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On some metabelian 2-group and applications II

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abstract: Let G be some metabelian 2-group such that $G/G' \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In this paper, we construct all the subgroups of G of index 2 or 4, we give the abelianization types of these subgroups and we compute the kernel of the transfer map. Then we apply these results to study the capitulation problem of the 2-ideal classes of some fields $\mathbf{k}$ satisfying the condition $Gal(\mathbf{k}_2^{(2)}/\mathbf{k}) \simeq G$, where $\mathbf{k}_2^{(2)}$ is the second Hilbert 2-class field of $\mathbf{k}$.

Key Words: 2-group, metabelian 2-group, capitulation, Hilbert class fields.

Contents

1. Introduction

Let k be an algebraic number field and let $\mathbf{C}l(k)$ denote its class group. Let $k^{(1)}$ be the Hilbert class field of k, that is the maximal abelian unramified extension of k. Let $k^{(2)}$ be the Hilbert class field of $k^{(1)}$ and put $G = Gal(k^{(2)}/k)$. Denote by F a finite extension of k and by H the subgroup of G which fixes F, then we say that an ideal class of k capitulates in F if it is in $\ker j_{k\to F}$, the kernel of the homomorphism:

$$j_{k\to F} : \mathbf{C}l(k) \longrightarrow \mathbf{C}l(F)$$

induced by extension of ideals from k to F. An important problem in Number Theory is to explicitly determine the kernel of $j_{k\to F}$, which is usually called the capitulation kernel. As $j_{k\to F}$ corresponds, by Artin reciprocity law, to the group theoretical transfer (for details see [14]):

$$V_{G\to H} : G/G' \longrightarrow H/H',$$

where G' (resp. H') is the derived group of G (resp. H). So, determining $\ker j_{k\to F}$ is equivalent to determine $\ker V_{G\to H}$, which transforms the capitulation problem to a problem of Group Theory. That is why the capitulation problem is completely solved if $G/G' \simeq (2, 2)$, since groups G such that $G/G' \simeq (2, 2)$ are determined and

2000 *Mathematics Subject Classification:* 11R11, 11R29, 11R32, 11R37 well classified (see [11,14]). If $G/G' \simeq (2, 2^n)$, for some integer $n \geq 2$, then G is metacyclic or not; in the first case the capitulation problem is completely solved, whereas in the second case the problem is open (see [6,7]). If $G/G' \simeq (2, 2, 2)$, then the structure of G is unknown in most cases, so the capitulation problem is also style open, in reality there are some studies which dealt with this problem in particular cases; see [1,2,3,9,10]. It is the purpose of this paper to provide answers to this problem in a particular case, it is the continuation of a project we started in [4,5]; we give some group theoretical results to solve the capitulation problem, in a particular case, if G satisfies the last condition. For this, we consider the family of groups defined, for integers $n \geq 1$ and $m \geq 2$, as follows

$$G_{n,m} = \langle \sigma,\ \tau,\ \rho: \quad \rho^4 = \tau^{2^{n+1}} = \sigma^{2^m} = 1,\ \rho^2 = \tau^{2^n}\sigma^{2^{m-1}}, \\ [\tau, \sigma] = 1, [\rho, \sigma] = \sigma^2,\ [\rho, \tau] = \rho^2 \rangle \tag{1.1}$$

In this paper, we construct all the subgroups of $G_{n,m}$ of index 2 or 4, we give the abelianization types of these subgroups and we compute the kernel of the transfer map $\mathrm{V}_{G\to H} : G_{n,m}/G'_{n,m} \to H/H'$, for any subgroup H of $G_{n,m}$, defined by the Artin map. Then we apply these results to study the capitulation problem of the 2-ideal classes of some fields $\mathbf{k}$ satisfying the condition $\mathrm{G}al(\mathbf{k}_2^{(2)}/\mathbf{k}) \simeq G_{n,m}$, where $\mathbf{k}_2^{(2)}$ is the second Hilbert 2-class field of $\mathbf{k}$. Finally, we illustrate our results by some examples which show that our group is realizable i.e. there is a field $\mathbf{k}$ such that $\mathrm{G}al(\mathbf{k}_2^{(2)}/\mathbf{k}) \simeq G_{n,m}$.

2. Main Results

Recall first that a group G is said to be *metabelian* if its derived group G' is abelian, and a subgroup H of a group G, not reduced to an element, is called maximal if it is the unique subgroup of G distinct from G containing H.

Let $G_{n,m}$ be the group family defined by the Formula (1.1). Since $[\tau, \sigma] = 1$, $[\rho, \sigma] = \sigma^2$ and $[\rho, \tau] = \rho^2 = \tau^{2^n}\sigma^{2^{m-1}}$, so $G'_{n,m} = \langle \sigma^2, \rho^2 \rangle = \langle \sigma^2, \tau^{2^n}\sigma^{2^{m-1}} \rangle = \langle \sigma^2, \tau^{2^n} \rangle$, which is abelian. Then $G_{n,m}$ is metabelian and $G_{n,m}/G'_{n,m} \simeq (2, 2, 2^n)$. Hence $G_{n,m}$ admits seven subgroups of index 2, denote them by $H_{i,2}$, and if $n = 1$ it admits also seven subgroups of index 4, we denote them by $H_{i,4}$, where $1 \leq i \leq 7$. These subgroups, their derived groups and the types of their abelianizations are given in Tables 1 and 2 below, where $b = \max(m, n + 1)$.

To check the Tables entries, we use the following lemmas.

Lemma 2.1 ([12], Prposition 5.1.5)**.** *Let x, y and z be elements of some group G, put $x^y = y^{-1}xy$. Then $[xy, z] = [x, z]^y[y, z]$ and $[x, yz] = [x, z][x, y]^z$.*

Lemma 2.2. *Let $G_{n,m} = \langle \sigma, \tau, \rho \rangle$ denote the group defined above, then*

1. *ρ^2 commutes with σ and τ.*
2. *$\rho^{-1}\sigma\rho = \sigma^{-1}$.*
3. *$\tau^{-1}\rho\tau = \rho^3$ and $\rho^{-1}\tau\rho = \tau\rho^2$.*

Table 1: Subgroups of $G_{n,m}$ of index 2

i		$H_{i,2}$	$H'_{i,2}$	$H_{i,2}/H'_{i,2}$
1		$\langle\sigma,\tau\rangle$	$\langle 1\rangle$	$(2^m, 2^{n+1})$
2		$\langle\sigma,\rho\rangle$	$\langle\sigma^2\rangle$	$(2,4)$
3	$m=2$	$\langle\tau,\rho\rangle$	$\langle\rho^2\rangle$	$(2,4)$
	$m\geq 3$	$\langle\tau,\rho,\sigma^2\rangle$	$\langle\rho^2,\sigma^4\rangle$	$(2,2,2^n)$
4	$n=1$ and $m=2$	$\langle\sigma\tau,\rho,\tau^2\rangle$	$\langle\tau^2\rangle$	$(2,4)$
	$n\geq 2$ and $m=2$	$\langle\sigma\tau,\rho\rangle$	$\langle\tau^{2^n}\rangle$	$(4,2^n)$
	$n=1$ and $m\geq 3$	$\langle\sigma\tau,\rho\rangle$	$\langle(\sigma\tau)^2\rangle$	$(2,4)$
	$n\geq 2$ and $m\geq 3$	$\langle\sigma\tau,\rho,\sigma^2\rangle$	$\langle\sigma^2\rho^2\rangle$	$(2,2^{n+1})$ if $n=m-1$; $(4,2^b)$ if $n\neq m-1$
5	$m=2$	$\langle\sigma\rho,\tau\rangle$	$\langle\rho^2\rangle$	$(2,2^{n+1})$
	$m\geq 3$	$\langle\sigma\rho,\tau,\sigma^2\rangle$	$\langle\rho^2,\sigma^4\rangle$	$(2,2,2^n)$
6		$\langle\tau\rho,\sigma\rangle$	$\langle\sigma^2\rangle$	$(2,2^{n+1})$
7		$\langle\sigma\rho,\tau\rho\rangle$	$\langle\sigma^2\rho^2\rangle$	$(4,2^n)$

Table 2: Subgroups of $G_{1,m}$ of index 4

i	$H_{i,4}$	$H'_{i,4}$	$H_{i,4}/H'_{i,4}$
1	$\langle\sigma,\tau^2\rangle$	$\langle 1\rangle$	$(2,2^m)$
2	$\langle\tau,\sigma^2\rangle$	$\langle 1\rangle$	$(4,2^{m-1})$
3	$\langle\rho,\sigma^2\rangle$	$\langle\sigma^4\rangle$	$(2,4)$
4	$\langle\sigma\tau,\tau^2\rangle$	$\langle 1\rangle$	$(2,2^m)$
5	$\langle\sigma\rho,\sigma^2\rangle$	$\langle\sigma^4\rangle$	$(2,4)$
6	$\langle\tau\rho,\sigma^2\rangle$	$\langle\sigma^4\rangle$	$(2,4)$
7	$\langle\sigma\tau\rho,\sigma^2\rangle$	$\langle\sigma^4\rangle$	$(2,4)$

4. $(\sigma\rho)^2=\rho^2$ *and* $(\sigma\tau\rho)^2=(\tau\rho)^2=\tau^2$.

5. $[\rho,\ \sigma\tau]=\rho^2\sigma^2$.

6. $[\rho,\ \tau^2]=1$ *and for all* $r\in\mathbb{N}$, $[\rho,\ \sigma^{2^r}]=\sigma^{2^{r+1}}$.

Proof: 1., 2. and 3. are obvious, since $\rho^2=\tau^{2^n}\sigma^{2^{m-1}}$, $[\rho,\ \sigma]=\sigma^2$ and $[\rho,\ \tau]=\rho^2$.
4. $(\sigma\rho)^2=\sigma\rho\sigma\rho=\sigma\rho^2\rho^{-1}\sigma\rho=\sigma\rho^2\sigma^{-1}=\rho^2$.
$(\sigma\tau\rho)^2=\sigma\tau\rho\sigma\tau\rho=\sigma\tau^2\tau^{-1}\rho\tau\sigma\rho=\sigma\tau^2\rho^{-1}\sigma\rho=\sigma\tau^2\sigma^{-1}=\tau^2$. We proceed similarly to prove the remaining result.
5. Obvious by Lemma 2.1.
6. $[\rho,\ \tau^2]=\rho^{-1}\tau^{-2}\rho\tau^2=\rho^{-1}\tau^{-1}\tau^{-1}\rho\tau\tau=\rho^{-1}\tau^{-1}\rho^3\tau=\rho^{-1}\rho^2\tau^{-1}\rho\tau=\rho^4=1$.
As $[\rho,\ \tau]=\tau^2$, so $[\rho,\ \tau^2]=\tau^4$. By induction, we show that for all $r\in\mathbb{N}^*$, $[\rho,\ \sigma^{2^r}]=\sigma^{2^{r+1}}$. □

Let us now prove some entries of the Tables, using Lemmas 2.1 and 2.2.

- For $H_{1,2}=\langle\sigma,\ \tau, G'_{n,m}\rangle=\langle\sigma,\ \tau\rangle$, we have $H'_{1,2}=\langle 1\rangle$, since $[\sigma,\tau]=1$. As $\sigma^{2^m}=\tau^{2^{n+1}}=1$, so $H_{1,2}/H'_{1,2}\simeq\left(2^m,2^{n+1}\right)$.

- For $H_{2,2}=\langle\sigma,\ \rho, G'_{n,m}\rangle=\langle\sigma,\ \rho,\ \tau^{2^n},\ \sigma^2\rangle=\langle\sigma,\ \rho,\ \tau^{2^n}\rangle$. As $\rho^2=\tau^{2^n}\sigma^{2^{m-1}}$, so $H_{2,2}=\langle\sigma,\ \rho\rangle$. Therefore, by Lemma 2.2, we get $H'_{2,2}=\langle\sigma^2\rangle$, thus $H_{2,2}/H'_{2,2}\simeq(2,4)$, since $\rho^4=1$.

- For $H_{4,2}=\langle\sigma\tau,\ \rho, G'_{n,m}\rangle=\langle\sigma\tau,\ \rho,\ \tau^{2^n},\ \sigma^2\rangle=\langle\sigma\tau,\ \rho,\ \sigma^2\rangle=\langle\sigma\tau,\ \rho,\ \tau^2\rangle$, since $\tau^2=(\sigma\tau)^2\sigma^{-2}$. We get
  - If $n=1$ and $m=2$, then $\rho^2=\tau^2\sigma^2$ and $\tau^4=\sigma^4=1$. Lemma 2.2 yields that $H'_{4,2}=\langle\tau^2\rangle$. Thus $H_{4,2}/H'_{4,2}\simeq(2,4)$ since $\rho^4=1$.
  - If $n\geq 2$ and $m=2$, then $\rho^2=\tau^{2^n}\sigma^2$ and $\sigma^4=1$, thus $\rho^2=(\sigma\tau)^{2^n}\sigma^2$; which implies that $\rho^2(\sigma\tau)^{-2^n}=\sigma^2$. Hence $H_{4,2}=\langle\sigma\tau,\ \rho\rangle$, and Lemma 2.2 yields that $H'_{4,2}=\langle\rho^2\sigma^2\rangle=\langle\tau^{2^n}\rangle$. Thus $H_{4,2}/H'_{4,2}\simeq(4,2^n)$.
  - If $n=1$ and $m\geq 3$, then $\rho^2=\tau^2\sigma^{2^{m-1}}=\tau^2(\sigma\tau)^{2^{m-1}}$ and $\tau^4=1$, hence $H_{4,2}=\langle\sigma\tau,\rho\rangle$, and Lemma 2.2 yields that $H'_{4,2}=\langle(\sigma\tau)^2\rangle$. Thus $H_{4,2}/H'_{4,2}\simeq(2,4)$.
  - If $n\geq 2$ and $m\geq 3$, then $H_{4,2}=\langle\sigma\tau,\rho,\sigma^2\rangle$, and Lemma 2.2 yields that $H'_{4,2}=\langle\rho^2\sigma^2,\sigma^4\rangle=\langle\rho^2\sigma^2\rangle$ since $\sigma^4=(\rho^2\sigma^2)^2$. On the other hand, $H_{4,2}=\langle\sigma\tau,\rho,\rho^2\sigma^2\rangle$, thus $H_{4,2}/H'_{4,2}=\langle\sigma\tau,\rho\rangle/H'_{4,2}$. We have two cases to distinguish. If $n=m-1$, then $\rho^2=(\sigma\tau)^{2^n}$; hence $H_{4,2}/H'_{4,2}\simeq(2,2^{n+1})=(2,2^m)$. If $n\neq m-1$, then $H_{4,2}/H'_{4,2}\simeq\left(4,2^{\max(n+1,m)}\right)$.

- For $H_{1,4}=\langle\sigma,\ G'_{n,m}\rangle=\langle\sigma,\ \sigma^2,\ \tau^2\rangle=\langle\sigma,\ \tau^2\rangle$, we have $H'_{1,4}=\langle 1\rangle$, since $[\sigma,\tau]=1$. As $\sigma^{2^m}=\tau^4=1$, so $H_{1,4}/H'_{1,4}\simeq(2,2^m)$.

- For $H_{2,4}=\langle\tau,\ G'_{n,m}\rangle=\langle\tau,\ \tau^2,\ \sigma^2\rangle=\langle\tau,\ \sigma^2\rangle$, we have $H'_{2,4}=\langle 1\rangle$, hence $H_{2,4}/H'_{2,4}\simeq\left(4,2^{m-1}\right)$.

The other entries of the Tables 1 and 2 are similarly checked.

Proposition 2.3. *Let $G_{n,m}$ be the group family defined by Formula (1.1), then*

1. *The order of $G_{n,m}$ is 2^{m+n+2} and that of $G'_{n,m}$ is 2^m.*
2. *The coclass of $G_{n,m}$ is $n+2$ and its nilpotency class is m.*
3. *The center, $Z(G)$, of G is of type $(2,2^n)$.*

Proof: 1. Since $\sigma^{2^m}=\tau^{2^{n+1}}=1$ and for all $0\leq i\leq m-1$ and $0\leq j\leq 1$ $\sigma^{2^i}\neq\tau^{2^j}$, then $\langle\sigma,\ \tau\rangle\simeq\left(2^m,2^{n+1}\right)$. Moreover, as $\rho^2=\tau^{2^n}\sigma^{2^{m-1}}$, so $\langle\sigma,\ \tau,\ \rho\rangle\simeq\left(2^m,2^{n+1},\ 2\right)$. Thus $|G_{n,m}|=2^{m+n+2}$. Similarly, we prove that $|G'_{n,m}|=2^m$, since $G'_{n,m}=\langle\sigma^2,\ \tau^{2^n}\rangle\simeq(2,2^{m-1})$.

2. The lower central series of $G_{n,m}$ is defined inductively by $\gamma_1(G_{n,m})=G_{n,m}$ and $\gamma_{i+1}(G_{n,m})=[\gamma_i(G_{n,m}),G_{n,m}]$, that is the subgroup of $G_{n,m}$ generated by the

set $\{[a,b]=a^{-1}b^{-1}ab|\ a\in\gamma_i(G_{n,m}), b\in G_{n,m}\}$, so the coclass of $G_{n,m}$ is defined to be $cc(G_{n,m})=h-c$, where $|G_{n,m}|=2^h$ and $c=c(G_{n,m})$ is the nilpotency class of $G_{n,m}$. We easily get
$\gamma_1(G_{n,m})=G_{n,m}$.
$\gamma_2(G_{n,m})=G'_{n,m}=\langle\sigma^2,\rho^2\rangle=\langle\sigma^2,\tau^{2^n}\rangle$.
$\gamma_3(G_{n,m})=[G'_{n,m},G_{n,m}]=\langle\sigma^4\rangle$ since $[\rho,\tau^2]=1$.
Then Lemma 2.2(6) yields that for $j\geq 2$, $\gamma_{j+1}(G_{n,m})=[\gamma_j(G_{n,m}),G_{n,m}]=\langle\sigma^{2^j}\rangle$. Hence $\gamma_{m+1}(G_{n,m})=\langle\sigma^{2^m}\rangle=\langle 1\rangle$ and $\gamma_m(G_{n,m})=\langle\sigma^{2^{m-1}}\rangle\neq\langle 1\rangle$, so $c(G_{n,m})=m$. Since $\mid G_{n,m}\mid=2^{m+n+2}$, then

$$cc(G_{n,m})=m+n+2-m=n+2.$$

3. To prove the last assertion, we use Lemma 12.12 of [13, pp. 204] which states that if G is a p-group and A is a normal abelian subgroup of G such that G/A is cyclic, then $A/A\cap Z(G)\simeq G'$. Let $A=H_{1,2}$, so A is abelian and $[G:A]=2$, thus $Z(G)\subset A$ and $A/Z(G)\simeq G'$. Hence $|G|=|A|[G:A]=2|G'||Z(G)|$, thus $|Z(G)|=\frac{1}{2}|G/G'|=2^{n+1}$. On the other hand, by Lemma 2.2 we have $[\rho,\sigma^{2^{m-1}}]=\sigma^{2^m}=1$ and $[\rho,\tau^2]=1$, so $\langle\sigma^{2^{m-1}},\tau^2\rangle\subset Z(G)$. As $\left|\langle\sigma^{2^{m-1}},\tau^2\rangle\right|=2^{n+1}$, so $\langle\sigma^{2^{m-1}},\tau^2\rangle=Z(G)\simeq(2,2^n)$. □

We continue with the following results.

Proposition 2.4 ([14]). *Let H be a normal subgroup of a group G. For $g\in G$, put $f=[\langle g\rangle.H:H]$ and let $\{x_1,x_2,\ldots,x_t\}$ be a set of representatives of $G/\langle g\rangle H$. The transfer map $\mathrm{V}_{G\to H}:G/G'\to H/H'$ is given by the following formula*

$$\mathrm{V}_{G\to H}(gG')=\prod_{i=1}^{t}x_i^{-1}g^f x_i.H'. \tag{2.1}$$

Easily, we prove the following corollaries.

Corollary 2.5. *Let H be a subgroup of $G_{n,m}$ of index 2. If $G_{n,m}/H=\{1,zH\}$, then* $\mathrm{V}_{G\to H}(gG'_{n,m})=\begin{cases} gz^{-1}gz.H'=g^2[g,\ z].H' & \text{if } g\in H,\\ g^2.H' & \text{if } g\notin H.\end{cases}$

Corollary 2.6. *Let H be a normal subgroup of $G_{n,m}$ of index 4. If $G_{n,m}/H=\{1,zH,z^2H,z^3H\}$, then*

$$\mathrm{V}_{G\to H}(gG'_{n,m})=\begin{cases} gz^{-1}gz^{-1}gz^{-1}gz^3.H' & \text{if } g\in H,\\ g^4.H' & \text{if } gH=zH,\\ g^2z^{-1}g^2z.H' & \text{if } g\notin H \text{ and } gH\neq zH.\end{cases}$$

Corollary 2.7. *Let H be a normal subgroup of $G_{n,m}$ of index 4. If $G_{n,m}/H=\{1,z_1H,z_2H,z_3H\}$ with $z_3=z_1z_2$, then*

$$\mathrm{V}_{G\to H}(gG'_{n,m})=\begin{cases} gz_1^{-1}gz_1z_2^{-1}gz_1^{-1}gz_1z_2.H' & \text{if } g\in H,\\ g^2z_i^{-1}g^2z_i.H' & \text{if } gH=z_jH \text{ with } i\neq j.\end{cases}$$

We can now establish our main result. Let $\ker \mathrm{V}_H$ denote the kernel of the transfer map $\mathrm{V}_{G_{n,m}\to H} : G_{n,m}/G'_{n,m} \to H/H'$, where H is a subgroup of $G_{n,m}$.

Theorem 2.8. *Keep the previous notations. Then*

1. $\ker \mathrm{V}_{H_{1,2}} = \langle \sigma G'_{n,m}\rangle$.
2. $\ker \mathrm{V}_{H_{2,2}} = \langle \sigma G'_{n,m}, \rho G'_{n,m}\rangle$.
3. $\ker \mathrm{V}_{H_{3,2}} = \begin{cases} \langle \tau\rho G'_{n,m}, \sigma\rho G'_{n,m}\rangle & \text{if } m=2 \text{ and } n=1,\\ \langle \tau G'_{n,m}, \sigma\rho G'_{n,m}\rangle & \text{if } m\geq 3 \text{ and } n=1,\\ \langle \sigma\rho G'_{n,m}\rangle & \text{otherwise}. \end{cases}$
4. $\ker \mathrm{V}_{H_{4,2}} = \begin{cases} \langle \tau G'_{n,m}, \rho G'_{n,m}\rangle & \text{if } m=2 \text{ and } n=1,\\ \langle \rho G'_{n,m}, \sigma\tau G'_{n,m}\rangle & \text{if } m\geq 3 \text{ and } n=1,\\ \langle \rho G'_{n,m}\rangle & \text{otherwise}. \end{cases}$
5. $\ker \mathrm{V}_{H_{5,2}} = \begin{cases} \langle \rho G'_{n,m}, \sigma\tau G'_{n,m}\rangle & \text{if } m=2 \text{ and } n=1,\\ \langle \rho G'_{n,m}, \tau G'_{n,m}\rangle & \text{if } m\geq 3 \text{ and } n=1,\\ \langle \rho G'_{n,m}\rangle & \text{otherwise}. \end{cases}$
6. $\ker \mathrm{V}_{H_{6,2}} = \begin{cases} \langle \tau\rho G'_{n,m}, \sigma G'_{n,m}\rangle & \text{if } n=1,\\ \langle \sigma G'_{n,m}\rangle & \text{if } n\geq 2. \end{cases}$
7. $\ker \mathrm{V}_{H_{7,2}} = \begin{cases} \langle \sigma\rho G'_{n,m}, \tau G'_{n,m}\rangle & \text{if } m=2 \text{ and } n=1,\\ \langle \sigma\rho G'_{n,m}, \tau\rho G'_{n,m}\rangle & \text{if } m\geq 3 \text{ and } n=1,\\ \langle \sigma\rho G'_{n,m}\rangle & \text{otherwise}. \end{cases}$
8. *If* $n=1$, *then for all* $1\leq i\leq 7$, $\ker \mathrm{V}_{H_{i,4}} = G_{1,m}/G'_{1,m}$.

Proof: We prove only some assertions, the others are similarly shown.
1. We know, from the Table 1, that $H_{1,2} = \langle \sigma, \tau\rangle$, then $G_{m,n}/H_{1,2} = \{1,\ \rho H_{1,2}\}$ and $H'_{1,2} = \langle 1\rangle$. Hence, by Corollary 2.5 and Lemma 2.2, we get

* $\mathrm{V}_{G_{m,n}\to H_{1,2}}(\sigma G'_{m,n}) = \sigma^2[\sigma,\ \rho]H'_{1,2} = \sigma^2\sigma^{-2}H'_{1,2} = H'_{1,2}$.
* $\mathrm{V}_{G_{m,n}\to H_{1,2}}(\tau G'_{m,n}) = \tau^2[\tau,\ \rho]H'_{1,2} = \tau^2\rho^{-2}H'_{1,2} = \tau^2\rho^2H'_{1,2} \neq H'_{1,2}$.
* $\mathrm{V}_{G_{m,n}\to H_{1,2}}(\rho G'_{m,n}) = \rho^2H'_{1,2} \neq H'_{1,2}$.
* $\mathrm{V}_{G_{m,n}\to H_{1,2}}(\sigma\tau G'_{m,n}) = (\sigma\tau)^2[\sigma\tau, \rho]H'_{1,2} = (\sigma\tau)^2\sigma^{-2}\rho^2H'_{1,2} = \tau^2\rho^2H'_{1,2} \neq H'_{1,2}$.
* $\mathrm{V}_{G_{m,n}\to H_{1,2}}(\sigma\rho G'_{m,n}) = (\sigma\rho)^2H'_{1,2} = \rho^2H'_{1,2} \neq H'_{1,2}$.
* $\mathrm{V}_{G_{m,n}\to H_{1,2}}(\tau\rho G'_{m,n}) = (\tau\rho)^2H'_{1,2} = \tau^2H'_{1,2} \neq H'_{1,2}$.
* $\mathrm{V}_{G_{m,n}\to H_{1,2}}(\sigma\tau\rho G'_{m,n}) = (\sigma\tau\rho)^2H'_{1,2} = \tau^2H'_{1,2} \neq H'_{1,2}$.

Therefore $\ker \mathrm{V}_{H_{1,2}} = \langle \sigma G'_{m,n}\rangle$.

3. Similarly, from the Table 1, we get $H_{3,2} = \begin{cases} \langle \tau, \rho\rangle & \text{if } m = 2, \\ \langle \tau, \rho, \sigma^2\rangle & \text{if } m \geq 3. \end{cases}$ Then $G_{m,n}/H_{3,2} = \{1,\ \sigma H_{3,2}\}$ and $H'_{3,2} = \begin{cases} \langle \rho^2\rangle = \langle (\sigma\tau)^2\rangle & \text{if } m = 2 \text{ and } n = 1, \\ \langle \rho^2\rangle = \langle \sigma^2\tau^{2^n}\rangle & \text{if } m = 2 \text{ and } n \geq 2, \\ \langle \rho^2, \sigma^4\rangle = \langle \sigma^4, \tau^2\rangle & \text{if } m \geq 3 \text{ and } n = 1, \\ \langle \rho^2, \sigma^4\rangle = \langle \sigma^4, \tau^{2^n}\rangle & \text{if } m \geq 3 \text{ and } n \geq 2. \end{cases}$

Hence, by Corollary 2.5 and Lemma 2.2, we get

1st case: m=2.

* $\mathrm{V}_{G_{m,n}\to H_{3,2}}(\sigma G'_{m,n}) = \sigma^2 H'_{3,2} \neq H'_{3,2}$.
* $\mathrm{V}_{G_{m,n}\to H_{3,2}}(\tau G'_{m,n}) = \tau^2[\tau,\ \sigma]H'_{3,2} = \tau^2 H'_{3,2} = H'_{3,2}$.
* $\mathrm{V}_{G_{m,n}\to H_{3,2}}(\rho G'_{m,n}) = \rho^2[\rho, \sigma]H'_{3,2} = \rho^2\sigma^2 H'_{3,2} \neq H'_{3,2}$.
* $\mathrm{V}_{G_{m,n}\to H_{3,2}}(\sigma\rho G'_{m,n}) = \rho^2 H'_{3,2} = H'_{3,2}$.
* $\mathrm{V}_{G_{m,n}\to H_{3,2}}(\sigma\tau G'_{m,n}) = (\sigma\tau)^2 H'_{3,2} = \begin{cases} H'_{3,2} & \text{if } n = 1, \\ (\sigma\tau)^2 H'_{3,2} \neq H'_{3,2} & \text{if } n \geq 2. \end{cases}$

Therefore $\ker \mathrm{V}_{H_{3,2}} = \begin{cases} \langle \tau\rho G'_{m,n}, \sigma\rho G'_{m,n}\rangle & \text{if } m = 2 \text{ and } n = 1, \\ \langle \sigma\rho G'_{m,n}\rangle & \text{if } m = 2 \text{ and } n \geq 2. \end{cases}$

2nd case: $m \geq 3$.

* $\mathrm{V}_{G_{m,n}\to H_{3,2}}(\sigma G'_{m,n}) = \sigma^2 H'_{3,2} \neq H'_{3,2}$.
* $\mathrm{V}_{G_{m,n}\to H_{3,2}}(\tau G'_{m,n}) = \tau^2[\tau,\ \sigma]H'_{3,2} = \tau^2 H'_{3,2} = \begin{cases} H'_{3,2} & \text{if } n = 1, \\ \tau^2 H'_{3,2} \neq H'_{3,2} & \text{if } n \geq 2. \end{cases}$
* $\mathrm{V}_{G_{m,n}\to H_{3,2}}(\rho G'_{m,n}) = \rho^2[\rho, \sigma]H'_{3,2} = \rho^2\sigma^2 H'_{3,2} \neq H'_{3,2}$.
* $\mathrm{V}_{G_{m,n}\to H_{3,2}}(\sigma\rho G'_{m,n}) = \rho^2 H'_{3,2} = H'_{3,2}$.
* $\mathrm{V}_{G_{m,n}\to H_{3,2}}(\sigma\tau G'_{m,n}) = (\sigma\tau)^2 H'_{3,2} \neq H'_{3,2}$.

Therefore, $\ker \mathrm{V}_{H_{3,2}} = \begin{cases} \langle \sigma\rho G'_{m,n}, \tau G'_{m,n}\rangle & \text{if } m \geq 3 \text{ and } n = 1, \\ \langle \sigma\rho G'_{m,n}\rangle & \text{if } m \geq 3 \text{ and } n \geq 2. \end{cases}$

8. We know, from the Table 2, that $H_{1,4} = \langle \sigma, \tau^2\rangle$, then $G_{n,m}/H_{1,4} = \{1,\ \tau H_{1,4},\ \rho H_{1,4},\ \tau\rho H_{1,4}\}$ and $H'_{1,4} = \langle 1\rangle$. Hence Corollary 2.7 and Lemma 2.2 yield that

* $\mathrm{V}_{G_{n,m}\to H_{1,4}}(\sigma G'_{n,m}) = \sigma\tau^{-1}\sigma\tau\rho^{-1}\sigma\tau^{-1}\sigma\tau\rho H'_{1,4} = \sigma^2\rho^{-1}\sigma^2\rho H'_{1,4} = H'_{1,4}$.
* $\mathrm{V}_{G_{n,m}\to H_{1,4}}(\tau G'_{n,m}) = \tau^2\rho^{-1}\tau^2\rho H'_{1,4} = \tau^4\tau^{-2}\rho^{-1}\tau^2\rho H'_{1,4} = \tau^4 H'_{1,4} = H'_{1,4}$.
* $\mathrm{V}_{G_{n,m}\to H_{1,4}}(\rho G'_{n,m}) = \rho^2\tau^{-1}\rho^{-2}\tau H'_{1,4} = \rho^4\rho^{-2}\tau^{-1}\rho^{-2}\tau H'_{1,4} = H'_{1,4}$.

Therefore, $\ker \mathrm{V}_{H_{1,4}} = \langle \sigma G'_{n,m},\ \tau G'_{n,m},\ \rho G'_{n,m}\rangle = G_{n,m}/G'_{n,m}$. □

3. Applications

Let $\mathbf{k}$ be a number field and $\mathrm{C}_{\mathbf{k},2}$ be its 2-class group, that is the 2-Sylow subgroup of the ideal class group $\mathrm{C}_{\mathbf{k}}$ of $\mathbf{k}$, in the wide sens. Let $\mathbf{k}_2^{(1)}$ be the Hilbert 2-class field of $\mathbf{k}$ in the wide sens. Then the Hilbert 2-class field tower of $\mathbf{k}$ is defined inductively by: $\mathbf{k}_2^{(0)} = \mathbf{k}$ and $\mathbf{k}_2^{(\ell+1)} = (\mathbf{k}_2^{(\ell)})^{(1)}$, where ℓ is a positive integer. Let $\mathbb{M}$ be an unramified extension of $\mathbf{k}$ and $\mathrm{C}_{\mathbb{M}}$ be the subgroup of $\mathrm{C}_{\mathbf{k}}$ associated to $\mathbb{M}$ by Class Field Theory. Denote by $j_{\mathbf{k}\to\mathbb{M}} : \mathrm{C}_{\mathbf{k}} \longrightarrow \mathrm{C}_{\mathbb{M}}$ the homomorphism that associates to the class of an ideal $\mathcal{A}$ of $\mathbf{k}$ the class of the ideal generated by $\mathcal{A}$ in $\mathbb{M}$, and by $\mathcal{N}_{\mathbb{M}/\mathbf{k}}$ the norm of the extension $\mathbb{M}/\mathbf{k}$.

Throughout all this section, assume that $Gal(\mathbf{k}_2^{(2)}/\mathbf{k}) \simeq G_{n,m}$. Hence, according to Class Field Theory, $\mathrm{C}_{\mathbf{k},2} \simeq G_{n,m}/G'_{n,m} \simeq (2,2,2^n)$, thus $\mathrm{C}_{\mathbf{k},2} = \langle \mathfrak{a}, \mathfrak{b}, \mathfrak{c}\rangle \simeq \langle \sigma G'_{n,m}, \tau G'_{n,m}, \rho G'_{n,m}\rangle$, where $(\mathfrak{a}, \mathbf{k}_2^{(2)}/\mathbf{k}) = \sigma G'_{n,m}$, $(\mathfrak{b}, \mathbf{k}_2^{(2)}/\mathbf{k}) = \tau G'_{n,m}$ and $(\mathfrak{c}, \mathbf{k}_2^{(2)}/\mathbf{k}) = \rho G'_{n,m}$, with $(\ .\ , \mathbf{k}_2^{(2)}/\mathbf{k})$ denotes the Artin symbol in $\mathbf{k}_2^{(2)}/\mathbf{k}$.

It is well known that each subgroup $H_{i,j}$, where $1 \le i \le 7$ and $j = 2$ or 4, of $\mathrm{C}_{\mathbf{k},2}$ is associated, by class field theory, to a unique unramified extension $\mathbf{K}_{i,j}$ of $\mathbf{k}_2^{(1)}$ such that $H_{i,j}/H'_{i,j} \simeq \mathrm{C}_{\mathbf{K}_{i,j},2}$.

Our goal is to study the capitulation problem of the 2-ideal classes of $\mathbf{k}$ in its unramified quadratic extensions $\mathbf{K}_{i,2}$ and in its unramified biquadratic extensions $\mathbf{K}_{i,4}$ if $n = 1$. By Class Field Theory, the kernel of $j_{\mathbf{k}\to\mathbb{M}}$, $\ker j_{\mathbf{k}\to\mathbb{M}}$, is determined by the kernel of the transfer map $\mathrm{V}_{G\to H} : G/G' \to H/H'$, where $G = \mathrm{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k})$ and $H = \mathrm{Gal}(\mathbb{M}_2^{(2)}/\mathbb{M})$.

Theorem 3.1. *Keep the previous notations.*

1. $\ker j_{\mathbf{k}\to\mathbf{K}_{1,2}} = \langle \mathfrak{a}\rangle$.
2. $\ker j_{\mathbf{k}\to\mathbf{K}_{2,2}} = \langle \mathfrak{a}, \mathfrak{c}\rangle$
3. $\ker j_{\mathbf{k}\to\mathbf{K}_{3,2}} = \begin{cases} \langle \mathfrak{bc}, \mathfrak{ac}\rangle & \textit{if } m = 2 \textit{ and } n = 1, \\ \langle \mathfrak{b}, \mathfrak{ac}\rangle & \textit{if } m \ge 3 \textit{ and } n = 1, \\ \langle \mathfrak{ac}\rangle & \textit{otherwise.} \end{cases}$
4. $\ker j_{\mathbf{k}\to\mathbf{K}_{4,2}} = \begin{cases} \langle \mathfrak{b}, \mathfrak{c}\rangle & \textit{if } m = 2 \textit{ and } n = 1, \\ \langle \mathfrak{ab}, \mathfrak{c}\rangle & \textit{if } m \ge 3 \textit{ and } n = 1, \\ \langle \mathfrak{c}\rangle & \textit{otherwise.} \end{cases}$
5. $\ker j_{\mathbf{k}\to\mathbf{K}_{5,2}} = \begin{cases} \langle \mathfrak{c}, \mathfrak{ab}\rangle & \textit{if } m = 2 \textit{ and } n = 1, \\ \langle \mathfrak{c}, \mathfrak{b}\rangle & \textit{if if } m \ge 3 \textit{ and } n = 1, \\ \langle \mathfrak{c}\rangle & \textit{otherwise.} \end{cases}$
6. $\ker j_{\mathbf{k}\to\mathbf{K}_{6,2}} = \begin{cases} \langle \mathfrak{bc}, \mathfrak{a}\rangle & \textit{if } n = 1, \\ \langle \mathfrak{a}\rangle & \textit{if if } n \ge 2. \end{cases}$
7. $\ker j_{\mathbf{k}\to\mathbf{K}_{7,2}} = \begin{cases} \langle \mathfrak{ac}, \mathfrak{b}\rangle & \textit{if } m = 2 \textit{ and } n = 1, \\ \langle \mathfrak{ac}, \mathfrak{bc}\rangle & \textit{if if } m \ge 3 \textit{ and } n = 1, \\ \langle \mathfrak{ac}\rangle & \textit{otherwise.} \end{cases}$

8. *If* $n=1$*, then for all* $1\leq i\leq 7$*,* $\ker j_{\mathbf{k}\to\mathbf{K}_{i,4}}=\mathrm{C}_{\mathbf{k},2}$*.*

9. *The* 2*-class group of* $\mathbf{k}_2^{(1)}$ *is of type* $\left(2,\ 2^{m-1}\right)$*.*

10. *The Hilbert* 2*-class field tower of* $\mathbf{k}$ *stops at* $\mathbf{k}_2^{(2)}$*.*

Proof: According to the Theorem 2.8, we have
1. Since $\ker \mathrm{V}_{H_{1,2}}=\langle\sigma G'_{n,m}\rangle$, so $\ker j_{\mathbf{k}\to\mathbf{K}_{1,2}}=\langle\mathfrak{a}\rangle$.
2. As $\ker \mathrm{V}_{H_{2,2}}=\langle\sigma G'_{n,m},\rho G'_{n,m}\rangle$, $\ker j_{\mathbf{k}\to\mathbf{K}_{3,2}}=\langle\mathfrak{a},\mathfrak{c}\rangle$.
3. Similarly, as $\ker \mathrm{V}_{H_{3,2}}=\begin{cases}\langle\tau\rho G'_{n,m},\sigma\rho G'_{n,m}\rangle & \text{if } m=2 \text{ and } n=1,\\ \langle\tau G'_{n,m},\sigma\rho G'_{n,m}\rangle & \text{if } m\geq 3 \text{ and } n=1,\\ \langle\sigma\rho G'_{n,m}\rangle & \text{otherwise}.\end{cases}$
Then $\ker j_{\mathbf{k}\to\mathbf{K}_{3,2}}=\begin{cases}\langle\mathfrak{bc},\mathfrak{ac}\rangle & \text{if } m=2 \text{ and } n=1,\\ \langle\mathfrak{b},\mathfrak{ac}\rangle & \text{if } m\geq 3 \text{ and } n=1,\\ \langle\mathfrak{ac}\rangle & \text{otherwise}.\end{cases}$

The other assertions are similarly proved.
8. It is well known, by class field theory, that $\mathrm{C}_{\mathbf{k}_2^{(1)},2}\simeq G'_{n,m}$, where $\mathrm{C}_{\mathbf{k}_2^{(1)},2}$ is the 2-class group of $\mathbf{k}_2^{(1)}$. As $G'_{n,m}=\langle\sigma^2,\ \tau^{2^n}\rangle\simeq\left(2,\ 2^{m-1}\right)$, since $\sigma^{2^m}=\tau^4=1$. So the result.
9. For every $n\geq 1$, we have $H_{1,4}=\langle\sigma,G'_{n,m}\rangle=\langle\sigma,\tau^{2^n}\rangle\simeq(2,2^n)$, $H_{2,4}=\langle\tau,\sigma^2\rangle\simeq(2^{n+1},2^{m-1})$ and $H_{4,4}=\langle\sigma\tau,G'_{n,m}\rangle=\langle\sigma\tau,\sigma^2\rangle\simeq\left(2^{\min(m-1,n)},2^{\max(m,n+1)}\right)$ are the three subgroups of index 2 of the group $H_{1,2}$, then $\mathbf{K}_{1,4}$, $\mathbf{K}_{2,4}$ and $\mathbf{K}_{4,4}$ are the three unramified quadratic extensions of $\mathbf{K}_{1,2}$. On the other hand, the 2-class groups of these fields are of rank 2, since, by Class Field Theory, $\mathrm{C}_{\mathbf{K}_{i,j},2}\simeq H_{i,j}/H'_{i,j}$ with $i=1$, 2 or 4 and $j=2$ or 4. Thus $\mathrm{C}_{\mathbf{K}_{1,2},2}\simeq(2^m,\ 4)$ and $\mathrm{C}_{\mathbf{K}_{2,4},2}\simeq\left(2^m,\ 2^{n+1}\right)$. Hence $h_2(\mathbf{K}_{2,4})=\frac{h_2(\mathbf{K}_{1,2})}{2}$, where $h_2(K)$ denotes the 2-class number of the field K. Therefore, we can apply Proposition 7 of [8], which says that $\mathbf{K}_{1,2}$ has an abelian 2-class field tower if and only if it has a quadratic unramified extension $\mathbf{K}_{2,4}/\mathbf{K}_{1,2}$ such that $h_2(\mathbf{K}_{2,4})=\frac{h_2(\mathbf{K}_{1,2})}{2}$. Thus $\mathbf{K}_{1,2}$ has abelian 2-class field tower which terminates at the first stage; this implies that the 2-class field tower of $\mathbf{k}$ terminates at $\mathbf{k}_2^{(2)}$, since $\mathbf{k}\subset\mathbf{K}_{1,2}$. Moreover, we know, from Proposition 2.3, that $|G_{n,m}|=2^{m+n+2}$ and $\left|G'_{n,m}\right|=2^m$, hence $\mathbf{k}_2^{(1)}\neq\mathbf{k}_2^{(2)}$. □

4. Example

Let $\mathbf{k}=\mathbb{Q}(\sqrt{d})$ be an imaginary quadratic number field with discriminant $d=-4pqq'$, where $p\equiv 5 \mod 8$, $q\equiv 3 \mod 8$ and $q'\equiv 7 \mod 8$ are primes such that $\left(\frac{q}{p}\right)=\left(\frac{q'}{p}\right)=-1$. Let $\mathbf{k}_2^{(1)}$ be the Hilbert 2-class field of $\mathbf{k}$, $\mathbf{k}_2^{(2)}$ its second Hilbert 2-class field and G be the Galois group of $\mathbf{k}_2^{(2)}/\mathbf{k}$. According to [10], $\mathbf{k}$ has an elementary abelian 2-class group $\mathbf{C}_{\mathbf{k},2}$ of rank 3, that is of type $(2,2,2)$. Denote by $h_2(-qq')$ the 2-class number of $\mathbb{Q}(\sqrt{-qq'})$, then by [15,10] $h_2(-qq')=2^m$ and the 2-class group of $\mathbb{Q}(\sqrt{-qq'})$ is of type $(2,2^{m-1})$ with $m\geq 2$. By [10, Theorem 1], we have $G\simeq G_{1,m}$. As $\mathbf{C}_{\mathbf{k},2}\simeq(2,2,2)$, then $\mathbf{k}$ has seven unramified quadratic extensions and seven unramified biquadratic extensions within his first Hilbert 2-class field $\mathbf{k}_2^{(1)}$. For more details about the results given in this section and about the following theorem the reader can see [10]. This theorem is given here to illustrate the results shown in the above sections.

Theorem 4.1. *Let* $\mathbf{k} = \mathbb{Q}(\sqrt{d})$ *be an imaginary quadratic number field with discriminant* $d = -4pqq'$, *where* $p \equiv 5 \mod 8$, $q \equiv 3 \mod 8$ *and* $q' \equiv 7 \mod 8$ *are primes such that* $\left(\frac{q}{p}\right) = \left(\frac{q'}{p}\right) = -1$. $\mathbf{k}$ *has fourteen unramified extensions within his first Hilbert* 2*-class field,* $\mathbf{k}_2^{(1)}$. *Denote by* $\mathbf{C}_{\mathbf{k},2}$ *the* 2*-class group of* $\mathbf{k}$. *Then the following assertions hold.*

1. $\mathbf{C}_{\mathbf{k},2}$ *is of type* $(2, 2, 2)$.
2. *Exactly four elements of* $\mathbf{C}_{\mathbf{k},2}$ *capitulate in each unramified quadratic extension of* $\mathbf{k}$ *except one where only* 2 *classes capitulate.*
3. *All the* 2*-classes of* $\mathbf{k}$ *capitulate in each unramified biquadratic extension of* $\mathbf{k}$.
4. *The Hilbert* 2*-class field tower of* $\mathbf{k}$ *stops at* $\mathbf{k}_2^{(2)}$.
5. $\mathbf{C}_{\mathbf{k}_2^{(1)},2} \simeq (2,\ 2^{m-1})$.
6. *The coclass of* G *is* 3 *and its nilpotency class is* m.
7. *The* 2*-class groups of the unramified quadratic extensions of* $\mathbf{k}$ *are of types* $(2,\ 4)$, $(2,\ 2,\ 2)$ *or* $(4,\ 2^m)$.
8. *The* 2*-class groups of the unramified biquadratic extensions of* $\mathbf{k}$ *are of types* $(2,\ 4)$, $(2,\ 2^m)$ *or* $(4,\ 2^{m-1})$.

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