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# Strong Insertion of a $\gamma-$continuous Function

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ABSTRACT: Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of a $\gamma-$continuous function between two comparable real-valued functions.

Key Words: Strong insertion, Strong binary relation, Preopen set, Semi-open set, $\gamma-$open set.

**Contents**



## 1. Introduction

The concept of a preopen set in a topological space was introduced by H. H. Corson and E. Michael in 1964 [4]. A subset $A$ of a topological space $(X, \tau)$ is called *preopen* or *locally dense* or *nearly open* if $A \subseteq Int(Cl(A))$. A set $A$ is called *preclosed* if its complement is preopen or equivalently if $Cl(Int(A)) \subseteq A$. The term ,preopen, was used for the first time by A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb [12], while the concept of a , locally dense, set was introduced by H. H. Corson and E. Michael [4].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [11]. A subset $A$ of a topological space $(X, \tau)$ is called *semi-open* [11] if $A \subseteq Cl(Int(A))$. A set $A$ is called *semi-closed* if its complement is semi-open or equivalently if $Int(Cl(A)) \subseteq A$.

Recall that a subset $A$ of a topological space $(X, \tau)$ is called $\gamma-$*open* if $A \cap S$ is preopen, whenever $S$ is preopen [1]. A set $A$ is called $\gamma-$closed if its complement is $\gamma-$open or equivalently if $A \cup S$ is preclosed, whenever $S$ is preclosed.

we have that if a set is $\gamma-$open then it is semi-open and preopen.

Recall that a real-valued function $f$ defined on a topological space $X$ is called $A-$continuous [15] if the preimage of every open subset of $\mathbb{R}$ belongs to $A$, where $A$ is a collection of subset of $X$. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [5,6].

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Hence, a real-valued function $f$ defined on a topological space $X$ is called precontinuous (resp. semi-continuous or $\gamma-$continuous) if the preimage of every open subset of $\mathbb{R}$ is preopen (resp. semi-open or $\gamma-$open) subset of $X$.

Precontinuity was called by V. Ptak nearly continuity [16].Nearly continuity or precontinuity is known also as almost continuity by T. Husain [7].Precontinuity was studied for real-valued functions on Euclidean space by Blumberg back in 1922 [2].

Results of Katětov [8,9] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [3], are used in order to give necessary and sufficient conditions for the strong insertion of a $\gamma-$continuous function between two comparable real-valued functions.

If $g$ and $f$ are real-valued functions defined on a space $X$, we write $g \leq f$ in case $g(x) \leq f(x)$ for all $x$ in $X$.

The following definitions are modifications of conditions considered in [10].

A property $P$ defined relative to a real-valued function on a topological space is a $\gamma-$*property* provided that any constant function has property $P$ and provided that the sum of a function with property $P$ and any $\gamma-$continuous function also has property $P$. If $P_1$ and $P_2$ are $\gamma-$property, the following terminology is used:(i) A space $X$ has the *weak* $\gamma-$*insertion property* for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g \leq f, g$ has property $P_1$ and $f$ has property $P_2$, then there exists a $\gamma-$continuous function $h$ such that $g \leq h \leq f$.(ii) A space $X$ has the *strong* $\gamma-$*insertion property* for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g \leq f, g$ has property $P_1$ and $f$ has property $P_2$, then there exists a $\gamma-$continuous function $h$ such that $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$.

In this paper, we give a sufficient condition for the weak $\gamma-$insertion property. Also for a space with the weak $\gamma-$insertion property, we give necessary and sufficient conditions for the space to have the strong $\gamma-$insertion property. Several insertion theorems are obtained as corollaries of these results. In addition, the insertion of a $\gamma-$continuous function has also considered by the author in [14]

## 2. The Main Result

Before giving a sufficient condition for insertability of a $\gamma-$continuous function, the necessary definitions and terminology are stated.

The abbreviations $pc$ , $sc$ and $\gamma c$ are used for precontinuous , semicontinuous and $\gamma-$continuous, respectively.

Let $(X, \tau)$ be a topological space, the family of all $\gamma-$open, $\gamma-$closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\gamma O(X, \tau)$, $\gamma C(X, \tau)$, $sO(X, \tau)$, $sC(X, \tau)$, $pO(X, \tau)$ and $pC(X, \tau)$, respectively.

**Definition 2.1.** *Let $A$ be a subset of a topological space $(X, \tau)$. Respectively, we define the $\gamma-$closure, $\gamma-$interior, s-closure, s-interior, p-closure and p-interior* *of a set $A$, denoted by $\gamma Cl(A), \gamma Int(A), sCl(A), sInt(A), pCl(A)$ and $pInt(A)$ as follows:*

$\gamma Cl(A) = \cap\{F : F \supseteq A, F \in \gamma C(X, \tau)\}$,
$\gamma Int(A) = \cup\{O : O \subseteq A, O \in \gamma O(X, \tau)\}$,
$sCl(A) = \cap\{F : F \supseteq A, F \in sC(X, \tau)\}$,
$sInt(A) = \cup\{O : O \subseteq A, O \in sO(X, \tau)\}$,
$pCl(A) = \cap\{F : F \supseteq A, F \in pC(X, \tau)\}$ *and*
$pInt(A) = \cup\{O : O \subseteq A, O \in pO(X, \tau)\}$.

*Respectively, we have $\gamma Cl(A), sCl(A), pCl(A)$ are $\gamma-$closed, semi-closed, preclosed and $\gamma Int(A), sInt(A), pInt(A)$ are $\gamma-$open, semi-open, preopen.*

The following first two definitions are modifications of conditions considered in [8,9].

**Definition 2.2.** *If $\rho$ is a binary relation in a set $S$ then $\bar{\rho}$ is defined as follows: $x \ \bar{\rho} \ y$ if and only if $y \ \rho \ \nu$ implies $x \ \rho \ \nu$ and $u \ \rho \ x$ implies $u \ \rho \ y$ for any $u$ and $v$ in $S$.*

**Definition 2.3.** *A binary relation $\rho$ in the power set $P(X)$ of a topological space $X$ is called a strong binary relation in $P(X)$ in case $\rho$ satisfies each of the following conditions:*

*1) If $A_i \ \rho \ B_j$ for any $i \in \{1, \dots, m\}$ and for any $j \in \{1, \dots, n\}$, then there exists a set $C$ in $P(X)$ such that $A_i \ \rho \ C$ and $C \ \rho \ B_j$ for any $i \in \{1, \dots, m\}$ and any $j \in \{1, \dots, n\}$.*

*2) If $A \subseteq B$, then $A \ \bar{\rho} \ B$.*

*3) If $A \ \rho \ B$, then $\gamma Cl(A) \subseteq B$ and $A \subseteq \gamma Int(B)$.*

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [3] as follows:

**Definition 2.4.** *If $f$ is a real-valued function defined on a space $X$ and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number $\ell$, then $A(f, \ell)$ is called a lower indefinite cut set in the domain of $f$ at the level $\ell$.*

We now give the following main result:

**Theorem 2.1.** *Let $g$ and $f$ be real-valued functions on a topological space $X$ with $g \leq f$. If there exists a strong binary relation $\rho$ on the power set of $X$ and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of $f$ and $g$ at the level $t$ for each rational number $t$ such that if $t_1 < t_2$ then $A(f, t_1) \ \rho \ A(g, t_2)$, then there exists a $\gamma-$continuous function $h$ defined on $X$ such that $g \leq h \leq f$.*

**Proof:** Let $g$ and $f$ be real-valued functions defined on $X$ such that $g \leq f$. By hypothesis there exists a strong binary relation $\rho$ on the power set of $X$ and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of $f$ and $g$ at the level $t$ for each rational number $t$ such that if $t_1 < t_2$ then $A(f, t_1) \ \rho \ A(g, t_2)$.

Define functions $F$ and $G$ mapping the rational numbers $\mathbb{Q}$ into the power set of $X$ by $F(t) = A(f,t)$ and $G(t) = A(g,t)$. If $t_1$ and $t_2$ are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2), G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [9] it follows that there exists a function $H$ mapping $\mathbb{Q}$ into the power set of $X$ such that if $t_1$ and $t_2$ are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2), H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$.

For any $x$ in $X$, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If $x$ is in $H(t)$ then $x$ is in $G(t')$ for any $t' > t$; since $x$ is in $G(t') = A(g,t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If $x$ is not in $H(t)$, then $x$ is not in $F(t')$ for any $t' < t$; since $x$ is not in $F(t') = A(f,t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers $t_1$ and $t_2$ with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = \gamma Int(H(t_2)) \setminus \gamma Cl(H(t_1))$. Hence $h^{-1}(t_1,t_2)$ is a $\gamma-$open subset of $X$, i. e., $h$ is a $\gamma-$continuous function on $X$. □

The above proof used the technique of proof of Theorem 1 of [8].

If a space has the strong $\gamma-$insertion property for $(P_1,P_2)$, then it has the weak $\gamma-$insertion property for $(P_1,P_2)$.The following result uses lower cut sets and gives a necessary and sufficient condition for a space satisfies that weak $\gamma-$insertion property to satisfy the strong $\gamma-$insertion property.

**Theorem 2.2.** *Let $P_1$ and $P_2$ be $\gamma-$property and $X$ be a space that satisfies the weak $\gamma-$insertion property for $(P_1,P_2)$. Also assume that $g$ and $f$ are functions on $X$ such that $g \leq f, g$ has property $P_1$ and $f$ has property $P_2$. The space $X$ has the strong $\gamma-$insertion property for $(P_1,P_2)$ if and only if there exist lower cut sets $A(f-g,2^{-n})$ and there exists a sequence $\{F_n\}$ of subsets of $X$ such that (i) for each $n, F_n$ and $A(f-g,2^{-n})$ are completely separated by $\gamma-$continuous functions, and (ii)$\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$.*

**Proof:** Theorem 3.1, of [13]. □

**Theorem 2.3.** *Let $P_1$ and $P_2$ be $\gamma-$properties and assume that the space $X$ satisfied the weak $\gamma-$insertion property for $(P_1,P_2)$. The space $X$ satisfies the strong $\gamma-$insertion property for $(P_1,P_2)$ if and only if $X$ satisfies the strong $\gamma-$insertion property for $(P_1,\gamma c)$ and for $(\gamma c,P_2)$.*

**Proof:** Theorem 3.2, of [13]. □

## 3. Applications

**Corollary 3.1.** *If for each pair of disjoint preclosed (resp. semi-closed) sets $F_1,F_2$ of $X$ , there exist $\gamma-$open sets $G_1$ and $G_2$ of $X$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then $X$ has the weak $\gamma-$insertion property for $(pc,pc)$ (resp. $(sc,sc)$).*

**Proof:** Let $g$ and $f$ be real-valued functions defined on the $X$, such that $f$ and $g$ are $pc$ (resp. $sc$), and $g \leq f$.If a binary relation $\rho$ is defined by $A\ \rho\ B$ in case $pCl(A) \subseteq pInt(B)$ (resp. $sCl(A) \subseteq sInt(B)$), then by hypothesis $\rho$ is a strong binary relation in the power set of $X$. If $t_1$ and $t_2$ are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a preclosed (resp. semi-closed) set and since $\{x \in X : g(x) < t_2\}$ is a preopen (resp. semi-open) set, it follows that $pCl(A(f, t_1)) \subseteq pInt(A(g, t_2))$ (resp. $sCl(A(f, t_1)) \subseteq sInt(A(g, t_2))$). Hence $t_1 < t_2$ implies that $A(f, t_1)\ \rho\ A(g, t_2)$. The proof follows from Theorem 2.1. □

**Corollary 3.2.** *If for each pair of disjoint preclosed (resp. semi-closed) sets $F_1, F_2$, there exist $\gamma-$open sets $G_1$ and $G_2$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then every precontinuous (resp. semi-continuous) function is $\gamma-$continuous.*

**Proof:** Let $f$ be a real-valued precontinuous (resp. semi-continuous) function defined on the $X$. Set $g = f$, then by Corollary 3.1, there exists a $\gamma-$continuous function $h$ such that $g = h = f$. □

**Corollary 3.3.** *If for each pair of disjoint preclosed (resp. semi-closed) sets $F_1, F_2$ of $X$ , there exist $\gamma-$open sets $G_1$ and $G_2$ of $X$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then $X$ has the strong $\gamma-$insertion property for $(pc, pc)$ (resp. $(sc, sc)$).*

**Proof:** Let $g$ and $f$ be real-valued functions defined on the $X$, such that $f$ and $g$ are $pc$ (resp. $sc$), and $g \leq f$. Set $h = (f + g)/2$, thus $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. Also, by Corollary 3.2, since $g$ and $f$ are $\gamma-$continuous functions hence $h$ is $\gamma-$continuous function. □

**Corollary 3.4.** *If for each pair of disjoint subsets $F_1, F_2$ of $X$ , such that $F_1$ is preclosed and $F_2$ is semi-closed, there exist $\gamma-$open subsets $G_1$ and $G_2$ of $X$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then $X$ have the weak $\gamma-$insertion property for $(pc, sc)$ and $(sc, pc)$.*

**Proof:** Let $g$ and $f$ be real-valued functions defined on the $X$, such that $g$ is $pc$ (resp. $sc$) and $f$ is $sc$ (resp. $pc$), with $g \leq f$.If a binary relation $\rho$ is defined by $A\ \rho\ B$ in case $sCl(A) \subseteq pInt(B)$ (resp. $pCl(A) \subseteq sInt(B)$), then by hypothesis $\rho$ is a strong binary relation in the power set of $X$. If $t_1$ and $t_2$ are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a semi-closed (resp. preclosed) set and since $\{x \in X : g(x) < t_2\}$ is a preopen (resp. semi-open) set, it follows that $sCl(A(f, t_1)) \subseteq$

$pInt(A(g, t_2))$ (resp. $pCl(A(f, t_1)) \subseteq sInt(A(g, t_2)))$. Hence $t_1 < t_2$ implies that $A(f, t_1)\ \rho\ A(g, t_2)$. The proof follows from Theorem 2.1. □

Before stating the consequences of Theorems 2.2, and 2.3, we state and prove the necessary lemmas.

**Lemma 3.1.** *The following conditions on the space $X$ are equivalent:*

*(i) For each pair of disjoint subsets $F_1, F_2$ of $X$, such that $F_1$ is preclosed and $F_2$ is semi-closed, there exist $\gamma-$open subsets $G_1, G_2$ of $X$ such that $F_1 \subseteq G_1, F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$.*

*(ii) If $F$ is a semi-closed (resp. preclosed) subset of $X$ which is contained in a preopen (resp. semi-open) subset $G$ of $X$, then there exists an $\gamma-$open subset $H$ of $X$ such that $F \subseteq H \subseteq \gamma Cl(H) \subseteq G$.*

**Proof:** (i) $\Rightarrow$ (ii) Suppose that $F \subseteq G$, where $F$ and $G$ are semi-closed (resp. preclosed) and preopen (resp. semi-open) subsets of $X$, respectively. Hence, $G^c$ is a preclosed (resp. semi-closed) and $F \cap G^c = \varnothing$.

By (i) there exists two disjoint $\gamma-$open subsets $G_1, G_2$ of $X$ s. t., $F \subseteq G_1$ and $G^c \subseteq G_2$. But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \varnothing \Rightarrow G_1 \subseteq G_2^c$$

hence

$$F \subseteq G_1 \subseteq G_2^c \subseteq G$$

and since $G_2^c$ is a $\gamma-$closed set containing $G_1$ we conclude that $\gamma Cl(G_1) \subseteq G_2^c$, i. e.,

$$F \subseteq G_1 \subseteq \gamma Cl(G_1) \subseteq G.$$

By setting $H = G_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that $F_1, F_2$ are two disjoint subsets of $X$, such that $F_1$ is preclosed and $F_2$ is semi-closed.

This implies that $F_2 \subseteq F_1^c$ and $F_1^c$ is a preopen subset of $X$. Hence by (ii) there exists a $\gamma-$open set $H$ s. t., $F_2 \subseteq H \subseteq \gamma Cl(H) \subseteq F_1^c$.

But

$$H \subseteq \gamma Cl(H) \Rightarrow H \cap (\gamma Cl(H))^c = \varnothing$$

and

$$\gamma Cl(H) \subseteq F_1^c \Rightarrow F_1 \subseteq (\gamma Cl(H))^c.$$

Furthermore, $(\gamma Cl(H))^c$ is a $\gamma-$open set of $X$. Hence $F_2 \subseteq H, F_1 \subseteq (\gamma Cl(H))^c$ and $H \cap (\gamma Cl(H))^c = \varnothing$. This means that condition (i) holds. □

**Lemma 3.2.** *Suppose that $X$ is a topological space. If each pair of disjoint subsets $F_1, F_2$ of $X$, where $F_1$ is preclosed and $F_2$ is semi-closed, can separate by $\gamma-$open subsets of $X$ then there exists a $\gamma-$continuous function $h : X \to [0, 1]$ s. t., $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.*

**Proof:** Suppose $F_1$ and $F_2$ are two disjoint subsets of $X$, where $F_1$ is preclosed and $F_2$ is semi-closed. Since $F_1 \cap F_2 = \varnothing$, hence $F_2 \subseteq F_1^c$. In particular, since $F_1^c$ is a preopen subset of $X$ containing semi-closed subset $F_2$ of $X$, by Lemma 3.1, there exists a $\gamma-$open subset $H_{1/2}$ of $X$ s. t.,

$$F_2 \subseteq H_{1/2} \subseteq \gamma Cl(H_{1/2}) \subseteq F_1^c.$$

Note that $H_{1/2}$ is also a preopen subset of $X$ and contains $F_2$, and $F_1^c$ is a preopen subset of $X$ and contains a semi-closed subset $\gamma Cl(H_{1/2})$ of $X$. Hence, by Lemma 3.1, there exists $\gamma-$open subsets $H_{1/4}$ and $H_{3/4}$ s. t.,

$$F_2 \subseteq H_{1/4} \subseteq \gamma Cl(H_{1/4}) \subseteq H_{1/2} \subseteq \gamma Cl(H_{1/2}) \subseteq H_{3/4} \subseteq \gamma Cl(H_{3/4}) \subseteq F_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0, 1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain $\gamma-$open subsets $H_t$ of $X$ with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function $h$ on $X$ by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin F_1$ and $h(x) = 1$ for $x \in F_1$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i. e., $h$ maps $X$ into [0,1]. Also, we note that for any $t \in D, F_2 \subseteq H_t$; hence $h(F_2) = \{0\}$. Furthermore, by definition, $h(F_1) = \{1\}$. It remains only to prove that $h$ is a $\gamma-$continuous function on $X$. For every $\beta \in \mathbb{R}$, we have if $\beta \leq 0$ then $\{x \in X : h(x) < \beta\} = \varnothing$ and if $0 < \beta$ then $\{x \in X : h(x) < \beta\} = \cup\{H_t : t < \beta\}$, hence, they are $\gamma-$open subsets of $X$. Similarly, if $\beta < 0$ then $\{x \in X : h(x) > \beta\} = X$ and if $0 \leq \beta$ then $\{x \in X : h(x) > \beta\} = \cup\{(\gamma Cl(H_t))^c : t > \beta\}$ hence, every of them is a $\gamma-$open subset of $X$. Consequently $h$ is a $\gamma-$continuous function. □

**Lemma 3.3.** *Suppose that $X$ is a topological space. If each pair of disjoint subsets $F_1, F_2$ of $X$, where $F_1$ is preclosed and $F_2$ is semi-closed, can separate by $\gamma-$open subsets of $X$, and $F_1$ (resp. $F_2$) is a countable intersection of $\gamma-$open subsets of $X$, then there exists a $\gamma-$continuous function $h : X \to [0, 1]$ s. t., $h^{-1}(0) = F_1$ (resp. $h^{-1}(0) = F_2$) and $h(F_2) = \{1\}$ (resp. $h(F_1) = \{1\}$).*

**Proof:** Suppose that $F_1 = \bigcap_{n=1}^{\infty} G_n$ (resp. $F_2 = \bigcap_{n=1}^{\infty} G_n$), where $G_n$ is a $\gamma-$open subset of $X$. We can suppose that $G_n \cap F_2 = \varnothing$ (resp. $G_n \cap F_1 = \varnothing$), otherwise we can substitute $G_n$ by $G_n \setminus F_2$ (resp. $G_n \setminus F_1$). By Lemma 3.2, for every $n \in \mathbb{N}$, there exists a $\gamma-$continuous function $h_n : X \to [0, 1]$ s. t., $h_n(F_1) = \{0\}$ (resp. $h_n(F_2) = \{0\}$) and $h_n(X \setminus G_n) = \{1\}$. We set $h(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x)$.

Since the above series is uniformly convergent, it follows that $h$ is a $\gamma-$ continuous function from $X$ to $[0, 1]$. Since for every $n \in \mathbb{N}, F_2 \subseteq X \setminus G_n$ (resp. $F_1 \subseteq X \setminus G_n$), therefore $h_n(F_2) = \{1\}$ (resp. $h_n(F_1) = \{1\}$) and consequently $h(F_2) = \{1\}$ (resp. $h(F_1) = \{1\}$). Since $h_n(F_1) = \{0\}$ (resp. $h_n(F_2) = \{0\}$), hence $h(F_1) = \{0\}$ (resp. $h(F_2) = \{0\}$). It suffices to show that if $x \notin F_1$ (resp. $x \notin F_2$), then $h(x) \neq 0$.

Now if $x \notin F_1$ (resp. $x \notin F_2$), since $F_1 = \bigcap_{n=1}^{\infty} G_n$ (resp. $F_2 = \bigcap_{n=1}^{\infty} G_n$), therefore there exists $n_0 \in \mathbb{N}$ s. t., $x \notin G_{n_0}$, hence $h_{n_0}(x) = 1$, i. e., $h(x) > 0$. Therefore $h^{-1}(0) = F_1$ (resp. $h^{-1}(0) = F_2$). □

**Lemma 3.4.** *Suppose that $X$ is a topological space such that every two disjoint semi-closed and preclosed subsets of $X$ can be separated by $\gamma-$open subsets of $X$. The following conditions are equivalent:*

*(i) For every two disjoint subsets $F_1$ and $F_2$ of $X$, where $F_1$ is preclosed and $F_2$ is semi-closed, there exists a $\gamma-$continuous function $h : X \to [0,1]$ s. t., $h^{-1}(0) = F_1$ (resp. $h^{-1}(0) = F_2$) and $h^{-1}(1) = F_2$ (resp. $h^{-1}(1) = F_1$).*

*(ii) Every preclosed (resp. semi-closed) subset of $X$ is a countable intersection of $\gamma-$open subsets of $X$.*

*(iii) Every preopen (resp. semi-open) subset of $X$ is a countable union of $\gamma-$closed subsets of $X$.*

**Proof:** (i) $\Rightarrow$ (ii) Suppose that $F$ is a preclosed (resp. semi-closed) subset of $X$. Since $\varnothing$ is a semi-closed (resp. preclosed) subset of $X$, by (i) there exists a $\gamma-$continuous function $h : X \to [0,1]$ s.t., $h^{-1}(0) = F$. Set $G_n = \{x \in X : h(x) < \frac{1}{n}\}$. Then for every $n \in \mathbb{N}$, $G_n$ is a $\gamma-$open subset of $X$ and $\bigcap_{n=1}^{\infty} G_n = \{x \in X : h(x) = 0\} = F$.

(ii) $\Rightarrow$ (i) Suppose that $F_1$ and $F_2$ are two disjoint subsets of $X$, where $F_1$ is preclosed and $F_2$ is semi-closed. By Lemma 3.3, there exists a $\gamma-$continuous function $f : X \to [0,1]$ s. t., $f^{-1}(0) = F_1$ and $f(F_2) = \{1\}$. Set $G = \{x \in X : f(x) < \frac{1}{2}\}$, $F = \{x \in X : f(x) = \frac{1}{2}\}$, and $H = \{x \in X : f(x) > \frac{1}{2}\}$. Then $G \cup F$ and $H \cup F$ are two $\gamma-$closed subsets of $X$ and $(G \cup F) \cap F_2 = \varnothing$. By Lemma 3.3, there exists a $\gamma-$continuous function $g : X \to [\frac{1}{2}, 1]$ s. t., $g^{-1}(1) = F_2$ and $g(G \cup F) = \{\frac{1}{2}\}$. Define $h$ by $h(x) = f(x)$ for $x \in G \cup F$, and $h(x) = g(x)$ for $x \in H \cup F$. Then $h$ is well-defined and a $\gamma-$continuous function, since $(G \cup F) \cap (H \cup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \cup F) \cup (H \cup F) = X$, hence $h$ defined on $X$ and maps to $[0,1]$. Also, we have $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$.

(ii) $\Leftrightarrow$ (iii) By De Morgan law and noting that the complement of every $\gamma-$open subset of $X$ is a $\gamma-$closed subset of $X$ and complement of every $\gamma-$closed subset of $X$ is a $\gamma-$open subset of $X$, the equivalence is hold. □

**Corollary 3.5.** *If for every two disjoint subsets $F_1$ and $F_2$ of $X$, where $F_1$ is preclosed (resp. semi-closed) and $F_2$ is semi-closed (resp. preclosed), there exists a $\gamma-$continuous function $h : X \to [0,1]$ s. t., $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$ then $X$ has the strong $\gamma-$insertion property for $(pc, sc)$ (resp. $(sc, pc)$).*

**Proof:** Since for every two disjoint subsets $F_1$ and $F_2$ of $X$, where $F_1$ is preclosed (resp. semi-closed) and $F_2$ is semi-closed (resp. preclosed), there exists a $\gamma-$continuous function $h : X \to [0,1]$ s. t., $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$, define $G_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $G_2 = \{x \in X : h(x) > \frac{1}{2}\}$. Then $G_1$ and $G_2$ are two disjoint $\gamma-$open subsets of $X$ that contain $F_1$ and $F_2$, respectively. Hence by Corollary 3.4, $X$ has the weak $\gamma-$insertion property for $(pc, sc)$ and $(sc, pc)$. Now, assume that $g$ and $f$ are functions on $X$ such that $g \leq f, g$ is $pc$ (resp. $sc$) and $f$ is $\gamma c$. Since $f - g$ is $pc$ (resp. $sc$), therefore the lower cut set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}$ is a preclosed (resp. semi-closed) subset of $X$. By Lemma 3.4, we can choose a sequence $\{F_n\}$ of $\gamma-$closed subsets of $X$ s. t., $\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$ and for every $n \in \mathbb{N}, F_n$ and $A(f-g, 2^{-n})$ are disjoint subsets of $X$. By Lemma 3.2, $F_n$ and $A(f-g, 2^{-n})$ can be completely separated by $\gamma-$continuous functions. Hence by Theorem 2.2, $X$ has the strong $\gamma-$insertion property for $(pc, \gamma c)$ (resp. $(sc, \gamma c)$).

By an analogous argument, we can prove that $X$ has the strong $\gamma-$insertion property for $(\gamma c, sc)$ (resp. $(\gamma c, pc)$). Hence, by Theorem 2.3, $X$ has the strong $\gamma-$insertion property for $(pc, sc)$ (resp. $(sc, pc)$). □

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