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# $X-$ dominating colour transversals in bipartite graphs

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abstract: Let $G = (X, Y, E)$ be a bipartite graph. A $X-$colouring of $G$ is a partition of $X$ into $k$ $X-$independent sets $\{X_1, X_2, \cdots, X_k\}$. The $X-$chromatic number $\chi_X(G)$ is the smallest order of an $X-$colouring of $G$. An X-dominating set $D \subseteq X$ is called a $X-$dominating colour transversal set of a graph $G$ if $D$ is a transversal of at least one $\chi_X-$partition of $G$. The minimum cardinality of a $X-$dominating colour transversal set is called $X-$dominating colour transversal number and is denoted by $\gamma_X dct(G)$. We find the bounds for $X-$dominating colour transversal number and characterize the graphs attaining these bounds.

Key Words: $X-$dominating set, $X-$dominating colour transversal set, $X-$chromatic partition.

## Contents



## 1. Introduction

Let $G = (V, E)$ be a graph of order $n$ and size $m$. By the neighborhood of a vertex $v$ of $G$ we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex $v$, denoted by $d_G(v)$, is the cardinality of its neighborhood.

A vertex of a graph is said to dominate itself and all its neighbors. A subset $D \subseteq V$ is a dominating set of $G$ if every vertex of $G$ is dominated by at least one vertex of $D$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. For a comprehensive survey of domination in graphs, see [2,3].

All graphs considered here are simple and undirected. Given any problem, say $P$, on an arbitrary graph $G$, there is a corresponding problem $Q$ on a bipartite graph $G^*$, such that a solution for $Q$ provides a solution for $P$. The bipartite theory of graphs was introduced by Stephen Hedetniemi and Renu Laskar in [4,5] and some parameters like $X-$dominating set, $Y-$dominating set, $X-$independent

2000 *Mathematics Subject Classification:* 05C69 set and $X-$colouring were defined. For more details on bipartite theory of graphs, see [6,7].

Colouring concepts and dominating sets are well studied topics in graph theory. The two concepts was combined by Gera et al introducing the parameter dominator colouring number [1] of a graph $G$. Here we define a new parameter called $X-$dominating colour transversal number of a bipartite graph.

## 2. Preliminaries

Let $G = (X, Y, E)$ denote a bipartite graph the partite set of which are $X$ and $Y$. Two vertices $u, v$ of $X$ are $X-$adjacent if they have a common neighbor in $Y$. If $x \in X$, then the set $N_Y(x) = \{u \in X : u$ and $x$ are $X-$adjacent $\}$. The $X-$degree of $x$, denoted by $d_Y(x)$, is the cardinality of the set $N_Y(x)$. The minimum $X-$degree is denoted by $\delta_Y(G)$.

A subset $D$ of $X$ is an $X-$dominating set [4] if every vertex in $X \setminus D$ is $X-$adjacent to at least one vertex in $D$. The minimum cardinality of an $X$–dominating set of $G$ is called the $X-$domination number of $G$ and is denoted by $\gamma_X(G)$.

The $X-$ chromatic number $\chi_X(G)$ of a graph $G$ is the minimum number of colours required to colour the vertices of $X(G)$ in such a way that no two $X-$adjacent vertices of $G$ receive the same colour. A partition of $X$ into $\chi_X(G)$ $X-$independent sets[4] ( called $X-$colour classes) is said to be a $\chi_X-$partition of $G$.

Let $S \subseteq X$ and let $u \in S$. The vertex $u$ is called an $Y-$isolate of $S$ if there exists no vertex $v \in S \setminus \{u\}$ such that $u$ and $v$ are $X-$adjacent. A vertex $v \in X \setminus S$ is called a $Y-$private neighbor of $u$ with respect to $S$ if $u$ is the only point in $S$ such that $u$ and $v$ are $X-$adjacent.

## 3. $X-$Dominating colour transversal sets

Unless otherwise stated, we consider bipartite graphs $G = (X, Y, E)$ with $|X| = p$ and $|Y| = q$.

**Definition 3.1.** *Let $\Pi$ be a $\chi_X-$partition of a graph $G$. A subset $D$ of $X$ is said to be a transversal of $\Pi$ if $D$ intersects every $X-$colour class of $\Pi$.*

We now define the $X-$dominating colour transversal number of a graph $G$.

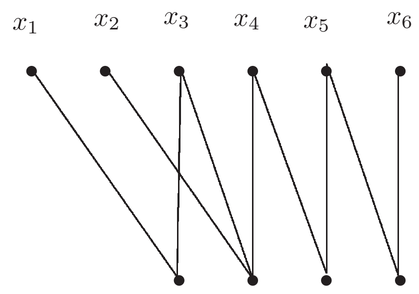
**Definition 3.2.** *A X-dominating set $D \subseteq X(G)$ is called a $X-$dominating colour transversal set ($X-$ dct set) of a graph $G$ if $D$ is a transversal of at least one $\chi_X-$partition of $G$.*

An $X-$dct set is minimal if none of its proper subsets is an $X-$dct set. The minimum cardinality of a $X-$dct set is called $X-$dominating colour transversal number and is denoted by $\gamma_X dct(G)$.

**Observation 3.3.** *For any graph $G$,*
*(i) $\gamma_X(G) \leq \gamma_X dct(G)$.*
*(ii) $\chi_X(G) \leq \gamma_X dct(G)$.*

**Example 3.4.**

Consider the graph $G$




The $\chi_X-$partition of $G$ is $\{\{x_1, x_2\}, \{x_3, x_5\}, \{x_4, x_6\}\}$. Consider the sets $D_1 = \{x_1, x_3, x_6\}$ and $D_2 = \{x_1, x_2, x_5, x_6\}$. The two sets are $X-$dct sets. The set $D_1$ is a minimal $X-$dct set.

The following theorem provides existence of graphs for which $\gamma_X(G) = a$ and $\gamma_X dct(G) = b$.

**Theorem 3.5.** *Given any two positive integers $a$ and $b$ where $a \leq b$, then there exists a bipartite graph $G$ with $\gamma_X(G) = a$ and $\gamma_X dct(G) = b$.*

**Proof:** Consider $K_{b,1}$ and attach a path of length six to some of $a-1$ vertices of $X(K_{b,1})$. The resulting graph is $G$. Let $X(G) = \{x_1, x_2, \cdots, x_b, x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, \cdots, x_{(a-1)1}, x_{(a-1)2}, x_{(a-1)3}\}$. Let $Y(G) = \{y, y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23}, \cdots, y_{(a-1)1}, y_{(a-1)2}, y_{(a-1)3}\}$. The edges are $E(G) = \{x_i y : 1 \leq i \leq b\} \cup \{x_{i1} y_{i1} : 1 \leq i \leq a-1\} \cup \{x_i y_{i1} : 1 \leq i \leq a-1\} \cup \{x_{i1} y_{i2} : 1 \leq i \leq a-1\} \cup \{x_{i2} y_{i2} : 1 \leq i \leq a-2\} \cup \{x_{i2} y_{i3} : 1 \leq i \leq a-2\} \cup \{x_{i3} y_{i3} : 1 \leq i \leq a-1\}$.The set $\{x_{12}, x_{22}, ..., x_{(a-1)2}, x_a\}$ is a minimum $X-$dominating set. Therefore, $\gamma_X(G) = a$. The partition $\pi = \{\{x_1, x_{12}\}, \{x_2, x_{22}\}, \cdots, \{x_{a-1} x_{(a-1)2}\}, \{x_a, x_{11}, x_{13}, x_{21}, x_{23}, \cdots, x_{(a-1)1}, x_{(a-1)3}\}, \{x_{a+1}\}, \{x_{a+2}\}, \cdots, \{x_b\}\}$ is a $\chi_X-$partition of $G$. Therefore, $\chi_X(G) = b$ and $\gamma_X dct(G) \geq b$.
The set $D = \{x_{12}, x_{22}, x_{32}, \cdots, x_{(a-1)2}, x_a, x_{a+1}, x_{a+2}, \cdots, x_b\}$ is a $X-$ dct set. Hence, $\gamma_X dct(G) \leq |D| = b$. Therefore, $\gamma_X dct(G) = b$. □

We now show the existence of graphs for which $\gamma_X(G) = \chi_X(G) = \gamma_X dct(G) = a$.

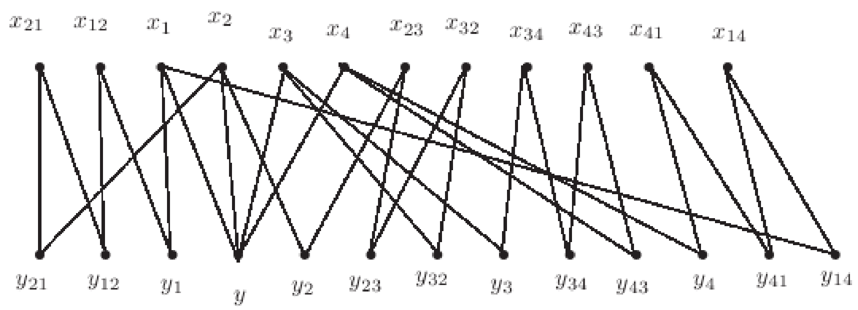
**Theorem 3.6.** *For any positive integer $a \geq 2$, there exists a graph $G$ with $\gamma_X(G) = \chi_X(G) = \gamma_X dct(G) = a$.*

**Proof:**We construct a graph $G$ as follows:
Consider $K_{a,1}$. Between any two vertices of $X(K_{a,1})$, attach a path of length six. This graph $G$ satisfies the property $\gamma_X(G) = \chi_X(G) = \gamma_X dct(G) = a$.
Let $X(G) = \{x_1, x_2, \cdots, x_a, x_{12}, x_{21}, x_{23}, x_{32}, \cdots, x_{a1}, x_{1a}\}$. Let $Y(G) = \{y, y_1, y_{12}, y_{21}, y_2, y_{23}, y_{32}, \cdots, y_a, y_{a1}, y_{1a}\}$ and $E(G) = \{x_i y : 1 \leq i \leq a\} \cup \{x_i y_i : 1 \leq i \leq a\} \cup \{x_{i(i+1)} y_i : 1 \leq i \leq a-1\} \cup \{x_{i(i+1)} y_{i(i+1)} : 1 \leq i \leq a-1\} \cup \{x_{(i+1)i} y_{i(i+1)} : 1 \leq i \leq a-1\} \cup \{x_{(i+1)i} y_{(i+1)i} : 1 \leq i \leq a-1\} \cup \{x_i y_{i(i-1)} : 2 \leq i \leq a\} \cup \{x_{a1} y_a, x_{a1} y_{a1}, x_{1a} y_{a1}, x_{1a} y_{1a}, x_1 y_{1a}\}$. Clearly, $D = \{x_{i(i+1)} : 1 \leq i \leq a-1\} \cup \{x_{a1}\}$ is a $X-$dominating set with minimum cardinality. Therefore, $\gamma_X(G) = a$.

Since the vertex $x_1$ is $X$–adjacent to $a$ vertices, $\chi_X(G) \geq a$. Let $X_1 = \{x_1, x_{21}, x_{a1}\}$, $X_i = \{x_i, x_{(i+1)i}, x_{(i-1)i}\}(2 \leq i \leq a-1)$ and $X_a = \{x_a, x_{1a}, x_{(a-1)a}\}$. Then, $\{X_1, X_2, ..., X_a\}$ is a $X-$chromatic partition of $G$. Therefore, $\chi_X(G) \leq a$. Hence, $\chi_X(G) = a$.
Since $\gamma_X(G) \leq \gamma_X dct(G)$, we have $\gamma_X dct(G) \geq a$. The set $D = \{x_{12}, x_{23}, x_{34}, ..., x_{a1}\}$ is a transversal of the above partition and is also a $X$–dominating set. Hence, $\gamma_X dct(G) \leq a$. Therefore, $\gamma_X dct(G) = a$. □

**Example 3.7.**

Graph with $\gamma_X(G) = \chi_X(G) = \gamma_X dct(G) = 4$.




The set $D = \{x_{12}, x_{23}, x_{34}, x_{41}\}$ is a minimum $X-$ dominating set of $G$. The $\chi_X-$ partition of $G$ is $\Pi = \{\{x_1, x_{21}, x_{41}\}, \{x_2, x_{32}, x_{12}\}, \{x_3, x_{43}, x_{23}\}, \{x_4, x_{14}, x_{34}\}\}$ and so $D$ is also a minimum $X-$dct set of $G$.

We characterize minimal $X-$dct sets through the following theorem.

**Theorem 3.8.** *An $X-$ dct set $D$ is minimal if and only if for every $u \in D$ any one of the following holds:*
*(i) $u$ is an $Y-$isolate of $D$.*
*(ii) there exists a $v \in X \setminus D$ such that $v$ is a $Y-$private neighbor of $u$ with respect to $D$.*
*(iii) For every $\chi_X-$partition, $\Pi = \{X_1, X_2, ..., X_\chi\}$ there exists one $X_i$ such that $X_i \cap D = \{u\}$ or $\phi$.*

**Proof:** Let $D$ be an $X-$dct set. If $D$ is minimal, then $D \setminus \{u\}$ is not an $X-$dct set for every $u \in D$. This implies that either $D \setminus \{u\}$ is not a $X-$dominating set or not a transversal of every $\chi_X-$partition of $G$.
Case (i): Suppose $D \setminus \{u\}$ is not a transversal for every $\chi_X-$partition $\{X_1, X_2, \cdots, X_\chi\}$. This implies that $(D \setminus \{u\}) \cap X_i = \phi$ for some $i$. That is $X_i \cap D = \{u\}$ or $\phi$ for some $i$. Hence (iii) is satisfied.
Case (ii): Suppose $D \setminus \{u\}$ is not a $X-$ dominating set. Hence, some vertex $v \in X \setminus D \cup \{u\}$ is not $X-$adjacent to any vertex in $D \setminus \{u\}$. Then either $v = u$ in which case $u$ is an $Y-$isolate of $D$ which is condition (i) or $v \in X \setminus D$ and $v$ is

not $X-$adjacent to any vertex of $X \setminus D \cup \{u\}$. That is $v$ is a $Y-$private neighbor of $u$ which is (ii).

Conversely, assume any one of the three conditions holds.
Suppose $D$ is not a minimal $X-$dct set of $G$.
There exists $u \in D$ such that $D \setminus \{u\}$ is a $X-$dct set of $G$ .

Let $\{X_1, X_2, \cdots, X_\chi\}$ be a $\chi_X-$partition of $X$ for which $D \setminus \{u\}$ and $D$ are transversals. Then $D \setminus \{u\}) \cap X_i \neq \phi$ and $D \cap X_i \neq \phi$ for every $i$. This implies $D \cap X_i \neq \{u\}$ or $\phi$ contradicting condition (iii).

Since $D - \{u\}$ is $X-$dct set for some $u \in D$, $D \setminus \{u\}$ is a X-dominating set of $G$. Hence $u$ is $X$-adjacent to at least one vertex in $D \setminus \{u\}$, and so condition (i) does not hold for $D$. Also every vertex in $X \setminus D$ is $X$-adjacent to at least one vertex in $D \setminus \{u\}$, and so condition (ii) does not hold for $u$. Thus $D$ does not satisfy (i) and (ii). □

## 4. $X-$dct number of certain known family of graphs

**Theorem 4.1.** $\gamma_X dct(K_{m,n}) = m$.

**Proof:** Every vertex in $X(K_{m,n})$ is $X-$ adjacent to other vertices in $X$. Therefore, $\chi_X(K_{m,n}) = m$. We know that $\chi_X(K_{m,n}) \leq \gamma_X dct(K_{m,n})$. Therefore, $\gamma_X dct(k_{m,n}) \geq m$. $X$ is itself a $X-$dct set. Hence, $\gamma_X dct(K_{m,n}) \leq m$. Therefore, $\gamma_X dct(K_{m,n}) = m$. □

**Notation:** Let $S_p$ be a bipartite graph $(X, Y, E)$, $|X| = p$; $|Y| = p - 1$ with a vertex $x$ in $X$ such that $x$ is $X$-adjacent to all other vertices of $X$ through different $y \in Y$ and all vertices in $X \setminus \{x\}$ are end vertices.

**Theorem 4.2.** $\gamma_X dct(S_p) = 2$.

**Proof:** The $\chi_X-$partition of $S_p$ is $\{\{x\}, X - \{x\}\}$. The vertex $x$ $X-$dominates other vertices in $X$. Therefore, $\{x, z\}$ is a minimal $X-$dct set of $S_p$ for any vertex $z$ in $X \setminus \{x\}$. Hence, $\gamma_X dct(S_p) = 2$. □

## 5. Bounds and characterization theorems

It is known that $1 \leq \gamma_X dct(G) \leq p$. We shall now characterize bipartite graphs with $\gamma_X dct(G) = p$. Let $x \in X$. Then $d_Y(x)$ is the number of vertices $X-$adjacent to $x$.

**Theorem 5.1.** *For a connected bipartite graph $G$, $\gamma_X dct(G) = p$ if and only if $d_Y(x) = p - 1$, $\forall x \in X$.*

**Proof:** Let $d_Y(x) = p - 1$, $\forall x \in X$. Then $\chi_X(G) = p$. Any $X-$ dct set is a transversal of $\chi_X-$partition, we have $\gamma_X dct(G) = p$.

Conversely, assume $\gamma_X dct(G) = p$.
**Claim:** $d_Y(x) = p - 1$, $\forall x \in X$.
If $d_Y(x) \neq p - 1$ for some $x \in X$, then there exists at least one vertex $u \in X$ such that $u$ and $x$ are non $X-$adjacent and $D = X \setminus \{x\}$ is a $X-$dominating set. We can form a $\chi_X-$partition with $\{x, u\}$ as a $X-$colour class. Thus we have a $\chi_X-$

partition in which $\{x\}$ is not a $X$–colour class, for which $D$ is a transversal. Hence, $D$ is a $X-$dct set of $G$. So, $\gamma_X dct(G) \leq |D| = p-1$, which is a contradiction to $\gamma_X dct(G) = p$. Thus, $\gamma_X dct(G) = p$ if and only if $d_Y(x) = p-1$, $\forall x \in X$. □
Now we give the upper bound of $\gamma_X dct(G)$ in terms of $X$-domination number and $X$-chromatic number.

**Theorem 5.2.** *For every graph $G$, $\gamma_X dct(G) \leq \gamma_X(G) + \chi_X(G) - 1$ and the bound is sharp.*

**Proof:** Let $D$ be a $\gamma_X-$set in $G$. Since $D$ is nonempty, $D$ intersects at least one colour class of every $\chi_X-$partition. For any $\chi_X-$partition, by choosing at the most $\chi_X - 1$ vertices, $D$ can be enlarged to an $X-$dct set. Hence,$\gamma_X dct(G) \leq \gamma_X(G) + \chi_X(G) - 1$.

Consider $K_{m,n}$. We have $\gamma_X(K_{m,n}) = 1$, $\chi_X(K_{m,n}) = m$ and $\gamma_X dct(K_{m,n}) = m$. Hence the bound is sharp. □

**Theorem 5.3.** *For a graph $G$, $\gamma_X dct(G) = \gamma_X(G) + \chi_X(G) - 1$ implies that one $X-$colour class of every $\chi_X-$partition is a $\gamma_X-$set.*

**Proof:** Let $\gamma_X dct(G) = \gamma_X(G) + \chi_X(G) - 1$. Suppose no colour class of some $\chi_X-$partition $\pi$ is a $\gamma_X-$set. Since no $\gamma_X-$set can be a proper subset of any $X-$colour class, it follows that any $\gamma_X-$set $D$ has to intersect at least two colour classes of $\Pi$. From the remaining $\chi_X - 2$ $X-$colour classes, choosing one vertex, we get an $X-$dct set with $\gamma_X + \chi_X - 2$ vertices, which implies that $\gamma_X dct(G) \leq \gamma_X(G) + \chi_X(G) - 2$, a contradiction. □

**Observation 5.4.** *Converse of the above theorem need not be true.*

Consider the cycle $C_8$. $\gamma_X dct(c_8) = 2 = \chi_X(C_8) = \gamma_X(C_8)$. Therefore, $\gamma_X dct(G) = \gamma_X(G) + \chi_X(G) - 1$ is not satisfied.

**Theorem 5.5.** *If $G$ is a graph, then $\gamma_X dct(G) = \gamma_X(G) + \chi_X(G) - 1$ implies that every $\gamma_X-$set is a colour class of every $\chi_X-$partition of $G$.*

**Proof:** Let $D$ be a $\gamma_X-$set. Let $\Pi$ be any $\chi_X-$partition of $G$. Then $D$ can neither be a proper subset of one $X$–colour class of $\Pi$ nor intersects two or more $X$–colour classes of $\Pi$. Hence, $D$ is exactly a $X-$colour class of $\Pi$. □

**Theorem 5.6.** *Let $G$ be a graph. Then $\gamma_X dct(G) = \gamma_X(G) + \chi_X(G) - 1$ if and only if every $\gamma_X-$set is a $X-$colour class of every $\chi_X-$partition of $G$ and is contained in a $\gamma_X dct-$set of $G$.*

**Proof:** Let $\gamma_X dct(G) = \gamma_X(G) + \chi_X(G) - 1$. Then by the above theorem, every $\gamma_X-$set is a $X-$ colour class of every $\chi_X-$partition. Let $D$ be a $\gamma_X-$set of $G$. Then $D$ is a $X-$colour class of every $\chi_X-$partition of $G$. By choosing one element from each of the remaining $(\chi_X - 1)$ colour classes, we get an $X-$dct set $D'$ of cardinality $\gamma_X(G) + \chi_X(G) - 1$. Hence, $D'$ is a $\gamma_X dct-$set containing $D$.

Conversely, let $D$ be a $\gamma_X-$set such that $D$ is a $X-$ colour class of every $\chi_X-$partition of $G$ and also is contained in some $\gamma_X dct-$set say $D'$. Hence,

$\chi_X(G) - 1 \leq \left|D^{'} - D\right| = \gamma_X dct(G) - \gamma_X(G)$. Therefore, $\gamma_X(G) + \chi_X(G) - 1 \leq \gamma_X dct(G)$. But $\gamma_X dct(G) \leq \gamma_X(G) + \chi_X(G) - 1$. Hence, $\gamma_X dct(G) = \gamma_X(G) + \chi_X(G) - 1$. □

## References


1. Gera.R, Horton.S and Ramussen. C, *Dominator coloring and safe clique Partitions*,Congressus Numerantium, Volume **181 (2006)**,19-32.
2. T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in graphs, Marcel Dekker, New York, 1998.
3. T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs - Advanced Topics, Marcel Dekker, New York, 1998.
4. Stephen Hedetniemi, Renu Laskar, *A Bipartite theory of graphs I*, Congressus Numerantium, Volume **55**; December 1986, 5–14.
5. Stephen Hedetniemi, Renu Laskar, *A Bipartite theory of graphs II*, Congressus Numerantium, Volume **64**; November 1988, 137-146.
6. V. Swaminathan and Y.B. Venkatakrishnan, *Bipartite theory of irredundant set*, Proyecciones Journal of Mathematics, Volume **30**;2011, 19-28.
7. V. Swaminathan and Y.B. Venkatakrishnan, *Bipartite theory on domination in complement of a graph*, International Journal of Computational and Mathematical Sciences, Volume **3**;2009,96-97.



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