Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 34** 2 (2016): 237–261.
ISSN-00378712 in press
doi:10.5269/bspm.v34i2.28146

Limit analysis of an elastic thin oscillating layer

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Abstract: The aim of this paper, is to study the limit behavior of the solution of a convex elasticity problem with a negative power type, of a containing structure, an elastic thin oscillating layer of thickness and periodicity parameter depending of a small enough parameter ε. The epi-convergence method is considered to find the limit problems with interface conditions.

Key Words: Limit behavior, elasticity problem, epi-convergence method, limit problems

Contents

1. Introduction

The problem of inclusion of an elastic thin layer between two elastic bodies, where the thin layer obeys to a nonlinear elastic law of power type, is widely studied in many works. Here we can mention some authors who treated the elasticity and thermal problems on the structure containing a thin plate layer, in one hand the thermal case was treated by Brillard and Sanchez-Palencia et al. in [7,12]. In the other hand the elasticity case was studied by Ait Moussa et al., Brillard et al., Geymonat et al. and Lenci et al. in [2,9,10,11]. The plastic plate case was treated by Messaho et al. in [4]. From the mechanical point of view for the thin layer and the paper of Acerbi et al. [1], we are motivated to studying the elasticity problems on a structure containing a thin oscillating layer. It is, therefore of interest to study the limit behavior of thin layer with an oscillating boundary, with a small enough periodicity parameter, between the two adherents when the thickness, rigidity and

2000 *Mathematics Subject Classification:* 35B40, 82B24, 76M50 periodicity parameters depending on a small enough parameter intended to tend towards 0 where the rigidity parameter is great enough.

In this present work, we consider a structure containing a thin oscillating layer of thickness, rigidity and periodicity parameter depending on ε being a parameter intended to tend towards 0. In a such structure we have treated the scalar case for a thermal conductivity problem in Messaho et al in [3]. The aim of this work is to study the limit behavior of an elasticity problem where a convex energy functional defined in a such structure.

This paper is organized in the following way. In section 2, we express the problem to study, and we give some notations and we define functional spaces for this study in the section 3. In the section 4, we study the problem (4.1). The section 5 is devoted to the determination of the limits problems and our main result.

2. Statement of the problem

We consider a structure, occupying a bonded domain $\Omega \subset \mathbb{R}^3$ with lipschitzian boundary $\partial\Omega$. It is constituted of two elastics bodies joined together by a thin layer with oscillating boundary (see figure 1), the latter obeys to a nonlinear elastic law of power type. More precisely the stress field is related to the field of displacement by

$$\sigma^\varepsilon \quad = \quad \frac{1}{\varepsilon^\alpha}|e(u^\varepsilon)|^{p-2}e(u^\varepsilon),\ p>1,\ \alpha \geq 0.$$

The structure occupying the domain Ω is subjected to a density of forces of volume f, $f\colon \Omega \to \mathbb{R}^3$, and it is fixed on the boundary $\partial\Omega$. Equations which relate the stress field σ^ε,$\sigma^\varepsilon\colon \Omega \to \mathbb{R}^9_S$, and the field of displacement u^ε, $u^\varepsilon\colon \Omega \to \mathbb{R}^3$ are

$$\begin{cases} \operatorname{div} \sigma^\varepsilon + f &= 0 & \text{in } \Omega, \\ \sigma^\varepsilon_{ij} &= a_{ijkh} e_{kh}(u^\varepsilon) & \text{in } \Omega_\varepsilon, \\ \sigma^\varepsilon &= \frac{1}{\varepsilon^\alpha}|e(u^\varepsilon)|^{p-2}e(u^\varepsilon) & \text{in } B_\varepsilon, \\ u^\varepsilon &= 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Where ε being a positive parameter intended to tend towards zero, $p > 1$, $\alpha \geq 0$, a_{ijkh} being the elasticity coefficients and $\mathbb{R}^9_S$ is the vector space of the square symmetrical matrices of three order . $e_{ij}(u)$ components of the linearized tensor of deformation $e(u)$.

φ_ε being a bounded real function and $]0, \varepsilon[^2$-periodic. In the sequel, we assume that the elasticity coefficients a_{ijkh} satisfy to the following hypotheses

$$a_{ijkh} \in L^\infty(\Omega), \tag{2.2}$$

$$a_{ijkh} = a_{jikh} = a_{khij}, \tag{2.3}$$

$$a_{ijkh}\tau_{ij}\tau_{kh} \geq C\tau_{ij}\tau_{ij},\ \forall \tau \in \mathbb{R}^9_S. \tag{2.4}$$

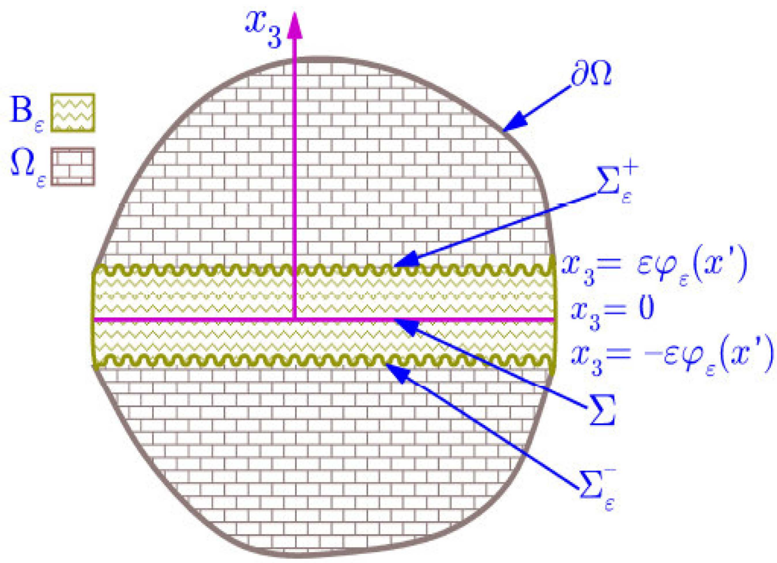

Figure 1: The domain Ω.

3. Notation and functional setting

3.1. Notations

Here is the notation that will be used in the sequel:
$x = (x', x_3)$ where $x' = (x_1, x_2), \tau\otimes\zeta = (\tau_i\zeta_j)_{1\le i,j\le 3}$ $\tau\otimes_S\zeta = \frac{\tau\otimes\zeta+\zeta\otimes\tau}{2}$ $\forall\tau,\zeta\in\mathbb{R}^3$,
$Y =]0,1[\times]0,1[$, $\varrho_T = (I-\nu\otimes\nu)\varrho((I-\nu\otimes\nu)$ $\forall(\varrho,\nu)\in\mathbb{R}^9\times\mathbb{R}^3$,
$\varphi\colon \mathbb{R}^2 \to [a_1, a_2]$ where φ is $Y-$ periodic and $a_2 \ge a_1 > 0, \lambda = 1, 2$,
$\varphi_\varepsilon(x') = \varphi(\frac{x'}{\varepsilon})$, $\frac{\partial\varphi}{\partial x_\lambda} \in \mathcal{C}(\Sigma)\cap L^\infty(\Sigma)$, $m(\varphi) = (\frac{1}{\int_Y dx'})\int_Y \varphi(x')dx'$,
$\nabla'. = (\frac{\partial.}{\partial x_1}, \frac{\partial.}{\partial x_2}, 0)$, $D(.) = (\frac{\partial(.)_i}{\partial x_j})_{1\le i,j\le 3}$, $\eta(t) = \lim_{\varepsilon\to 0}\varepsilon^{1-t}$, $\beta(t) = \lim_{\varepsilon\to 0}\varepsilon^{1+p-t}$ with $t > 0$. $1 < r \le \min(2, p)$ and r' is the conjugate of r (satisfies to $\frac{1}{r} + \frac{1}{r'} = 1$).

In the following C will denote any constant with respect to ε. Also we use the convention $0. + \infty = 0$.

3.2. Functional setting

First, we introduce the following space :

$$V^\varepsilon \quad = \quad \left\{ \ u \in W_0^{1,r}(\Omega,\mathbb{R}^3) \ \middle| \ e(u) \in L^2(\Omega_\varepsilon,\mathbb{R}^9_S) \text{ and } e(u) \in L^p(B_\varepsilon,\mathbb{R}^9_S) \ \right\},$$

we show easily that V^ε is a Banach space with the following norm

$$u \longmapsto \|e(u)\|_{L^2(\Omega_\varepsilon,\mathbb{R}^9_S)} + \|e(u)\|_{L^p(B_\varepsilon,\mathbb{R}^9_S)}.$$

Let

$$\begin{aligned} V^p(\Sigma) &= \left\{ u \in H_0^1(\Omega,\mathbb{R}^3) : e_T(u_{|\Sigma}) \in L^p(\Sigma,\mathbb{R}^9_S) \right\}, \\ V_0^p(\Sigma) &= \left\{ u \in H_0^1(\Omega,\mathbb{R}^3) : e_T(u_{|\Sigma}) = 0 \right\}, \\ V^{1,p}(\Sigma) &= \left\{ u \in V_0^p(\Sigma) : u_{3|\Sigma} \in W^{2,p}(\Sigma) \right\}, \\ V^C(\Sigma) &= \left\{ u \in V_0^p(\Sigma) : \nabla'(u_{3|\Sigma}) = C \right\}. \end{aligned}$$

$V_0^p(\Sigma)$ and $V^C(\Sigma)$ are two Banach spaces, provided with the norm of $H_0^1(\Omega,\mathbb{R}^3)$. $V^p(\Sigma)$ and $V^{1,p}(\Sigma)$ are two Banach spaces, provided respectively with the following norms

$$u \longmapsto \|e(u)\|_{L^2(\Omega,\mathbb{R}^9_S)} + \left\|e_T(u_{|\Sigma})\right\|_{L^p(\Sigma,\mathbb{R}^9_S)},$$

$$u \longmapsto \|e(u)\|_{L^2(\Omega,\mathbb{R}^9_S)} + \left\|D(\nabla' u_{3|\Sigma})\right\|_{L^p(\Sigma,\mathbb{R}^2)}.$$

Let us

$$\mathbb{G}^{\alpha,p} = \begin{cases} \left\{u \in H_0^1(\Omega,\mathbb{R}^3) : \eta(\alpha) e_T(u_{|\Sigma}) \in L^p(\Sigma,\mathbb{R}^9_S)\right\} & \text{if } \alpha \le 1, \\ \left\{u \in V_0^p(\Sigma) : \beta(\alpha) u^*_{3|\Sigma} \in W^{2,p}(\Sigma)\right\} & \text{if } 1 < \alpha \le p+1, \\ V^C(\Sigma) & \text{if } \alpha > p+1. \end{cases}$$

$$\mathbb{D}^{\alpha,p} = \begin{cases} \mathcal{D}(\Omega,\mathbb{R}^3) & \text{if } \alpha \le 1, \\ \mathcal{D}(\Omega,\mathbb{R}^3) \cap V_0^p(\Sigma) & \text{if } 1 < \alpha \le p+1, \\ \mathcal{D}(\Omega,\mathbb{R}^3) \cap V^C(\Sigma) & \text{if } \alpha > p+1. \end{cases}$$

It was known that

$$\overline{\mathbb{D}^{\alpha,p}} = \mathbb{G}^{\alpha,p}.$$

Our goal in this work is to study the problem (2.1), and its limit behavior when ε tends to zero.

4. Study of the problem (2.1)

We remark that, the problem (2.1) is equivalent to the minimization problem

$$\inf_{v \in V^\varepsilon} \left\{ \frac{1}{2} \int_{\Omega_\varepsilon} a_{ijhk} e_{hk}(v) e_{ij}(v) + \frac{1}{p\varepsilon^\alpha} \int_{B_\varepsilon} |e(v)|^p - \int_\Omega f.v \right\} \tag{4.1}$$

In order to study the problem (2.1), we will interest to the study of the minimization problem (4.1). The existence of the solution of (4.1) is given in the following proposition.

Proposition 4.1. *Under the hypotheses (2.2), (2.3), (2.4) and for $f \in L^{r'}(\Omega,\mathbb{R}^3)$, the problem (4.1) admits an unique solution u^ε in V^ε.*

The proof of this proposition is based on classical convexity arguments see for example [6].

Lemma 4.2. *Under the hypothesis* (2.4) *and for* $f \in L^{r'}(\Omega, \mathbb{R}^3)$*, the solution of* (4.1) u^ε *satisfies*

$$\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon, \mathbb{R}^9_S)} \leq C, \tag{4.2}$$

$$\frac{1}{\varepsilon^\alpha}\|e(u^\varepsilon)\|^p_{L^p(B_\varepsilon, \mathbb{R}^9_S)} \leq C, \tag{4.3}$$

moreover u^ε *is bounded in* $W^{1,r}_0(\Omega, \mathbb{R}^3)$.

Proof: Since u^ε is the solution of the problem (4.1) , we have

$$\int_{\Omega_\varepsilon} a_{ijhk} e_{hk}(u^\varepsilon) e_{ij}(v) + \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} |e(u^\varepsilon)|^{p-2} e(u^\varepsilon) e(v) = \int_\Omega f\ v, \quad \forall v \in V^\varepsilon.$$

In particular for $v = u^\varepsilon$, we obtain

$$\int_{\Omega_\varepsilon} a_{ijhk} e_{hk}(u^\varepsilon) e_{ij}(u^\varepsilon) + \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} |e(u^\varepsilon)|^p = \int_\Omega f\ u^\varepsilon.$$

According to the inequalities of Hölder and Young, we have

$$\begin{aligned}
\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon, \mathbb{R}^9_S)} + \frac{1}{\varepsilon^\alpha}\|e(u^\varepsilon)\|^p_{L^p(B_\varepsilon, \mathbb{R}^9_S)} &\leq C\|e(u^\varepsilon)\|_{L^r(\Omega, \mathbb{R}^9_S)} \\
&\leq C(\|e(u^\varepsilon)\|_{L^r(\Omega_\varepsilon, \mathbb{R}^9_S)} + \|e(u^\varepsilon)\|_{L^r(B_\varepsilon, \mathbb{R}^9_S)}) \\
&\leq C(\|e(u^\varepsilon)\|_{L^2(\Omega_\varepsilon, \mathbb{R}^9_S)} + \|e(u^\varepsilon)\|_{L^p(B_\varepsilon, \mathbb{R}^9_S)}) \\
&\leq C + \frac{1}{2}\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon, \mathbb{R}^9_S)} \\
&\quad + \frac{1}{p}\|e(u^\varepsilon)\|^p_{L^p(B_\varepsilon, \mathbb{R}^9_S)} \\
&\leq C + \frac{1}{2}\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon, \mathbb{R}^9_S)} \\
&\quad + \frac{1}{\varepsilon^\alpha}\frac{1}{p}\|e(u^\varepsilon)\|^p_{L^p(B_\varepsilon, \mathbb{R}^9_S)}
\end{aligned}$$

So that

$$\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon, \mathbb{R}^9_S)} + \frac{1}{\varepsilon^\alpha}\|e(u^\varepsilon)\|^p_{L^p(B_\varepsilon, \mathbb{R}^9_S)} \leq C$$

Therefore, we will have (4.2) and (4.3).
Since $r \leq \min(2, p)$ and according to (4.2) and (4.3), so for a small enough ε the solution (u^ε) is bounded in $W^{1,r}_0(\Omega, \mathbb{R}^3)$. □

Let $u : \Omega \to \mathbb{R}^3$, we define the tangential derivative of u, noted δu by

$$\delta u = Du(I - e_3 \otimes e_3) \text{ where } e_3 = (0, 0, 1)^T, \tag{4.4}$$

so (4.4) becomes

$$\delta u = Du - \frac{\partial u}{\partial x_3} \otimes e_3, \tag{4.5}$$

from (4.5) we have

$$(\delta u)_{i3} = 0, \ \forall i \in \{1,2,3\}.$$

We give some lemmas that will be used in the sequel.

Lemma 4.3 (See [1]). *Let $f, g \in \mathcal{C}^\infty(\Sigma, \mathbb{R}^3)$, with f is to compact support, then we have*

$$\int_\Sigma f \delta_j g = -\int_\Sigma g \delta_j f, \tag{4.6}$$

where $\delta_j f = \frac{\partial f}{\partial x_j}$, $j = 1, 2$.

From the Green formula we have the following lemme

Lemma 4.4. *Let $\sigma \in \mathcal{C}^\infty(\Sigma, \mathbb{R}^9)$ and $u \in \mathcal{D}(\Sigma, \mathbb{R}^3)$, so we have*

$$\int_\Sigma \sigma e(u) = -\int_\Sigma div_T(\sigma) u, \tag{4.7}$$

with $div_T(\sigma) = div(\frac{\sigma + \sigma^T}{2})$

Lemma 4.5. *Let u be a regular function defined in a neighborhood of Σ, then*

$$\delta_j \left(\int_0^{\varepsilon\varphi_\varepsilon} u \right) = \varepsilon u(x', \varepsilon\varphi_\varepsilon) \delta_j \varphi_\varepsilon + \int_0^{\varepsilon\varphi_\varepsilon} \delta_j u. \tag{4.8}$$

This lemma is a consequence of [1, proposition 2].
Let us

$$w_\varepsilon = \frac{1}{2\varepsilon\varphi_\varepsilon} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} u^\varepsilon.$$

Lemma 4.6. *The solution u^ε of the problem (4.1) possess a cluster point u^* in $W_0^{1,r}(\Omega, \mathbb{R}^3)$ with respect to the weak topology, such that ${u^*}_{|\Sigma}$ is a weak cluster point of w_ε in $L^r(\Sigma, \mathbb{R}^3)$.*

Proof: According to the lemma 4.2, (4.2) and (4.3), for a small enough ε, u^ε is bounded in $W_0^{1,r}(\Omega, \mathbb{R}^3)$, so for a subsequences of u^ε, still denoted by u^ε, there exists $u^* \in W_0^{1,r}(\Omega, \mathbb{R}^3)$ such that

$$u^\varepsilon \rightharpoonup u^* \text{ in } W_0^{1,r}(\Omega, \mathbb{R}^3).$$

We have

$$\begin{aligned}\int_\Sigma \left|w_\varepsilon - u^\varepsilon{}_{|\Sigma}\right|^r &\leq \int_\Sigma \frac{1}{2\varepsilon\varphi_\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \left|u^\varepsilon(x)-u^\varepsilon(x',0)\right|^r \\ &\leq \frac{C}{\varepsilon}\int_\Sigma\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\left|\int_0^{x_3}\frac{\partial u^\varepsilon}{\partial x_3}\right|^r \\ &\leq \frac{C}{\varepsilon}\int_\Sigma\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}(x_3)^{r-1}\int_0^{x_3}\left|\frac{\partial u^\varepsilon}{\partial x_3}\right|^r \\ &\leq C\varepsilon^{r-1}\int_{B_\varepsilon}\left|Du^\varepsilon\right|^r \\ &\leq C\varepsilon^{r-1}\int_\Omega\left|Du^\varepsilon\right|^r.\end{aligned}$$

Thanks to the lemma 4.2 and the Korn's inequality, so we have

$$\begin{aligned}\int_\Sigma \left|w_\varepsilon - u^\varepsilon{}_{|\Sigma}\right|^r &\leq C\varepsilon^{r-1}\left(\int_{\Omega_\varepsilon}\left|e(u^\varepsilon)\right|^r + \int_{B_\varepsilon}\left|e(u^\varepsilon)\right|^r\right) \\ &\leq C\left(\varepsilon^{r-1}+\varepsilon^{r-1+\frac{\alpha}{r}}\right),\end{aligned}$$

then $\lim_{\varepsilon\to 0}\int_\Sigma \left|w_\varepsilon - u^\varepsilon{}_{|\Sigma}\right|^r = 0$, since $u^\varepsilon_{|\Sigma} \rightharpoonup u^*{}_{|\Sigma}$ in $L^r(\Sigma,\mathbb{R}^3)$, so $w_\varepsilon \rightharpoonup u^*{}_{|\Sigma}$ in $L^r(\Sigma,\mathbb{R}^3)$. □

Let us

$$v_\varepsilon = \frac{1}{(\varepsilon\varphi_\varepsilon)^{p+1}}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}|x_3|^p\, u^\varepsilon.$$

Lemma 4.7. *The sequence v_ε possess a weak cluster point $\frac{1}{p+1}u^*{}_{|\Sigma}$ in $L^r(\Sigma,\mathbb{R}^3)$.*

The proof of this lemma is based on the same technic used in the proof of the lemma 4.6.

Note by $e_T(u) = (I - e_3\otimes e_3)\delta u$ the tangential part of the tensor $e(u)$. In order to apply the epiconvergence method, we need to characterize the topological spaces containing any cluster point of the solution of the problem (4.1) with respect to the used topology, therefore the weak topology to use is insured by the lemma 4.2. So the topological spaces characterization is given in the following proposition

Proposition 4.8. *The solution u^ε of the problem (4.1), possess a weak cluster point $u^* \in W_0^{1,r}(\Omega,\mathbb{R}^3)$ and u^* satisfies*

1. *$u^* \in H_0^1(\Omega,\mathbb{R}^3)$,*
2. *If $\alpha = 1$, we have $e_T(u^*{}_{|\Sigma}) \in L^p(\Sigma,\mathbb{R}^9_S)$,*
3. *If $1 < \alpha < p+1$, we have $e_T(u^*{}_{|\Sigma}) = 0$,*

4. *If $\alpha = p+1$, we have $e_T(u^*{}_{|\Sigma}) = 0$ and $u^*_{3|\Sigma} \in W^{2,p}(\Sigma)$,*

5. *If $\alpha > p+1$, we have $e_T(u^*{}_{|\Sigma}) = 0$ and $\delta u^*_{3|\Sigma} = C$.*

Proof:

1. Thanks to the lemma 4.6, for a subsequence of u^ε, still denoted by u^ε, there exists $u^* \in W_0^{1,r}(\Omega, \mathbb{R}^3)$ such that

$$u^\varepsilon \rightharpoonup u^* \text{ in } W_0^{1,r}(\Omega, \mathbb{R}^3),$$

so that

$$\lim_{\varepsilon \to 0} \int_\Omega e(u^\varepsilon)v = \int_\Omega e(u^*)v, \forall v \in L^{r'}(\Omega, \mathbb{R}^9_S),$$

according to (4.2), we have $\chi_{\Omega_\varepsilon} e(u^\varepsilon)$ is bounded in $L^2(\Omega, \mathbb{R}^9_S)$, so for a subsequence of $\chi_{\Omega_\varepsilon} e(u^\varepsilon)$, still denoted by $\chi_{\Omega_\varepsilon} e(u^\varepsilon)$, there exists $w \in L^2(\Omega, \mathbb{R}^9_S) \subset L^r(\Omega, \mathbb{R}^9_S)$, such that

$$\chi_{\Omega_\varepsilon} e(u^\varepsilon) \rightharpoonup w \text{ in } L^2(\Omega, \mathbb{R}^9_S),$$

then

$$\chi_{\Omega_\varepsilon} e(u^\varepsilon) \rightharpoonup w \text{ in } L^r(\Omega, \mathbb{R}^9_S).$$

Let $v \in L^{r'}(\Omega, \mathbb{R}^9_S)$, then

$$\int_\Omega e(u^\varepsilon)v = \int_\Omega e(u^\varepsilon)\chi_{\Omega_\varepsilon} v + \int_\Omega e(u^\varepsilon)\chi_{B_\varepsilon} v.$$

Passing to the limit, we have

$$\int_\Omega e(u^*)v = \int_\Omega wv + \lim_{\varepsilon \to 0} \int_\Omega e(u^\varepsilon)\chi_{B_\varepsilon} v,$$

since $\chi_{B_\varepsilon} v \rightharpoonup 0$, so that

$$\int_\Omega e(u^*)v = \int_\Omega wv,$$

in particular for $v = |e(u^*) - w|^{r-2}(e(u^*) - w)$, it follows that

$$\int_\Omega |e(u^*) - w|^r = 0,$$

then $e(u^*) = w$, hence $e(u^*) \in L^2(\Omega, \mathbb{R}^9_S)$, and according to the classical result (see, [13, proposition 1.2, p. 16]), we have $u^* \in H_0^1(\Omega, \mathbb{R}^3)$.

2. If $\alpha = 1$, let us pose

$$v_\varepsilon = \frac{1}{\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} e_T(u^\varepsilon),$$

we have

$$\int_\Sigma |v_\varepsilon|^p = \int_\Sigma \frac{1}{\varepsilon^p}\left|\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} e_T(u^\varepsilon)\right|^p, \tag{4.9}$$

thanks to the inequality of Hölder and $a_1 \le \varphi \le a_2$, (4.9) becomes

$$\begin{aligned}\int_\Sigma |v_\varepsilon|^p &\le \frac{C}{\varepsilon}\int_{B_\varepsilon} |e_T(u^\varepsilon)|^p\,,\\ &\le \frac{C}{\varepsilon}\int_{B_\varepsilon} |e(u^\varepsilon)|^p\,,\end{aligned}$$

from (4.3), we have

$$\int_\Sigma |v_\varepsilon|^p \le C\varepsilon^{\alpha-1}, \tag{4.10}$$

since $\alpha = 1$, then v_ε is bounded in $L^p(\Sigma, \mathbb{R}^9_S)$, so for a subsequence of v_ε, still denoted by v_ε, there exists $v \in L^p(\Sigma, \mathbb{R}^9_S)$ such that $v_\varepsilon \rightharpoonup v$ in $L^p(\Sigma, \mathbb{R}^9_S)$. Let $g \in \mathcal{D}(\Sigma, \mathbb{R}^9)$, according to the lemma 4.5 we obtain

$$\begin{aligned}\int_\Sigma g\,\frac{1}{\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} e_T(u^\varepsilon) &= \int_\Sigma g\,\frac{1}{\varepsilon}\Big\{e_T\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} u^\varepsilon\right) - \varepsilon\delta\varphi_\varepsilon \otimes_S \Big(u^\varepsilon(x', \varepsilon\varphi_\varepsilon)\\ &\quad + u^\varepsilon(x', -\varepsilon\varphi_\varepsilon)\Big)\Big\},\end{aligned}$$

to simplify the writing, note $U^\varepsilon = u^\varepsilon(x', \varepsilon\varphi_\varepsilon) + u^\varepsilon(x', -\varepsilon\varphi_\varepsilon)$.
Thanks to the lemma 4.4, we have

$$\int_\Sigma g\,\frac{1}{\varepsilon}e_T\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} u^\varepsilon\right) = -\int_\Sigma div_T g\left(\frac{1}{\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} u^\varepsilon\right),$$

thanks to the lemma 4.6 and $\varphi_\varepsilon \to m(\varphi)$ in $L^{r'}(\Sigma)$, so passing to limit, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Sigma g\,e_T\left(\frac{1}{\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} u^\varepsilon\right) &= -2m(\varphi)\int_\Sigma div_T g\; {u^*}_{|\Sigma}\\ &= 2m(\varphi)\int_\Sigma g\,e_T\left({u^*}_{|\Sigma}\right).\end{aligned}$$

Now, to continue our proof, we need to establish the following lemma

Lemma 4.9. *for all $g \in \mathcal{D}(\Sigma, \mathbb{R}^9_S)$, we have*

$$\lim_{\varepsilon\to 0}\int_\Sigma g\ \delta\varphi_\varepsilon \otimes_S U^\varepsilon = 0.$$

The proof of this lemma

Proof: Indeed, we have

$$\int_\Sigma g\ \delta\varphi_\varepsilon \otimes_S U^\varepsilon = \frac{1}{2}\int_\Sigma g\ \delta\varphi_\varepsilon \otimes U^\varepsilon + \frac{1}{2}\int_\Sigma g\ U^\varepsilon \otimes \delta\varphi_\varepsilon.$$

First, show that $\beta_\varepsilon = \int_\Sigma g\ \delta\varphi_\varepsilon \otimes U^\varepsilon \to 0$ and the second term will be shown with the same way.
We have

$$\begin{aligned}\beta_\varepsilon &= \sum_{i,j}\int_\Sigma g_{ij}\frac{\partial\varphi_\varepsilon}{\partial x_i}U^\varepsilon_j\\ &= \sum_{i,j}\left\langle \frac{\partial\varphi_\varepsilon}{\partial x_i}U^\varepsilon_j, g_{ij}\right\rangle_{\mathcal{D}',\mathcal{D}(\Sigma)}\\ &= \sum_{i,j}\left\langle U^\varepsilon_j, \frac{\partial\varphi_\varepsilon}{\partial x_i}g_{ij}\right\rangle_{\mathcal{D}',\mathcal{D}(\Sigma)},\end{aligned}$$

we show easily that $U^\varepsilon \rightharpoonup 2u^*{}_{|\Sigma}$ in $L^r(\Sigma, \mathbb{R}^3)$.
We have

$$\beta_\varepsilon = \sum_{i,j}\left\langle U^\varepsilon_j - 2u^*_{j\,|\Sigma}, \frac{\partial\varphi_\varepsilon}{\partial x_i}g_{ij}\right\rangle_{\mathcal{D}',\mathcal{D}(\Sigma)} + \sum_{i,j}\left\langle 2u^*_{j\,|\Sigma}, \frac{\partial\varphi_\varepsilon}{\partial x_i}g_{ij}\right\rangle_{\mathcal{D}',\mathcal{D}(\Sigma)}$$

Let $\theta_{\varepsilon,n} \in \mathcal{D}(\Sigma, \mathbb{R}^3)$ such that $\theta^i_{\varepsilon,n} \to \frac{\partial\varphi_\varepsilon}{\partial x_i}$ in $L^r(\Sigma)$, so there exists $h_{\varepsilon,n}\to 0$ when $n \to +\infty$, such that

$$\beta_\varepsilon = \sum_{i,j}\left\langle U^\varepsilon_j - 2u^*_{j\,|\Sigma}, \theta^i_{\varepsilon,n}g_{ij}\right\rangle_{\mathcal{D}',\mathcal{D}(\Sigma)} + h_{\varepsilon,n} + \sum_{i,j}\left\langle 2u^*_{j\,|\Sigma}, \frac{\partial\varphi_\varepsilon}{\partial x_i}g_{ij}\right\rangle_{\mathcal{D}',\mathcal{D}(\Sigma)},$$

since $\frac{\partial\varphi}{\partial x_i}$ is bonded on Σ, so for a greater enough n, for each $x' \in \Sigma$ we have

$$\begin{aligned}\left|\theta^i_{\varepsilon,n}(x')\right| &\le \frac{3}{2}\left|\frac{\partial\varphi_\varepsilon}{\partial x_i}(x')\right|\\ &\le \frac{C}{\varepsilon},\end{aligned}$$

according to the Hölder inequality, we have

$$\beta_\varepsilon \le \frac{C}{\varepsilon}\sum_{i,j}\left\|U^\varepsilon_j - 2u^*_{j\,|\Sigma}\right\|_{L^r(\Sigma)}\|g_{ij}\|_{L^{r'}(\Sigma)} + h_{\varepsilon,n} + \sum_{i,j}\left\langle 2u^*_{j\,|\Sigma}, \frac{\partial\varphi_\varepsilon}{\partial x_i}g_{ij}\right\rangle_{\mathcal{D}',\mathcal{D}(\Sigma)}$$

So for each $\gamma > 0$, there exists ε_0 such that for $\varepsilon < \varepsilon_0$ we have

$$\beta_\varepsilon \leq \frac{C\gamma}{\varepsilon} \sum_{i,j} \|g_{ij}\|_{L^{r'}(\Sigma)} + h_{\varepsilon,n} + \sum_{i,j} \left\langle 2u^*_{j\,|\Sigma}, \frac{\partial \varphi_\varepsilon}{\partial x_i} g_{ij} \right\rangle_{\mathcal{D}',\mathcal{D}(\Sigma)}.$$

Let $\phi_l \in \mathcal{D}(\Sigma, \mathbb{R}^3)$ such that $\phi_l \to 2u^*{}_{|\Sigma}$ in $L^r(\Sigma, \mathbb{R}^3)$, so there exists $a_{\varepsilon,l} \to 0$ when $l \to +\infty$ such that

$$\begin{aligned} \beta_\varepsilon &\leq \frac{C\gamma}{\varepsilon} \sum_{i,j} \|g_{ij}\|_{L^{r'}(\Sigma)} + h_{\varepsilon,n} + \sum_{i,j} \left\langle \phi_l, \frac{\partial \varphi_\varepsilon}{\partial x_i} g_{ij} \right\rangle_{\mathcal{D}',\mathcal{D}(\Sigma)} + a_{\varepsilon,l} \\ &\leq \frac{C\gamma}{\varepsilon} \sum_{i,j} \|g_{ij}\|_{L^{r'}(\Sigma)} + h_{\varepsilon,n} + \sum_{i,j} \left\langle \frac{\partial \varphi_\varepsilon}{\partial x_i}, \phi_l g_{ij} \right\rangle_{\mathcal{D}',\mathcal{D}(\Sigma)} + a_{\varepsilon,l}. \end{aligned}$$

We show easily that $\lim\limits_{\varepsilon\to 0} a_{\varepsilon,l} = 0$, so passing to the limit in γ to 0, n to $+\infty$, then in ε to 0, we obtain

$$\beta_\varepsilon \to 0.$$

Hence the proof of the lemma 4.9 is complete. □

So thanks to the lemma 4.9, we obtain

$$\lim_{\varepsilon\to 0} \int_\Sigma g \, \frac{1}{\varepsilon} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} e_T(u^\varepsilon) = \int_\Sigma g \, \left(2m(\varphi) e_T\left(u^*{}_{|\Sigma}\right)\right)$$

Since $v_\varepsilon \rightharpoonup v$ in $L^p(\Sigma, \mathbb{R}^9_S)$, and according to $v_\varepsilon \rightharpoonup 2m(\varphi)e_T\left(u^*{}_{|\Sigma}\right)$ in $\mathcal{D}'(\Sigma, \mathbb{R}^9_S)$, so it follows that

$$e_T\left(u^*{}_{|\Sigma}\right) \in L^p(\Sigma, \mathbb{R}^9_S).$$

3. $1 < \alpha < p+1$
From (4.10), $v_\varepsilon \to 0$ and according to $v_\varepsilon \rightharpoonup 2m(\varphi)e_T\left(u^*{}_{|\Sigma}\right)$ in $\mathcal{D}'(\Sigma, \mathbb{R}^9_S)$, so

$$e_T\left(u^*{}_{|\Sigma}\right) = 0.$$

4. $\alpha = p+1$, let us pose

$$\xi_\varepsilon = \frac{1}{\varepsilon^{p+1}} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^{p-2} x_3 e_T(u^\varepsilon),$$

we have

$$\int_\Sigma |\xi_\varepsilon|^p = \int_\Sigma \frac{1}{\varepsilon^{p^2+p}} \left| \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^{p-2} x_3 e_T(u^\varepsilon) \right|^p, \tag{4.11}$$

from Hôlder's inequality and the fact that $a_1 \leq \varphi_\varepsilon \leq a_2$, (4.11) becomes

$$\begin{aligned}\int_\Sigma |\xi_\varepsilon|^p &\leq \frac{C}{\varepsilon^{p+1}} \int_\Sigma \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |e_T(u^\varepsilon)|^p \\ &\leq \frac{C}{\varepsilon^{p+1}} \int_{B_\varepsilon} |e_T(u^\varepsilon)|^p \\ &\leq \frac{C}{\varepsilon^{p+1}} \int_{B_\varepsilon} |e(u^\varepsilon)|^p \\ &\leq C\varepsilon^{\alpha-p-1},\end{aligned}$$

for $\alpha \geq p+1$, ξ_ε is bounded $L^p(\Sigma, \mathbb{R}^9_S)$, so for a subsequence of ξ_ε, still denoted by ξ_ε, there exists $\xi \in L^p(\Sigma, \mathbb{R}^9_S)$ such that

$$\begin{cases} \xi_\varepsilon \rightharpoonup \xi \text{ in } L^p(\Sigma, \mathbb{R}^9_S) & \text{if} \quad \alpha = p+1, \\ \xi_\varepsilon \to 0 \text{ in } L^p(\Sigma, \mathbb{R}^9_S) & \text{if} \quad \alpha > p+1. \end{cases}$$

Let $g \in \mathcal{D}(\Sigma, \mathbb{R}^9_S)$, from the lemma 4.5, we have

$$\begin{aligned}\int_\Sigma g \, \frac{1}{\varepsilon^{p+1}} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^{p-2} x_3 e_T(u^\varepsilon) \quad = \quad & \int_\Sigma g \, \frac{1}{\varepsilon^{p+1}} \Big\{ e_T\Big(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^{p-2} x_3 u^\varepsilon \Big) \\ & -\varepsilon(\varepsilon\varphi_\varepsilon)^{p-1} \delta\varphi_\varepsilon \otimes_S [u^\varepsilon] \Big\},\end{aligned}$$

thanks to the lemma 4.4, we have

$$\int_\Sigma g \, \frac{1}{\varepsilon^{p+1}} e_T \left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^{p-2} x_3 u^\varepsilon \right) = - \int_\Sigma div_T g \left(\frac{1}{\varepsilon^{p+1}} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^{p-2} x_3 u^\varepsilon \right),$$

so that

$$\begin{aligned}\int_\Sigma g \, \frac{1}{\varepsilon^{p+1}} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^{p-2} x_3 e_T(u^\varepsilon) \quad &= - \int_\Sigma div_T g \, \left(\frac{1}{\varepsilon^{p+1}} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^{p-2} x_3 u^\varepsilon \right) \\ &\quad - \int_\Sigma g \, (\varphi_\varepsilon)^{p-1} \delta\varphi_\varepsilon \otimes_S \frac{1}{\varepsilon} [u^\varepsilon] \\ &= - \int_\Sigma div_T g \, \left(\frac{1}{\varepsilon^{p+1}} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^{p-2} x_3 u^\varepsilon \right) \\ &\quad - \int_\Sigma g \, (\varphi_\varepsilon)^{p-1} \delta\varphi_\varepsilon \otimes_S \frac{1}{\varepsilon} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \frac{\partial u^\varepsilon}{\partial x_3} \\ &= \quad I_1 + I_2,\end{aligned}$$

with

$$\begin{aligned} I_1 &= -\int_\Sigma div_{T} g \left(\frac{1}{\varepsilon^{p+1}} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^{p-2} x_3 u^\varepsilon \right) \\ &= -\int_\Sigma div_{T} g \, \frac{1}{\varepsilon^{p+1}} \left(\frac{(\varepsilon\varphi_\varepsilon)^p}{p} [u^\varepsilon] - \frac{1}{p} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^p \frac{\partial u^\varepsilon}{\partial x_3} \right) \\ &= -\int_\Sigma div_{T} g \, \frac{(\varphi_\varepsilon)^p}{p\varepsilon} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \frac{\partial u^\varepsilon}{\partial x_3} + \int_\Sigma div_{T} g \, \frac{1}{p\varepsilon^{p+1}} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^p \frac{\partial u^\varepsilon}{\partial x_3} \\ &= I_{1,1} + I_{1,2}, \end{aligned}$$

and

$$\begin{aligned} I_{1,1} &= -\int_\Sigma div_{T} g \, \frac{(\varphi_\varepsilon)^p}{p\varepsilon} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \frac{\partial u^\varepsilon}{\partial x_3} \\ &= -\int_\Sigma div_{T} g \, \frac{(\varphi_\varepsilon)^p}{p\varepsilon} \left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} (\frac{\partial u^\varepsilon}{\partial x_3} + \delta u_3^\varepsilon) - \delta u_3^\varepsilon \right) \\ &= -\int_\Sigma div_{T} g \, \frac{(\varphi_\varepsilon)^p}{p\varepsilon} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} (e(u^\varepsilon)e_3 - e_{33}(u^\varepsilon)e_3) \\ &\quad + \int_\Sigma div_{T} g \, \frac{(\varphi_\varepsilon)^p}{p\varepsilon} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \delta u_3^\varepsilon, \end{aligned}$$

let

$$\beta_\varepsilon = \frac{(\varphi_\varepsilon)^p}{p\varepsilon} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} (e(u^\varepsilon)e_3 - e_{33}(u^\varepsilon)e_3),$$

from (4.3), we have $\beta_\varepsilon \to 0$ so that $L^p(\Sigma, \mathbb{R}^3)$, then it follows that

$$\lim_{\varepsilon \to 0} \left\{ -\int_\Sigma div_{T} g \, \frac{(\varphi_\varepsilon)^p}{p\varepsilon} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} (e(u^\varepsilon)e_3 - e_{33}(u^\varepsilon)e_3) \right\} = 0,$$

thanks to the lemma 4.5, we obtain

$$\begin{aligned} \int_\Sigma div_{T} g \, \frac{(\varphi_\varepsilon)^p}{p\varepsilon} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \delta u_3^\varepsilon &= \int_\Sigma div_{T} g \, \frac{(\varphi_\varepsilon)^p}{p\varepsilon} \Big(\delta(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} u_3^\varepsilon) \\ &\quad -\varepsilon\delta\varphi_\varepsilon \left\{ u_3^\varepsilon(x', \varepsilon\varphi_\varepsilon) + u_3^\varepsilon(x', -\varepsilon\varphi_\varepsilon) \right\} \Big), \end{aligned}$$

note that $U_3^\varepsilon = u_3^\varepsilon(x', \varepsilon\varphi_\varepsilon) + u_3^\varepsilon(x', -\varepsilon\varphi_\varepsilon)$, we obtain

$$\begin{aligned}\int_\Sigma div_T g \, \frac{(\varphi_\varepsilon)^p}{p\varepsilon} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \delta u_3^\varepsilon &= \int_\Sigma div_T g \, \frac{(\varphi_\varepsilon)^p}{p\varepsilon} \delta (\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} u_3^\varepsilon) \\ &\quad - \int_\Sigma div_T g \, \frac{(\varphi_\varepsilon)^p}{p} \delta\varphi_\varepsilon U_3^\varepsilon \\ &= - \int_\Sigma div(div_T g \, (\varphi_\varepsilon)^p) \frac{1}{p\varepsilon} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} u_3^\varepsilon \\ &\quad - \int_\Sigma div_T g \, \frac{(\varphi_\varepsilon)^p}{p} \delta\varphi_\varepsilon U_3^\varepsilon,\end{aligned}$$

we show in the same way like in the proof of the lemma 4.9, that

$$\lim_{\varepsilon\to 0} \int_\Sigma div_T g \, \frac{(\varphi_\varepsilon)^p}{p} \delta\varphi_\varepsilon U_3^\varepsilon = 0,$$

and

$$\lim_{\varepsilon\to 0} \int_\Sigma div(div_T g \, (\varphi_\varepsilon)^p) \frac{1}{p\varepsilon} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} u_3^\varepsilon = 0.$$

So that $\lim_{\varepsilon\to 0} I_{1,1} = 0$. We have

$$\begin{aligned} I_{1,2} &= \int_\Sigma div_T g \, \frac{1}{p\varepsilon^{p+1}} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^p \, \frac{\partial u^\varepsilon}{\partial x_3} \\ &= \int_\Sigma div_T g \, \frac{1}{p\varepsilon^{p+1}} \left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^p \, (e(u^\varepsilon).e_3 - e_{33}(u^\varepsilon)e_3) - |x_3|^p \, \delta u_3^\varepsilon \right),\end{aligned}$$

let us pose $a_\varepsilon = \frac{1}{p\varepsilon^{p+1}} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^p \, (e(u^\varepsilon).e_3 - e_{33}(u^\varepsilon)e_3)$, from (4.3) we obtain

$$\begin{aligned}\int_\Sigma |a_\varepsilon|^p &\leq \frac{C}{\varepsilon} \int_{B_\varepsilon} |e(u^\varepsilon)|^p \\ &\leq C\varepsilon^{\alpha-1},\end{aligned}$$

then $a_\varepsilon \to 0$ in $L^p(\Sigma, \mathbb{R}^3)$, so it follows that

$$\lim_{\varepsilon\to 0} \int_\Sigma div_T g \, \frac{1}{p\varepsilon^{p+1}} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^p \, (e(u^\varepsilon).e_3 - e_{33}(u^\varepsilon)e_3) = 0.$$

According to the lemma 4.5, we find that

$$\begin{aligned}\int_\Sigma div_T g \, \frac{1}{p\varepsilon^{p+1}} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^p \, \delta u_3^\varepsilon &= \int_\Sigma div_T g \, \frac{1}{p\varepsilon^{p+1}} \delta \left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^p \, u_3^\varepsilon \right) \\ &\quad - \int_\Sigma div_T g \, (\varphi_\varepsilon)^p \delta\varphi_\varepsilon U_3^\varepsilon \\ &= - \int_\Sigma div(div_T g) \, \frac{1}{p\varepsilon^{p+1}} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3|^p \, u_3^\varepsilon \\ &\quad - \int_\Sigma div_T g \, (\varphi_\varepsilon)^p \delta\varphi_\varepsilon U_3^\varepsilon,\end{aligned}$$

we show easily, like in the proof of the lemma 4.9 and thanks to the lemma 4.6, that

$$\frac{1}{\varepsilon^{p+1}}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}|x_3|^p\, u_3^\varepsilon \rightharpoonup \frac{2}{p+1}m(\varphi^{p+1})u^*_{3|\Sigma} \text{ in } L^r(\Sigma),$$

and

$$\lim_{\varepsilon\to 0}\int_\Sigma div_T g\, (\varphi_\varepsilon)^p \delta\varphi_\varepsilon U_3^\varepsilon = 0.$$

So that

$$\begin{aligned}\lim_{\varepsilon\to 0} I_{1,2} &= \int_\Sigma div(div_T g)\,\frac{2m(\varphi^{p+1})}{p(p+1)}u^*_{3|\Sigma}\\ &= -\int_\Sigma div_T g\,\frac{2m(\varphi^{p+1})}{p(p+1)}\delta u^*_{3|\Sigma}\\ &= \int_\Sigma g\,\frac{2m(\varphi^{p+1})}{p(p+1)}e_T(\delta u^*_{3|\Sigma})\end{aligned}$$

Let us show that $\lim\limits_{\varepsilon\to 0} I_2 = 0$, indeed

$$\begin{aligned}I_2 &= -\int_\Sigma g\,(\varphi_\varepsilon)^{p-1}\delta\varphi_\varepsilon\otimes_S\frac{1}{\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\frac{\partial u^\varepsilon}{\partial x_3}\\ &= -\int_\Sigma g\,(\varphi_\varepsilon)^{p-1}\delta\varphi_\varepsilon\otimes_S\frac{1}{\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\left(\frac{\partial u^\varepsilon}{\partial x_3}+\delta u_3^\varepsilon\right)-\delta u_3^\varepsilon\\ &= -\int_\Sigma g\,(\varphi_\varepsilon)^{p-1}\delta\varphi_\varepsilon\otimes_S\frac{1}{\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}(e(u^\varepsilon)e_3-e_{33}(u^\varepsilon)e_3)\\ &\quad+\int_\Sigma g\,(\varphi_\varepsilon)^{p-1}\delta\varphi_\varepsilon\otimes_S\frac{1}{\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\delta u_3^\varepsilon\end{aligned}$$

According to the lemma 4.5, and the fact that $\frac{1}{\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}(e(u^\varepsilon)e_3 - e_{33}(u^\varepsilon)e_3)$ converges to 0 in $L^p(\Sigma,\mathbb{R}^3)$, and while redoing the same way like in the proof of the lemma 4.9, we obtain

$$\lim_{\varepsilon\to 0} I_2 = 0.$$

So

$$\lim_{\varepsilon\to 0}\int_\Sigma g\frac{1}{\varepsilon^{p+1}}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}|x_3|^{p-2}\,x_3 e_T(u^\varepsilon) = \int_\Sigma g\,\frac{2m(\varphi^{p+1})}{p(p+1)}e_T(\delta u^*_{3|\Sigma}),$$

moreover $\xi_\varepsilon \rightharpoonup \xi$ in $L^p(\Sigma,\mathbb{R}^9_S)$, then $\xi = \frac{2m(\varphi^{p+1})}{p(p+1)}e_T(\delta u^*_{3|\Sigma})$, it follows that $\delta u^*_{3|\Sigma}\in W^{1,p}(\Sigma,\mathbb{R}^3)$, hence $u^*_{3|\Sigma}\in W^{2,p}(\Sigma)$.

5. If $\alpha > p+1$, we have
$e_T(u^*{}_{|\Sigma}) = 0$, from the case (4), (ie $\alpha = p+1$), we obtain $\xi_\varepsilon \to 0$ in $L^p(\Sigma,\mathbb{R}^9_S)$, so that $\frac{2m(\varphi^{p+1})}{p(p+1)}e_T(\delta u^*_{3|\Sigma}) = 0$, as $e_T(\delta u^*_{3|\Sigma}) = \delta\delta u^*_{3|\Sigma}$, then $\delta\delta u^*_{3|\Sigma} = 0$, it follows that $\delta u^*_{3|\Sigma} = C$.

□

Remark 4.10. The proposition 4.8 remains true for any weak cluster point u of a sequence u_ε, in $W_0^{1,r}(\Omega,\mathbb{R}^3)$, satisfies (4.2) and (4.3).

To study the limit behavior of the problem (4.1), we'll use the epi-convergence method , (see Annex, definition 6.1).

5. Limit behavior

Let

$$F^\varepsilon(u)=\begin{cases}\dfrac{1}{2}\displaystyle\int_{\Omega_\varepsilon} a_{ijkh}e_{kh}(u)e_{ij}(u)+\frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon}|e(u)|^p & \text{if } u\in V^\varepsilon\\ +\infty & \text{if } u\in W_0^{1,r}(\Omega,\mathbb{R}^3)\setminus V^\varepsilon\end{cases}\tag{5.1}$$

$$G(u)=-\int_\Omega f\ u, \forall u\in W_0^{1,r}(\Omega,\mathbb{R}^3)\tag{5.2}$$

We design by τ_f the weak topology on the space $W_0^{1,r}(\Omega,\mathbb{R}^3)$. In sequel, we shall characterize, according to the values of α, the epi-limit of the functional energy given by (5.1) in the following theorem.

Theorem 5.1. *According to the values of α, there exists a functional $F^\alpha\colon W_0^{1,r}(\Omega,\mathbb{R}^3)\to\mathbb{R}\cup\{+\infty\}$ such that*

$$\tau_f - lim_e F^\varepsilon = F^\alpha \ \ in\ W_0^{1,r}(\Omega,\mathbb{R}^3)\tag{5.3}$$

where F^α is given by

1. *If $\alpha\neq 1$ and $\alpha\neq p+1$:*

$$F^\alpha(u)=\begin{cases}\dfrac{1}{2}\displaystyle\int_\Omega a_{ijkh}e_{kh}(u)e_{ij}(u) & \text{if } u\in\mathbb{G}^{\alpha,p},\\ +\infty & \text{if } u\in W_0^{1,r}(\Omega,\mathbb{R}^3)\setminus\mathbb{G}^{\alpha,p}.\end{cases}$$

2. *If $\alpha=1$:*

$$F^\alpha(u)=\begin{cases}\dfrac{1}{2}\displaystyle\int_\Omega a_{ijkh}e_{kh}(u)e_{ij}(u)+\frac{2m(\varphi)\ \eta(\alpha)}{p}\int_\Sigma\left|e_T(u_{|\Sigma})\right|^p & \text{if } u\in\mathbb{G}^{\alpha,p},\\ +\infty & \text{if } u\in W_0^{1,r}(\Omega,\mathbb{R}^3)\setminus\mathbb{G}^{\alpha,p}.\end{cases}$$

3. *If $\alpha=p+1$:*

$$F^\alpha(u)=\begin{cases}\dfrac{1}{2}\displaystyle\int_\Omega a_{ijkh}e_{kh}(u)e_{ij}(u)+\frac{2m(\varphi^{p+1})\ \beta(\alpha)}{p(p+1)}\int_\Sigma\left|\delta\delta u_{3|\Sigma})\right|^p & \text{if } u\in\mathbb{G}^{\alpha,p},\\ +\infty & \text{if } u\in W_0^{1,r}(\Omega,\mathbb{R}^3)\setminus\mathbb{G}^{\alpha,p}.\end{cases}$$

Proof:

a) We are now in position to determine the upper epi-limit
Let $u \in \mathbb{G}^{\alpha,p} \subset W_0^{1,r}(\Omega,\mathbb{R}^3)$, so there exists a sequence (u^n) in $\mathbb{D}^{\alpha,p}$ such that $u^n \to u$ in $\mathbb{G}^{\alpha,p}$ when $n \to +\infty$, so $u^n \to u$ in $W_0^{1,r}(\Omega,\mathbb{R}^3)$.
Let θ be a regular function satisfies

$$\theta(x_3) = 1 \text{ if } |x_3| \leq 1,\ \theta(x_3) = 0 \text{ if } |x_3| \geq 2 \text{ and } \left|\theta'(x_3)\right| \leq 2\ \forall x_3 \in \mathbb{R},$$

and

$$\theta_\varepsilon(x) = \theta(\frac{x_3}{\varepsilon\varphi_\varepsilon});$$

it's clear that $\theta_\varepsilon \to 0$ in $L^2(\Omega)$.
we define

$$u^{\varepsilon,n} = \theta_\varepsilon(x)(u^n{}_{|\Sigma} - x_3\delta u^n_{3\,|\Sigma}) + (1-\theta_\varepsilon(x))u^n,$$

we have $u^{\varepsilon,n} \in V^\varepsilon$ and we prove easily that $u^{\varepsilon,n} \rightharpoonup u^n$ in $\mathbb{G}^{\alpha,p}$ when $\varepsilon \longrightarrow 0$.
As

$$F^\varepsilon(u^{\varepsilon,n}) \quad = \quad \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijkh}e_{kh}(u^{\varepsilon,n})e_{ij}(u^{\varepsilon,n}) + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} |e(u^{\varepsilon,n})|^p.$$

It implies that

$$\begin{aligned}
F^\varepsilon(u^{\varepsilon,n}) &= \frac{1}{2}\int_{|x_3|>2\varepsilon\varphi_\varepsilon} a_{ijkh}e_{kh}(u^{\varepsilon,n})e_{ij}(u^{\varepsilon,n}) + \frac{1}{2}\int_{\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon} a_{ijkh}e_{kh}(u^{\varepsilon,n})e_{ij}(u^{\varepsilon,n}) \\
&\quad + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} |e(u^{\varepsilon,n})|^p \\
&= \frac{1}{2}\int_{|x_3|>2\varepsilon\varphi_\varepsilon} a_{ijkh}e_{kh}(u^n)e_{ij}(u^n) + \frac{1}{2}\int_{\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon} a_{ijkh}e_{kh}(u^{\varepsilon,n})e_{ij}(u^{\varepsilon,n}) \\
&\quad \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} \left|e\Big(u^n{}_{|\Sigma} - x_3\delta u^n_{3\,|\Sigma}\Big)\right|^p \\
&= S_1 + S_2 + S_3.
\end{aligned}$$

Since φ_ε is bounded, so it follows that

$$\begin{aligned}
\lim_{\varepsilon\longrightarrow 0} S_2 &= 0, \\
\lim_{\varepsilon\longrightarrow 0} S_1 &= \frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u^n)e_{ij}(u^n).
\end{aligned}$$

we have

$$S_3 = \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} \left|e_T\left(u^n{}_{|\Sigma}\right) + x_3 e_T(\delta u^n_{3\,|\Sigma})\right|^p, \tag{5.4}$$

1. If $0 \leq \alpha \leq 1$, we remark that $\gamma_\varepsilon = S_3 - \dfrac{1}{p\varepsilon^\alpha}\displaystyle\int_{B_\varepsilon} \left|e_{_T}({u^n}_{|\Sigma})\right|^p$, vanishes when $\varepsilon \to 0$. Indeed :
As

$$\gamma_\varepsilon \leq \frac{C}{p\varepsilon^\alpha}\int_{B_\varepsilon} |x_3 e_{_T}(\varphi)| \left(1 + \left|e_{_T}({u^n}_{|\Sigma}) + x_3 e_{_T}(\varphi)\right|^{p-1} + \left|e_{_T}({u^n}_{|\Sigma})\right|^{p-1}\right),$$

so that

$$\begin{aligned}\gamma_\varepsilon &\leq \frac{C\varepsilon^{1-\alpha}}{p}\int_{B_\varepsilon} |e_{_T}(\varphi)|,\\ &\leq \omega_\varepsilon,\end{aligned}$$

with $\omega_\varepsilon \to 0$ when $\varepsilon \to 0$. So passing to the upper limit, we obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n}) &= \frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u^n)e_{ij}(u^n) + \limsup_{\varepsilon\to 0}\frac{2\varepsilon^{1-\alpha}}{p}\int_\Sigma \varphi_\varepsilon \left|e_{_T}({u^n}_{|\Sigma})\right|^p,\\ &= \frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u^n)e_{ij}(u^n) + \frac{2\eta(\alpha)m(\varphi)}{p}\int_\Sigma \left|e_{_T}({u^n}_{|\Sigma})\right|^p.\end{aligned}$$

2. If $1 < \alpha \leq p+1$ (5.4) becomes

$$\begin{aligned}S_3 &= \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} |x_3 e_{_T}(\delta u_3^n)|^p,\\ &= \frac{2\varepsilon^{p+1-\alpha}}{p(p+1)}\int_\Sigma \varphi_\varepsilon^{p+1}\left|\delta\delta {u_3^n}_{|\Sigma}\right|^p,\end{aligned}$$

As $\varphi_\varepsilon^{p+1} \rightharpoonup^* m(\varphi^{p+1})$ in $L^\infty(\Sigma)$, so passing to the upper limit we have

$$\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n}) = \frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u^n)e_{ij}(u^n) + \frac{2\beta(\alpha)m(\varphi^{p+1})}{p(p+1)}\int_\Sigma \left|\delta\delta {u_3^n}_{|\Sigma}\right|^p.$$

3. If $\alpha > p+1$ (5.4) becomes

$$\begin{aligned}S_3 &= \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} |x_3 e_{_T}(\delta u_3^n)|^p\\ &= \frac{2\varepsilon^{p+1-\alpha}}{p(p+1)}\int_\Sigma \varphi_\varepsilon^{p+1}\left|\delta\delta {u_3^n}_{|\Sigma}\right|^p\\ &= 0,\end{aligned}$$

so passing to the upper limit we obtain

$$\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n}) = \frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u^n)e_{ij}(u^n),$$

Since $u^n \to u$ in $\mathbb{G}^{\alpha,p}$ when $n \longrightarrow +\infty$, therefore according to a classic result, diagonalization's lemma , (see, [5, Lemma 1.15 p. 32]), there exists a function $n(\varepsilon) : \mathbb{R}^+ \to \mathbb{N}$ increasing to $+\infty$ when $\varepsilon \to 0$ such that $u^{\varepsilon,n(\varepsilon)} \rightharpoonup u$ in $\mathbb{G}^{\alpha,p}$ when $\varepsilon \to 0$. and while $n \to +\infty$, consequently we have

$$\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n(\varepsilon)}) \quad \le \quad \limsup_{n\to+\infty}\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n}),$$

1. If $\alpha = 1$, we obtain

$$\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n(\varepsilon)}) \quad \le \quad \frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u)e_{ij}(u) + \frac{2\eta(\alpha)m(\varphi)}{p}\int_\Sigma \left|e_T(u_{|\Sigma})\right|^p,$$

2. If $\alpha = p+1$, we have

$$\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n(\varepsilon)}) \quad \le \quad \frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u)e_{ij}(u) + \frac{2\beta(\alpha)m(\varphi^{p+1})}{p(p+1)}\int_\Sigma \left|\delta\delta u_{3|\Sigma}\right|^p,$$

3. If $\alpha \neq 1$ and $\alpha \neq p+1$, we have

$$\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n(\varepsilon)}) \quad \le \quad \frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u)e_{ij}(u).$$

For $u \in W_0^{1,r}(\Omega,\mathbb{R}^3) \setminus \mathbb{G}^{\alpha,p}$, so for any sequence $u^\varepsilon \rightharpoonup u$ in $W_0^{1,r}(\Omega,\mathbb{R}^3)$ we obtain

$$\limsup_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) \le +\infty.$$

b) We are now in position to determine the lower epi-limit
Let $u \in \mathbb{G}^{\alpha,p}$ and $(u^\varepsilon) \subset V^\varepsilon$ such that $u^\varepsilon \rightharpoonup u$ in $W_0^{1,r}(\Omega,\mathbb{R}^3)$, then

$$\chi_{\Omega_\varepsilon} e(u^\varepsilon) \rightharpoonup e(u) \quad \text{ in } L^r(\Omega,\mathbb{R}^9_S) \tag{5.5}$$

1. If $\alpha \neq 1$ and $\alpha \neq p+1$
   As
   $$F^\varepsilon(u^\varepsilon) \ge \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijkh}e_{kh}(u^\varepsilon)e_{ij}(u^\varepsilon)$$
   From the subdifferentiability's inequality of $u \to \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijkh}e_{kh}(u)e_{ij}(u)$, and passing to the lower limit, we obtain
   $$\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) \ge \frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u)e_{ij}(u).$$
2. If $\alpha = 1$
   If $\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) = +\infty$, there is nothing to prove, because
   $$\frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u)e_{ij}(u) + \frac{2m(\varphi)\eta(\alpha)}{p}\int_\Sigma \left|e_T(u_{|\Sigma})\right|^p \le +\infty.$$

otherwise, $\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) < +\infty$, there exists a subsequence of $F^\varepsilon(u^\varepsilon)$, still denoted by $F^\varepsilon(u^\varepsilon)$ and a constant $C > 0$, such that $F^\varepsilon(u^\varepsilon) \leq C$, which implies that

$$\frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} |e(u^\varepsilon)|^p \leq C. \tag{5.6}$$

Let v_ε be the sequence defined in the proof of the proposition 4.8, according to the last and from (5.6), we obtain

$$v_\varepsilon \rightharpoonup 2m(\varphi)e_T(u_{|\Sigma}) \text{ in } L^p(\Sigma, \mathbb{R}^9_S)$$

We have

$$\begin{aligned}
F^\varepsilon(u^\varepsilon) &= \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijkh}e_{kh}(u^\varepsilon)e_{ij}(u^\varepsilon) + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} |e(u^\varepsilon)|^p \\
&\geq \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijkh}e_{kh}(u^\varepsilon)e_{ij}(u^\varepsilon) + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} |e_T(u^\varepsilon)|^p \\
&\geq \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijkh}e_{kh}(u^\varepsilon)e_{ij}(u^\varepsilon) + \frac{1}{p\varepsilon^\alpha}\int_{\Sigma} \frac{\varepsilon}{(2\varphi_\varepsilon)^{p-1}} |v_\varepsilon|^p \\
&\geq \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijkh}e_{kh}(u^\varepsilon)e_{ij}(u^\varepsilon) + \frac{\varepsilon^{1-\alpha}}{p}\int_{\Sigma} \frac{1}{(2\varphi_\varepsilon)^{p-1}} |v_\varepsilon|^p,
\end{aligned}$$

From the subdifferentiability's inequality of

$$v \to \frac{\varepsilon^{1-\alpha}}{p}\int_\Sigma \frac{1}{(2\varphi_\varepsilon)^{p-1}}|v|^p, \ \forall v \in L^p(\Sigma, \mathbb{R}^9_S).$$

we have

$$\begin{aligned}
F^\varepsilon(u^\varepsilon) &\geq \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijkh}e_{kh}(u^\varepsilon)e_{ij}(u^\varepsilon) + \frac{\varepsilon^{1-\alpha}}{p}\int_\Sigma \frac{1}{(2\varphi_\varepsilon)^{p-1}}\left|2m(\varphi)e_T(u_{|\Sigma})\right|^p \\
&+ \frac{\varepsilon^{1-\alpha}}{p}\int_\Sigma \frac{1}{(2\varphi_\varepsilon)^{p-1}}\left|2m(\varphi)e_T(u_{|\Sigma})\right|^{p-2}(2m(\varphi)e_T(u_{|\Sigma}))(v_\varepsilon - 2m(\varphi)e_T(u_{|\Sigma})),
\end{aligned}$$

thanks to the lemma 6.4, we have $\frac{1}{\varphi_\varepsilon^{p-1}} \to m(\frac{1}{\varphi^{p-1}})$ in $L^{p'}(\Sigma)$ and the fact that $m(\frac{1}{\varphi^{p-1}})\, m(\varphi)^{p-1} \geq 1$, so from (5.5) and passing to the limit, we deduce that

$$\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) \geq \frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u)e_{ij}(u) + \frac{2\eta(\alpha)m(\varphi)}{p}\int_\Sigma \left|e_T(u_{|\Sigma})\right|^p.$$

3. If $\alpha = p+1$
If $\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) = +\infty$, there is nothing to prove, because

$$\frac{1}{2}\int_\Omega a_{ijkh}e_{kh}(u)e_{ij}(u) + \frac{2\beta(\alpha)m(\varphi^{p+1})}{p(p+1)}\int_\Sigma \left|\delta\delta u_{3|\Sigma}\right|^p \leq +\infty.$$

otherwise, $\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) < +\infty$, there exists a subsequence of $F^\varepsilon(u^\varepsilon)$ still denoted by $F^\varepsilon(u^\varepsilon)$ and a constant $C > 0$, such that $F^\varepsilon(u^\varepsilon) \leq C$, which implies that

$$\frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} |e(u^\varepsilon)|^p \leq C. \tag{5.7}$$

Let ξ_ε be the sequence defined in the proof of the proposition 4.8, according to this last and from (5.7), we have

$$\xi_\varepsilon \rightharpoonup \frac{2}{p+1} m(\varphi^{p+1})\, \delta\delta u_{3|_\Sigma} \text{ in } L^p(\Sigma, \mathbb{R}^9_S)$$

To simplify the writing let us pose $\xi = \frac{2}{p+1} m(\varphi^{p+1})\delta\delta u_{3|_\Sigma}$, which implies that

$$\begin{aligned}
F^\varepsilon(u^\varepsilon) &\geq \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijkh} e_{kh}(u^\varepsilon) e_{ij}(u^\varepsilon) + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} |e_T(u^\varepsilon)|^p \\
&\geq \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijkh} e_{kh}(u^\varepsilon) e_{ij}(u^\varepsilon) + \frac{1}{p\varepsilon^\alpha}\int_\Sigma \Big(\frac{2}{p+1}\Big)^{p-1} \frac{\varepsilon^{p+1}}{(\varphi_\varepsilon)^{p^2-1}} |\xi_\varepsilon|^p \\
&\geq \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijkh} e_{kh}(u^\varepsilon) e_{ij}(u^\varepsilon) + \frac{\varepsilon^{p+1-\alpha}}{p}\int_\Sigma \Big(\frac{2}{p+1}\Big)^{p-1} \frac{1}{(\varphi_\varepsilon)^{p^2-1}} |\xi_\varepsilon|^p,
\end{aligned}$$

According to the subdifferentiability's inequality of

$$v \to \frac{\varepsilon^{p+1-\alpha}}{p}\int_\Sigma \Big(\frac{2}{p+1}\Big)^{p-1} \frac{1}{(\varphi_\varepsilon)^{p^2-1}} |v|^p, \ \forall v \in L^p(\Sigma, \mathbb{R}^9_S).$$

We have

$$\begin{aligned}
F^\varepsilon(u^\varepsilon) \geq{}& \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijkh} e_{kh}(u^\varepsilon) e_{ij}(u^\varepsilon) + \frac{\varepsilon^{p+1-\alpha}}{p}\int_\Sigma \Big(\frac{2}{p+1}\Big)^{p-1} \frac{1}{(\varphi_\varepsilon)^{p^2-1}} |\xi|^p \\
&+ \frac{\varepsilon^{p+1-\alpha}}{p}\int_\Sigma \Big(\frac{2}{p+1}\Big)^{p-1} \frac{1}{(\varphi_\varepsilon)^{p^2-1}} |\xi|^{p-2}\, \xi(v_\varepsilon - \xi),
\end{aligned}$$

from the lemma 6.4, we have $\frac{1}{\varphi_\varepsilon^{p^2-1}} \to m(\frac{1}{\varphi^{p^2-1}})$ in $L^{p'}(\Sigma)$ and the fact that $m(\frac{1}{\varphi^{p^2-1}})\ m(\varphi^{p+1})^{p-1} \geq 1$, so from (5.5) and passing to the limit, consequently we have

$$\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) \geq \frac{1}{2}\int_\Omega a_{ijkh} e_{kh}(u) e_{ij}(u) + \frac{2\beta(\alpha) m(\varphi^{p+1})}{p(p+1)}\int_\Sigma |\delta\delta u_{3|_\Sigma}|^p.$$

For $u \in W_0^{1,r}(\Omega, \mathbb{R}^3) \setminus \mathbb{G}^{\alpha,p}$ et $u^\varepsilon \in V^\varepsilon$, such that $u^\varepsilon \rightharpoonup u$ in $W_0^{1,r}(\Omega, \mathbb{R}^3)$. Assume that

$$\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) < +\infty.$$

So there exists a constant $C > 0$ and a subsequence of $F^\varepsilon(u^\varepsilon)$, still denoted by $F^\varepsilon(u^\varepsilon)$, such that

$$F^\varepsilon(u^\varepsilon) < C. \tag{5.8}$$

So u^ε verifies the following evaluations (4.2) and (4.3), as $u^\varepsilon \rightharpoonup u$ in $W_0^{1,r}(\Omega, \mathbb{R}^3)$, thanks to the remark 4.10, we have $u \in \mathbb{G}^{\alpha,p}$, what contradicts the fact that $u \in W_0^{1,r}(\Omega, \mathbb{R}^3) \setminus \mathbb{G}^{\alpha,p}$, consequently we have

$$\liminf_{\varepsilon \to 0} F^\varepsilon(u^\varepsilon) = +\infty.$$

Hence the proof of the theorem 5.1 is complete. □

In the sequel, we are interested to the limit problem determination linked to the problem (4.1), when ε approaches to zero. Thanks to the epi-convergence results, (see Annex, theorem 6.3, proposition 6.2) and the theorem 5.1, according to the τ_f-continuity of the functional G in $W_0^{1,r}(\Omega, \mathbb{R}^3)$, we have
$F^\varepsilon + G$ τ_f-epi-converges to $F^\alpha + G$ in $W_0^{1,r}(\Omega, \mathbb{R}^3)$.

Proposition 5.2. *For any $f \in L^{r'}(\Omega, \mathbb{R}^3)$ and according to the values of the parameter α, there exists $u^* \in W_0^{1,r}(\Omega, \mathbb{R}^3)$ satisfies*

$$u^\varepsilon \rightharpoonup u^* \text{ in } W_0^{1,r}(\Omega, \mathbb{R}^3),$$
$$F^\alpha(u^*) + G(u^*) = \inf_{v \in \mathbb{G}^{\alpha,p}} \left\{ \ F^\alpha(v) + G(v). \ \right\}$$

Proof: Thanks to the lemma 4.2, the family u^ε is bounded in $W_0^{1,r}(\Omega, \mathbb{R}^3)$, therefore it possess a τ_f−cluster point u^* in $W_0^{1,r}(\Omega, \mathbb{R}^3)$. And thanks to a classical epi-convergence result, (see Annex, theorem 6.3), it follows that u^* is a solution of the limit problem

$$\inf_{v \in W_0^{1,r}(\Omega, \mathbb{R}^3)} \Big\{ F^\alpha(v) + G(v) \Big\}. \tag{5.9}$$

As F^α equals $+\infty$ on $W_0^{1,r}(\Omega, \mathbb{R}^3) \setminus \mathbb{G}^{\alpha,p}$, so (5.9) becomes

$$(\Pi^\alpha) \inf_{v \in \mathbb{G}^\alpha} \Big\{ F^\alpha(v) + G(v) \Big\}.$$

According to the uniqueness of solutions of the problem (5.9), so u^ε admits an unique τ_f-cluster point u^*, and therefore $u^\varepsilon \rightharpoonup u^*$ in $W_0^{1,r}(\Omega, \mathbb{R}^3)$. □

Conclusion. We showed that the structure, constituted of two elastic bodies joined together by an elastic thin oscillating layer of thickness and rigidity and periodicity parameter depending on a small enough parameter ε, obeying to a nonlinear elastic law whose parameters depend on the negative powers of ε, behaves at the limit like an elastic body embedded on the boundary and subjected to a density of forces of volume f, according to the powers of ε, the layer behaves like a rather rigid nonlinear elastic material surface with membrane effect, too rigid inextensible material surface, a material surface with effect of inflection or the structure is embedded on the interface Σ.

6. Annex

Definition 6.1 ([5, Definition 1.9]). *Let* $(\mathbb{X}, \tau)$ *be a metric space and* $(F^\varepsilon)_\varepsilon$ *and* F *be functionals defined on* $\mathbb{X}$ *and with value in* $\mathbb{R} \cup \{+\infty\}$. F^ε *epi-converges to* F *in* $(\mathbb{X}, \tau)$, *noted* $\tau - lim_e F^\varepsilon = F$, *if the following assertions are satisfied*

- *For all* $x \in \mathbb{X}$, *there exists* x^0_ε, $x^0_\varepsilon \xrightarrow{\tau} x$ *such that* $\limsup_{\varepsilon \to 0} F^\varepsilon(x^0_\varepsilon) \leq F(x)$.
- *For all* $x \in \mathbb{X}$ *and all* x_ε *with* $x_\varepsilon \xrightarrow{\tau} x$, $\liminf_{\varepsilon \to 0} F^\varepsilon(x_\varepsilon) \geq F(x)$.

Note the following stability result of the epi-convergence.

Proposition 6.2 ([5, p. 40]). *Suppose that* F^ε *epi-converges to* F *in* $(\mathbb{X}, \tau)$ *and that* $G \colon \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$, *is* $\tau -$ *continuous. Then* $F^\varepsilon + G$ *epi-converges to* $F + G$ *in* $(\mathbb{X}, \tau)$

This epi-convergence is a special case of the $\Gamma-$convergence introduced by De Giorgi (1979) [8]. It is well suited to the asymptotic analysis of sequences of minimization problems since one has the following fundamental result.

Theorem 6.3 ([5, theorem 1.10]). *Suppose that*

1. F^ε *admits a minimizer on* $\mathbb{X}$,
2. *The sequence* $(\overline{u}^\varepsilon)$ *is* τ*-relatively compact,*
3. *The sequence* F^ε *epi-converges to* F *in this topology* τ.

Then every cluster point $\overline{u}$ *of the sequence* $(\overline{u}^\varepsilon)$ *minimizes* F *on* $\mathbb{X}$ *and*

$$\lim_{\varepsilon' \to 0} F^{\varepsilon'}(\overline{u}^{\varepsilon'}) = F(\overline{u}),$$

if $(\overline{u}^{\varepsilon'})_{\varepsilon'}$ *denotes the subsequence of* $(\overline{u}^\varepsilon)_\varepsilon$ *which converges to* $\overline{u}$.

Lemma 6.4. *Let* $\varphi \in L^\infty(\Sigma)$, *a* Y*-periodic,* $Y =]0,1[\times]0,1[$. *Let*

$$\varphi_\varepsilon(x) = \varphi(\frac{x}{\varepsilon}), \text{ for a small enough } \varepsilon > 0.$$

So that

$$\begin{aligned} \varphi_\varepsilon \to m(\varphi) \quad & \text{in } L^s(\Sigma) \text{ for } 1 \leq s < \infty, \\ \varphi_\varepsilon \rightharpoonup^* m(\varphi) \quad & \text{in } L^\infty(\Sigma). \end{aligned}$$

Proof: Since φ_ε is a εY-periodic, so one has

$$\begin{aligned} \varphi_\varepsilon \rightharpoonup m(\varphi) \quad & \text{in } L^s(\Sigma) \text{ for } 1 \leq s < \infty, \\ \varphi_\varepsilon \rightharpoonup^* m(\varphi) \quad & \text{in } L^\infty(\Sigma). \end{aligned} \tag{6.1}$$

Since φ is bounded a.e. in Σ, so for every $s \geq 1$, there exists a constant $C > 0$, such that

$$\begin{aligned}\int_\Sigma |\varphi_\varepsilon - m(\varphi)|^s &\leq C \int_\Sigma |\varphi_\varepsilon - m(\varphi)| \\ &\leq C\Big(\int_{\varphi \geq m(\varphi)} (\varphi_\varepsilon - m(\varphi)) - \int_{\varphi \leq m(\varphi)} (\varphi_\varepsilon - m(\varphi))\Big).\end{aligned} \tag{6.2}$$

Passing to the limit in (6.2), one has $\varphi_\varepsilon \to m(\varphi)$ in $L^s(\Sigma)$ for $1 \leq s < \infty$. □

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