + 2D([x, y], [x, z]) - xy - xz \in Z(R) \text{ for all } x, y, z \in L. \tag{3.20}$$

Using (3.19), we obtain

$$2D([x, y], [x, z]) \in Z(R) \text{ for all } x, y, z \in L. \tag{3.21}$$

Since R is 2-torsion free, (3.21) yields that

$$D([x, y], [x, z]) \in Z(R) \text{ for all } x, y, z \in L. \tag{3.22}$$

In particular, if we substitute y for z in (3.22), then we have $f([x, y]) \in Z(R)$ for all $x, y \in L$. Again using (3.19), we get $xy \in Z(R)$ for all $x, y \in L$. This implies that $[x, y] \in Z(R)$. Arguing in the similar manner as in Theorem 3.1, we get the result.

Similarly we can prove the result if $f([x, y]) + xy \in Z(R)$ for all $x, y \in L$. □

Theorem 3.8. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f([x, y]) \mp yx \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: The proof runs on the same parallel lines as of Theorem 3.7. □

Theorem 3.9. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f(xy) \mp f(x) \mp [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: Suppose

$$f(xy) - f(x) - [x, y] \in Z(R) \text{ for all } x, y \in L. \tag{3.23}$$

Replacing y by $y+z$, we get $f(xy)+f(xz)+2D(xy,xz)-f(x)-[x,y]-[x,z]\in Z(R)$ for all $x,y,z\in L$. Using (3.23), we obtain

$$f(xz)+2D(xy,xz)-[x,z]\in Z(R) \text{ for all } x,y,z\in L. \tag{3.24}$$

Substituting $-z$ for z in (3.24), we get

$$f(xz)-2D(xy,xz)+[x,z]\in Z(R) \text{ for all } x,y,z\in L. \tag{3.25}$$

Adding (3.24) and (3.25), we obtain

$$2f(xz)\in Z(R) \text{ for all } x,y,z\in L. \tag{3.26}$$

Since R is 2-torsion free, we have $f(xy)\in Z(R)$ for all $x,y\in L$. Using (3.23), we get

$$f(x)-[x,y]\in Z(R) \text{ for all } x,y\in L. \tag{3.27}$$

Replacing x by $x+z$, in (3.27), we have

$$f(x)+f(z)+2D(x,z)-[x,y]-[z,y]\in Z(R) \text{ for all } x,z\in L. \tag{3.28}$$

Again using (3.27) and using 2-torsion freeness of R, we find $D(x,z)\in Z(R)$. In particular $f(x)=D(x,x)\in Z(R)$ for all $x\in L$. Since $f(xz)\in Z(R)$ and $f(x)\in Z(R)$, we have $f(xy)-f(x)\in Z(R)$ for all $x,y\in L$. Using (3.23), we get $[x,y]\in Z(R)$ for all $x,y\in L$. Arguing in the similar manner as in Theorem 3.1, we get the result.

Similarly we can prove the result if $f(xy)+f(x)+[x,y]\in Z(R)$ for all $x,y\in L$.□

Theorem 3.10. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.): R\times R\longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f(xy)\mp f(y)\mp[x,y]\in Z(R)$ for all $x,y\in L$, then $L\subseteq Z(R)$.*

Proof: Let

$$f(xy)-f(y)-[x,y]\in Z(R) \text{ for all } x,y\in L. \tag{3.29}$$

Replacing y by $y+z$, we have $f(xy)+f(xz)+2D(xy,xz)-f(y)-f(z)-2D(y,z)-[x,y]-[x,z]\in Z(R)$ for all $x,y,z\in L$. Using (3.29), we get

$$2(D(xy,xz)-D(y,z))\in Z(R) \text{ for all } x,y,z\in L. \tag{3.30}$$

Substituting y for z in (3.30) and using the fact that R is 2-torsion free, we find

$$f(xy)-f(y)\in Z(R) \text{ for all } x,y\in L. \tag{3.31}$$

This implies that $[x,y]\in Z(R)$ for all $x,y\in L$. Arguing in the similar manner as in Theorem 3.1, we get the result.

Similarly we can prove the result if $f(xy)+f(y)+[x,y]\in Z(R)$ for all $x,y\in L$.□

Theorem 3.11. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f([x, y]) \mp f(x) \mp [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: Suppose

$$f([x, y]) - f(x) - [x, y] \in Z(R) \text{ for all } x, y \in L. \tag{3.32}$$

Replacing x by $x + z$ in (3.32), we obtain

$$\begin{aligned} &f([x, y]) + f([z, y]) + 2D([x, y], [z, y]) - f(x) - f(z) \\ &-2D(x, z) - [x, y] - [z, y] \in Z(R) \text{ for all } x, y, z \in L. \end{aligned} \tag{3.33}$$

Using (3.32),we have

$$2(D([x, y], [z, y]) - D(x, z)) \in Z(R) \text{ for all } x, y, z \in L. \tag{3.34}$$

Substituting x for z in (3.34) and using the fact that R is 2-torsion free, we obtain

$$f([x, y]) - f(x) \in Z(R) \text{ for all } x, y \in L. \tag{3.35}$$

Again using (3.32) and (3.35), we have $[x, y] \in Z(R)$ for all $x, y \in L$. Arguing in the similar manner as in Theorem 3.1, we get the result .

Similarly we can prove the Theorem if $f([x, y]) + f(x) + [x, y] \in Z(R)$ for all x, $y \in L$. □

Theorem 3.12. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f([x, y]) \mp f(y) \mp [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: Suppose

$$f([x, y]) - f(y) - [x, y] \in Z(R) \text{ for all } x, y \in L. \tag{3.36}$$

Replacing y by $y + z$, we get

$$\begin{aligned} &f([x, y]) + f([x, z]) + 2D([x, y], [x, z]) - f(y) - f(z) - 2D(y, z) \\ &-[x, y] - [x, z] \in Z(R) \text{ for all } x, y, z \in L. \end{aligned} \tag{3.37}$$

In view of (3.36), (3,37) yields that

$$2(D([x, y], [x, z]) - D(y, z)) \in Z(R) \text{ for all } x, y, z \in L. \tag{3.38}$$

Substituting y for z in (3.38) and using the fact that R is 2-torsion free, we obtain

$$f([x, y]) - f(y) = D([x, y], [x, y]) - D(y, y)) \in Z(R) \text{ for all } x, y \in L. \tag{3.39}$$

Using (3.36) and (3.39), we have $[x,y] \in Z(R)$ for all $x, y \in L$. Arguing in the similar manner as in Theorem 3.1, we get the result .

Similarly we can prove the Theorem if $f([x,y]) + f(y) + [x,y] \in Z(R)$ for all x, $y \in L$. □

Using the similar techniques as we have used in the proof of Theorem 3.11 and 3.12, we can prove the following:

Theorem 3.13. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f([x,y]) \mp f(x) \mp [y,x] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Theorem 3.14. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f([x,y]) \mp f(y) \mp [y,x] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Theorem 3.15. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f([x,y]) \mp f(xy) \mp [x,y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: Let

$$f([x,y]) - f(xy) - [x,y] \in Z(R) \text{ for all } x, y \in L. \tag{3.40}$$

Replacing y by $y + z$ in (3.40), we get

$$\begin{aligned} &f([x,y]) + f([x,z]) + 2D([x,y],[x,z]) - f(xy) - f(xz) - 2D(xy,xz) \\ &-[x,y] - [x,z] \in Z(R) \text{ for all } x, y, z \in L. \end{aligned} \tag{3.41}$$

Using (3.40) and (3.41), we obtain

$$2(D([x,y],[x,z]) - D(xy,xz) \in Z(R) \text{ for all } x, y, z \in L. \tag{3.42}$$

Since R is 2-torsion free, we have

$$D([x,y],[x,z]) - D(xy,xz) \in Z(R) \text{ for all } x, y, z \in L. \tag{3.43}$$

Substituting y for z in (3.43), we get

$$f([x,y]) - f(xy) \in Z(R) \text{ for all } x, y \in L. \tag{3.44}$$

Using (3.40), we have $[x,y] \in Z(R)$ for all $x, y \in L$. Arguing in the similar manner as in Theorem 3.1, we get the result.

The proof is same if $f([x, y]) + f(xy) + [x, y] \in Z(R)$ for all $x, y \in L$. □

Similarly we can prove the following:

Theorem 3.16. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f([x, y]) \mp f(xy) \mp [y, x] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Theorem 3.17. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f(x)f(y) \mp [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: Suppose

$$f(x)f(y) - [x, y] \in Z(R) \text{ for all } x, y \in L. \tag{3.45}$$

Substituting $y + z$ for y in (3.45), we have

$$f(x)f(y) + f(x)f(z) + 2f(x)D(y, z) - [x, y] - [x, z] \in Z(R) \text{ for all } x, y, z \in L. \tag{3.46}$$

Using (3.45), we find

$$2f(x)D(y, z) \in Z(R) \text{ for all } x, y, z \in L. \tag{3.47}$$

Since R is 2-torsion free, we have

$$f(x)D(y, z) \in Z(R) \text{ for all } x, y, z \in L. \tag{3.48}$$

In particular if we replace z by y in (3.48), then

$$f(x)f(y) \in Z(R) \text{ for all } x, y \in L. \tag{3.49}$$

Hence using (3.49) and (3.45), we obtain $[x, y] \in Z(R)$ for all $x, y \in L$. Arguing in the similar manner as in Theorem 3.1, we get the result.

Similarly we can prove the case if $f(x)f(y) + [x, y] \in Z(R)$ for all $x, y \in L$. □

Theorem 3.18. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f(x)f(y) \mp [y, x] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: The proof runs on the same parallel lines as that of Theorem 3.17. □

Theorem 3.19. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f(x)f(y) \mp xy \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: Let

$$f(x)f(y) - xy \in Z(R) \text{ for all } x, y \in L. \tag{3.50}$$

Substituting $y + z$ for y in (3.50), we have

$$f(x)f(y) + f(x)f(z) + 2f(x)D(y, z) - xy - xz \in Z(R) \text{ for all } x, y, z \in L. \tag{3.51}$$

Applying (3.50), we find

$$2f(x)D(y, z) \in Z(R) \text{ for all } x, y, z \in L. \tag{3.52}$$

Since R is 2-torsion free, we have

$$f(x)D(y, z) \in Z(R) \text{ for all } x, y, z \in L \tag{3.53}$$

In particular replacing z by y in (3.53) and using (3.50), we find

$$f(x)f(y) \in Z(R) \text{ for all } x, y \in L. \tag{3.54}$$

This implies that $xy \in Z(R)$ for all $x, y \in L$. Arguing in the similar manner as in Theorem 3.1, we get the result.

Similarly we can prove the case if $f(x)f(y) + xy \in Z(R)$ for all $x, y \in L$. □

Theorem 3.20. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f(x)f(y) \mp yx \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: The proof runs on the same parallel lines as that of Theorem 3.19. □

Theorem 3.21. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f(x) \circ f(y) \mp [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: Suppose

$$f(x) \circ f(y) - [x, y] \in Z(R) \text{ for all } x, y \in L. \tag{3.55}$$

Replacing y by $y + z$ in (3.55), we get

$$\begin{aligned} &f(x) \circ f(y) + f(x) \circ f(z) + 2(f(x) \circ D(y, z)) - [x, y] - [x, z] \in \\ &Z(R) \text{ for all } x, y \in L. \end{aligned} \tag{3.56}$$

Comparing (3.55) and (3.56), we have

$$2(f(x) \circ D(y,z)) \in Z(R) \text{ for all } x,y,z \in L.$$

Since R is 2-torsion free, we find

$$f(x) \circ D(y,z) \in Z(R) \text{ for all } x,y,z \in L. \tag{3.57}$$

Replacing y by z in (3.57),we get

$$f(x) \circ f(y) \in Z(R) \text{ for all } x,y,z \in L. \tag{3.58}$$

From (3.55) and (3.58), we have

$$[x,y] \in Z(R) \text{ for all } x,y \in L.$$

Arguing in the similar manner as in Theorem 3.1, we get the result.

The proof is same if $f(x) \circ f(y) + [x,y] \in Z(R)$ for all $x,y \in L$. □

Theorem 3.22. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f(x) \circ f(y) \mp xy \in Z(R)$ for all $x,y \in L$, then $L \subseteq Z(R)$.*

Proof: Suppose

$$f(x) \circ f(y) - xy \in Z(R) \text{ for all } x,y \in L. \tag{3.59}$$

Replacing y by $y+z$, in (3.59), we have

$$f(x) \circ f(y) + f(x) \circ f(z) + 2(f(x) \circ D(y,z)) - xy - xz \in Z(R) \text{ for all } x,y,z \in L. \tag{3.60}$$

Comparing (3.59) and (3.60), we have

$$2(f(x) \circ D(y,z)) \in Z(R) \text{ for all } x,y,z \in L. \tag{3.61}$$

Substitute y for z in (3.61) and using 2-torsion freeness of R, we get

$$f(x) \circ f(y) \in Z(R) \text{ for all } x,y \in L. \tag{3.62}$$

Using (3.59) and (3.62), we obtain

$$xy \in Z(R) \text{ for all } x,y \in L. \tag{3.63}$$

Interchanging the role of x and y in (3.63) and subtracting from (3.63), we find

$$[x,y] \in Z(R) \text{ for all } x,y \in L. \tag{3.64}$$

Arguing in the similar manner as in Theorem 3.1, we get the result.

The prove is same for the case $f(x) \circ f(y) + xy \in Z(R)$ for all $x,y \in L$. □

Theorem 3.23. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f(x) \circ f(y) \mp yx \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: The proof runs on the same parallel lines as of Theorem 3.22. □

Theorem 3.24. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f(x)f(y) \mp x \circ y \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: Suppose

$$f(x)f(y) - x \circ y = 0 \text{ for all } x, y \in L. \tag{3.65}$$

Replacing y by $y + z$ in (3.65), we get

$$f(x)f(y) + f(x)f(z) + 2f(x)D(y,z) - x \circ y - x \circ z = 0 \text{ for all } x, y, z \in L. \tag{3.66}$$

From (3.65) and (3.66), we have

$$2f(x)D(y,z) = 0 \text{ for all } x, y, z \in L. \tag{3.67}$$

Using 2-torsion freeness of R and replacing y by z in (3.67), we get

$$f(x)f(y) = 0 \text{ for all } x, y \in L. \tag{3.68}$$

Using (3.68) and (3.65), we have

$$xy + yx = 0 \text{ for all } x, y \in L. \tag{3.69}$$

Replace y by ry in (3.69) and using (3.69), we get

$$[x,r]y = 0 \text{ for all } x, y \in L, r \in R. \tag{3.70}$$

A simple calculation yields that $[x,r]R[x,r] = 0$ for all $x, y \in L,\ r \in R$. Since R is semiprime, we have $[x,r] = 0$ for all $x \in L,\ r \in R$. Hence $L \subseteq Z(R)$.

Similarly we can prove if $f(x)f(y) + x \circ y \in Z(R)$ for all $x, y \in L$. □

Theorem 3.25. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $f(x) \circ f(y) \mp x \circ y = 0$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof: Suppose

$$f(x) \circ f(y) - x \circ y = 0 \text{ for all } x, y \in L. \tag{3.71}$$

Replace y by $y+z$ in (3.71), we have

$$f(x) \circ f(y) + f(x) \circ f(z) + 2f(x) \circ D(y,z) - x \circ y - x \circ z = 0 \text{ for all } x,y,z \in L. \tag{3.72}$$

Comparing (3.71) and (3.72), we get

$$2f(x) \circ D(y,z) = 0 \text{ for all } x,y,z \in L. \tag{3.73}$$

Using 2-torsion freeness of R and replacing z by y in (3.73), we obtain

$$f(x) \circ f(y) = 0 \text{ for all } x,y \in L. \tag{3.74}$$

Using (3.74) and (3.71), we have

$$x \circ y = 0 \text{ for all } x,y \in L. \tag{3.75}$$

Using the same argument as we have done in the proof of Theorem 3.24, we get the result. □

Theorem 3.26. *Let R be a 2-torsion free semiprime ring and L be a nonzero left ideal of R. Let $D(.,.) : R \times R \longrightarrow R$ be a symmetric biadditive mapping and f be the trace of D. If $[x,y] - f(xy) + f(yx) \in Z(R)$ for all $x,y \in L$, then $L \subseteq Z(R)$.*

Proof: Suppose that

$$[x,y] - f(xy) + f(yx) \in Z(R) \text{ for all } x,y \in L. \tag{3.76}$$

Replacing y by $y+z$ in (3.76), we get

$$\begin{aligned} &[x,y] + [x,z] - f(xy) - f(xz) - 2D(xy,xz) \\ &+ f(yx) + f(zx) + 2D(yx,zx) \in Z(R) \text{ for all } x,y,z \in L. \end{aligned} \tag{3.77}$$

This implies that

$$-2D(xy,xz) + 2D(yx,zx) \in Z(R) \text{ for all } x,y,z \in L. \tag{3.78}$$

Since R is 2-torsion free, we have

$$-D(xy,xz) + D(yx,zx) \in Z(R) \text{ for all } x,y,z \in L. \tag{3.79}$$

Replacing z by y in (3.79), we get

$$-f(xy) + f(yx) \in Z(R) \text{ for all } x,y \in L. \tag{3.80}$$

Comparing (3.76) and (3.80), we get $[x,y] \in Z(R)$ and arguing in the similar manner as we have done in the proof of Theorem 3.1, we get the result. □

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