a pie

Bol. Soc. Paran. Mat. ©SPM -ISSN-2175-1188 on LINE SPM: www.spm.uem.br/bspm (3s.) **v. 34** 2 (2016): 231–236. ISSN-00378712 in press doi:10.5269/bspm.v34i2.27177

μ -Compactness with Respect to a Hereditary Class

C. Carpintero, E. Rosas, M. Salas-Brown and J. Sanabria

ABSTRACT: We define and study the notion of compactness in generalized topological spaces with respect to a hereditary class: $\mu \mathcal{H}$ -compact spaces.

Key Words: Generalized topology, μ -compact space, hereditary classes.

Contents

1	Introduction	231
2	Preliminaries	231
3	μ -compactness with respect to an ideal	232

1. Introduction

The compactness is one of the most important concepts used, not only in general topology, but also in other areas of mathematics. Over recent years several authors have been working in formulate weak notions of open sets, in terms of these open sets those authors have extended and generalized the concept of compactness. In this article, we use the notions of generalized topology and hereditary class introduced by Csázár in [3] and [4], respectively, in order to define and characterize the $\mu\mathcal{H}$ -compact spaces, also some properties $\mu\mathcal{H}$ -compact spaces are obtained. The strategy of using generalized topologies and hereditary classes to extend classical topological concepts have been used by many authors such as [4], [7], [15], [17], among others.

Although the proofs presented in this articles are slight modifications of the standard proofs concern to the respective statements on usual compactness, the results presented in this article are of great interest since the definition of $\mu \mathcal{H}$ -compact spaces unifies and generalizes many of the notions of compactness obtained by many authors such as [1], [2], [6], [9], [10], [11], [12], [13], [14]. Also, our results generalizes and characterize several properties studied by the authors above cited. Moreover, the examples given in this note are not trivial and are different from the usual given by other authors.

2. Preliminaries

Let X be a nonempty set and 2^X be the power set of X. Then $\mu \subset 2^X$ is called a generalized topology (briefly GT) on X [3] if $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ implies $\bigcup_{i \in I} G_i \in \mu$. We call the pair (X, μ) a generalized topological space (briefly

Typeset by $\mathcal{B}^{\mathcal{S}}\mathcal{P}_{\mathcal{M}}$ style. © Soc. Paran. de Mat.

231

²⁰⁰⁰ Mathematics Subject Classification: 54C10, 54D10

GTS) on X. The elements of μ are called μ -open sets and the complements are called μ -closed sets. The generalized closure of a subset A of X, denoted by $c_{\mu}(A)$, is the intersection of all μ -closed sets containing A. And the generalized interior of A, denoted by $i_{\mu}(A)$, is the union of all μ -open sets contained in A. Let (X, μ_1) and (X, μ_2) be two GTS's, then a function $f : (X, \mu_1) \to (X, \mu_2)$ is said to be (μ_1, μ_2) -continuous iff $U \in \mu_2$ implies $f^{-1}(U) \in \mu_1$.

Let $A \subset X$. A collection \mathcal{F} of subsets of X, is said to be a μ -covering of A if \mathcal{F} is a covering of A by μ -open sets [9]. A subset A of X is said to be μ -compact if for every μ -covering $\{U_{\lambda} : \lambda \in \Lambda\}$ of A there exists a finite subcollection $\{U_{\lambda} : \lambda \in \Lambda_0\}$ that also covers A. X is said to be μ -compact if X is μ -compact as a subset [9].

A hereditary class \mathcal{H} on a nonempty set X [4] is a collection of subsets of X that satisfies the following property: If $A \in \mathcal{H}$ and $B \subset A$ then $B \in \mathcal{H}$. If the hereditary class \mathcal{H} satisfies the additional condition: If $A, B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$, then \mathcal{H} is called an ideal on X [8]. Let (X, μ) be a GTS and \mathcal{H} a hereditary class on X. For a subset A of X is defined the generalized local function of A with respect to \mathcal{H} and μ [4], as follows: $A^* = \{x \in X : U \cap A \notin \mathcal{H} \text{ for all } U \in \mu_x\}$, where $\mu_x = \{U : x \in U \text{ and } U \in \mu\}$. And for A a subset of X, is defined: $c^*_{\mu}(A) = A \cup A^*$. The collection $\mu^* = \{A \subset X : X \setminus A = c^*_{\mu}(X \setminus A)\}$ is a GT on X. The elements of μ^* are called μ^* -open and the complement of a μ^* -open set is called μ^* -closed set. It is clear that a subset A is μ^* -closed if and only if $A^* \subset A$.

Theorem 2.1. [4] Let (X, μ) be a GTS and \mathcal{H} be a hereditary class on X and A a subset of X, then $A^* \subset c_{\mu}(A)$.

Theorem 2.2. [4] Let (X, μ) be a GTS and \mathcal{H} be a hereditary class on X and A be a subset of X. If A is μ^* -open then for all $x \in A$ there exists $U \in \mu_x$ and $E \in \mathcal{H}$ such that $x \in U \setminus E \subset A$.

3. μ -compactness with respect to an ideal

In this section we introduce and study the notion of compactness with respect to a GTS and a hereditary class in order to obtain many well known generalized forms of compactness in the literature.

Definition 3.1. Let (X, μ) be a GTS and \mathcal{H} be a hereditary class on X. A subset A of X is said to be $\mu\mathcal{H}$ -compact if for every μ -covering $\{U_{\lambda} : \lambda \in \Lambda\}$ of A there exists a finite subcollection $\{U_{\lambda} : \lambda \in \Lambda_0\}$ such that $A \setminus \bigcup \{U_{\lambda} : \lambda \in \Lambda_0\} \in \mathcal{H}$. X is said to be a $\mu\mathcal{H}$ -compact space if X is $\mu\mathcal{H}$ -compact as a subset.

All μ -compact set is a $\mu \mathcal{H}$ -compact set. Observe that for particular cases of generalized topologies and hereditary classes, we recover well known classical concepts of compactness as is shown:

- 1. If μ is a topology and \mathcal{H} is an ideal then the $\mu\mathcal{H}$ -compact sets are the \mathcal{H} compact sets [14].
- 2. If μ is the collection of semiopen sets and \mathcal{H} is the empty collection then the $\mu\mathcal{H}$ -compact sets are the semi compact sets [6].

- 3. If μ is the collection of the semi open sets and \mathcal{H} is an ideal then the $\mu\mathcal{H}$ compact sets are the $S\mathcal{H}$ -compacts sets [1].
- 4. If μ is the collection of the preopen sets and \mathcal{H} is the empty collection then the $\mu\mathcal{H}$ -compact sets are the strongly-compact sets [10].
- 5. If μ is the collection of the α -open sets and \mathcal{H} is an ideal then the $\mu\mathcal{H}$ -compact sets are the $\alpha\mathcal{H}$ -compact sets [12].
- 6. If μ is a topology and \mathcal{H} is the empty collection then the $\mu\mathcal{H}$ -compact sets are the compact sets.
- 7. If μ is a GT and \mathcal{H} is the empty collection then the $\mu\mathcal{H}$ -compact sets are the μ -compact sets [9]. In this particular case we recover also the concept of γ -compact set introduced in [2].

Theorem 3.2. Let (X, μ) be a GTS and \mathcal{H} be an ideal on X, then the union of two $\mu\mathcal{H}$ -compact sets is a $\mu\mathcal{H}$ -compact set.

Proof: Let A, B be two $\mu \mathcal{H}$ -compact sets of X and let $\{U_{\lambda} : \lambda \in \Lambda\}$ be any μ covering of $A \cup B$. Then there exist two finite subsets $\Lambda_0, \Lambda_1 \subset \Lambda$ such that: $A \setminus \bigcup \{U_{\lambda} : \lambda \in \Lambda_0\} \in \mathcal{H}$ and $B \setminus \bigcup \{U_{\lambda} : \lambda \in \Lambda_1\} \in \mathcal{H}$. Observe that

$$A \cup B \setminus \left[J\{U_{\lambda} : \lambda \in \Lambda_0 \cup \Lambda_1\} \subset A \setminus \left[J\{U_{\lambda} : \lambda \in \Lambda_0\} \cup B \setminus \left[J\{U_{\lambda} : \lambda \in \Lambda_1\}\right] \right]$$

Since $\Lambda_0 \cup \Lambda_1$ is a finite subset of Λ and \mathcal{H} is an ideal on X, follows that $A \cup B \setminus \bigcup \{U_\lambda : \lambda \in \Lambda_0 \cup \Lambda_1\} \in \mathcal{H}$. In consequence $A \cup B$ is a $\mu \mathcal{H}$ -compact set of X. \Box

The following example shows that if the class \mathcal{H} is not an ideal then the union of two $\mu\mathcal{H}$ -compact subsets is not necessary $\mu\mathcal{H}$ -compact.

Example 3.3. Let \mathbb{R} be the set of real numbers, μ the usual topology and the hereditary class $\mathcal{H} = \{A \subset \mathbb{R} : A \subset (0,1) \text{ or } A \subset (1,2)\}$. Observe that A = (0,1) and B = (1,2) are $\mu\mathcal{H}$ -compact sets. But $A \cup B$ does not is $\mu\mathcal{H}$ -compact. Suppose that $A \cup B$ is $\mu\mathcal{H}$ -compact, then the collection $\{(1/n, 2 - 1/n) : n \in \mathbb{Z}_+\}$ is a μ - covering of $A \cup B$, follows that there exist a finite set $n_1, ..., n_k$ such that $(A \cup B) \setminus \bigcup_{i=1}^k (1/n_k, 2 - 1/n_k) \in \mathcal{H}$. If we take $N = max\{n_1, ..., n_2\}$, follows that $(A \cup B) \setminus (1/N, 2 - 1/N) \in \mathcal{H}$, but this is a contradiction, because $(A \cup B) \setminus (1/N, 2 - 1/N) = (0, 1/N] \cup [2 - 1/N, 2) \notin \mathcal{H}$.

Theorem 3.4. Let (X, μ) be a GTS and \mathcal{H} be a hereditary class on X. If A is a μ -closed subset of X and X is $\mu\mathcal{H}$ -compact then A is a $\mu\mathcal{H}$ -compact set.

Proof: Let $\{U_{\lambda} : \lambda \in \Lambda\}$ be a μ -covering of A, then the collection $\mathcal{F} = \{U_{\lambda} : \lambda \in \Lambda\} \cup (X \setminus A)$ is a μ -covering of X. By hypothesis there exist a finite subcollection $\hat{\mathcal{F}}$ of \mathcal{F} , such that $X \setminus \bigcup \{V_{\lambda} : V_{\lambda} \in \hat{\mathcal{F}}\} \in \mathcal{H}$. Now, we analyze the following two cases:

- 1. $\hat{\mathcal{F}} = \{U_{\lambda} : \lambda \in \Lambda_0\} \cup (X \setminus A),$
- 2. $\hat{\mathcal{F}} = \{U_{\lambda} : \lambda \in \Lambda_0\}$, where Λ_0 is a finite subset of Λ

Case 1. Then $X \setminus \{\bigcup \{U_{\lambda} : \lambda \in \Lambda_0\} \cup (X \setminus A)\} \in \mathcal{H}$, observe that:

$$X \setminus \{ \bigcup \{ U_{\lambda} : \lambda \in \Lambda_0 \} \cup (X \setminus A) \} = A \setminus \bigcup \{ U_{\lambda} : \lambda \in \Lambda_0 \}$$

and therefore, $A \setminus \bigcup \{ U_{\lambda} : \lambda \in \Lambda_0 \} \in \mathcal{H}.$

Case 2. Then $X \setminus \bigcup \{U_{\lambda} : \lambda \in \Lambda_0\} \in \mathcal{H}$ and therefore, $A \setminus \bigcup \{U_{\lambda} : \lambda \in \Lambda_0\} \in \mathcal{H}$. In both cases, we obtain that A is a $\mu \mathcal{H}$ -compact set. \Box

Theorem 3.5. Let (X, μ) be a GTS and \mathcal{H} be a hereditary class on X. If X is $\mu^*\mathcal{H}$ -compact then X is $\mu\mathcal{H}$ -compact. The converse is true if the class \mathcal{H} is an ideal.

Proof:

The first part of the theorem follows from the fact that all μ -closed set is μ^* -closed set by Theorem 2.1.

Now suppose that \mathcal{H} is an ideal and X is $\mu\mathcal{H}$ -compact. Given $\mathcal{A} = \{U_{\lambda} : \lambda \in \Lambda\}$ a μ^* -covering of X, then for each $x \in X$, $x \in U_{\lambda_x}$ for some $\lambda_x \in \Lambda$. By Theorem 2.2, there exist $V_{\lambda_x} \in \mu_x$ and $E_{\lambda_x} \in \mathcal{H}$ such that $x \in V_{\lambda_x} \setminus E_{\lambda_x} \subset U_{\lambda_x}$.

Observe that the collection $\{V_{\lambda_x} : \lambda_x \in \Lambda\}$ is a μ -covering of X, follows that there exists a finite subset $\Lambda_0 \subset \Lambda$ such that: $E = X \setminus \bigcup \{V_{\lambda_x} : \lambda_x \in \Lambda_0\} \in \mathcal{H}$.

Since \mathcal{H} is an ideal, $\bigcup \{E_{\lambda_x} : \lambda_x \in \Lambda_0\} \in \mathcal{H}$ and also $E \cup \{\bigcup \{E_{\lambda_x} : \lambda_x \in \Lambda_0\}\} \in \mathcal{H}$. Observe that $X \setminus \bigcup \{U_{\lambda_x} : \lambda_x \in \Lambda_0\} \subset E \cup \{\bigcup \{E_{\lambda_x} : \lambda_x \in \Lambda_0\}\}$. In consequence $X \setminus \bigcup \{U_{\lambda_x} : \lambda_x \in \Lambda_0\} \in \mathcal{H}$.

The following example shows that the converse of the above theorem may not be true if the condition \mathcal{H} is an ideal is removed.

Example 3.6. Let \mathbb{R} be the set of real numbers, the GT defined as $\mu = \{A \subset \mathbb{R} : A \text{ is infinite}\} \cup \{\emptyset\}$. The hereditary class on \mathbb{R} , $\mathcal{H} = \{\mathbb{R} \setminus A : A \in \mu\}$. Observe that \mathcal{H} does not is an ideal. \mathbb{R} is $\mu\mathcal{H}$ -compact. To proof this, take $\{U_{\lambda} : \lambda \in \Lambda\}$ any μ -covering of \mathbb{R} . Any finite subcollection $\{U_{\lambda} : \lambda \in \Lambda_0\}$ of $\{U_{\lambda} : \lambda \in \Lambda\}$, where $\Lambda_0 \subset \Lambda$, we obtain that, $\mathbb{R} \setminus \bigcup \{U_{\lambda} : \lambda \in \Lambda_0\} \subset \mathbb{R} \setminus U_{\lambda} \in \mathcal{H}$. In consequence, \mathbb{R} is $\mu\mathcal{H}$ -compact. Observe that for each $x \in \mathbb{R}$, $\{x\}$ is μ^* -open. Follows that $\{\{x\} : x \in \mathbb{R}\}$ is a μ^* -covering of \mathbb{R} . Suppose now that there exist $x_1, x_2, ..., x_n \in \mathbb{R}$ such that $\mathbb{R} \setminus \bigcup_{i=1}^n \{x_i\} \in \mathcal{H}$ and this is not possible. Therefore, \mathbb{R} is not $\mu^*\mathcal{H}$ -compact.

Theorem 3.7. Let (X, μ) be a GTS and \mathcal{H} be a hereditary class on X, X is $\mu\mathcal{H}$ -compact if and only if for any collection $\{F_{\lambda} : \lambda \in \Lambda\}$ of μ -closed sets of X such that $\bigcap\{F_{\lambda} : \lambda \in \Lambda\} = \emptyset$ there exists a finite subset $\Lambda_0 \subset \Lambda$ such that $\bigcap\{F_{\lambda} : \lambda \in \Lambda_0\} \in \mathcal{H}$.

234

Proof: Let $\{F_{\lambda} : \lambda \in \Lambda\}$ be any collection of μ -closed sets of X such that $\bigcap \{F_{\lambda} : \lambda \in \Lambda\} = \emptyset$, then $\{X \setminus F_{\lambda} : \lambda \in \Lambda\}$ is a μ -covering of X. By hypothesis, there exists a finite subset $\Lambda_0 \subset \Lambda$ such that $X \setminus \bigcup \{X \setminus F_{\lambda} : \lambda \in \Lambda_0\} \in \mathcal{H}$. Hence $\bigcap \{F_{\lambda} : \lambda \in \Lambda_0\} \in \mathcal{H}$.

Conversely, let $\{U_{\lambda} : \lambda \in \Lambda\}$ be any μ -covering of X. Then $\{X \setminus U_{\lambda} : \lambda \in \Lambda\}$ is a collection of μ -closed sets such that $\bigcap \{X \setminus U_{\lambda} : \lambda \in \Lambda\} = \emptyset$. By hypothesis, there exists a finite subset $\Lambda_0 \subset \Lambda$ such that $\bigcap \{X \setminus U_{\lambda} : \lambda \in \Lambda_0\} \in \mathcal{H}$. Consequently, we obtain that, $X \setminus \bigcup \{U_{\lambda} : \lambda \in \Lambda_0\} \in \mathcal{H}$ and therefore, X is $\mu \mathcal{H}$ -compact. \Box

Lemma 3.8. Let $f : (X, \mu) \to (Y, \mu')$ be a function. If \mathcal{H} is a hereditary class on X, then $f(\mathcal{H}) = \{f(E) : E \in \mathcal{H}\}$ is a hereditary class on Y.

Proof: Let $E_1 \in \mathcal{H}$ and $V \subset f(E_1)$. The set $E_2 = \{x_y \in E_1 : f(x_y) \in V\}$ is a subset of E_1 , therefore, $E_2 \in \mathcal{H}$. Since $f(E_2) = V$, follows that $V \in f(\mathcal{H})$. \Box

Theorem 3.9. Let $f : (X, \mu) \to (Y, \mu')$ be a (μ, μ') -continuous and surjective function and \mathcal{H} be a hereditary class on X. If X is $\mu\mathcal{H}$ -compact then Y is $\mu'f(\mathcal{H})$ -compact.

Proof: Let $\{U_{\lambda} : \lambda \in \Lambda\}$ a μ' -covering of Y, then $\{f^{-1}(U_{\lambda}) : \lambda \in \Lambda\}$ is a μ -covering of X. Hence there exists a finite subset $\Lambda_0 \subset \Lambda$ such that $X \setminus \bigcup \{f^{-1}(U_{\lambda}) : \lambda \in \Lambda_0\} \in \mathcal{H}$. Since f is a surjective function, $Y \setminus \bigcup \{U_{\lambda} : \lambda \in \Lambda_0\} \subset f(X \setminus \bigcup \{f^{-1}(U_{\lambda}): \lambda \in \Lambda_0\}) \in f(\mathcal{H})$. In consequence, Y is $\mu' f(\mathcal{H})$ -compact. \Box

References

- Abd El-Monsef M. E., Lashien E. F. and Nasef A. A., S-compactness via ideals, Tamkang Jour. of Math. 24 (1993), no. 4 431–443.
- 2. Császár Á., γ-Compact Spaces, Acta Math. Hungar. 87 (2000), Vol 1-2, 99–107.
- Császár Á., Generalized topology, generalized continuity, Acta Math. Hungar. 96 (2002), 351–357.
- Császár Á., Modification of generalized topologies via hereditary classes, Acta Math. Hungar. 115 (2007), Vol 1-2, 29–36.
- 5. Császár Á., Weak structures, Acta Math. Hungar. 131 (2011), Vol 1-2, 193–195
- Dorsett Ch., Semi compactness, semi separation axioms, and product spaces, Bull. Malaysian Math. Soc. 2 4 (1981), 21–28.
- Kim Y. K., Min W. K., On operations induced by hereditary classes on generalized topological spaces. Acta Math. Hungar. 137 (2012), no. 1-2, 130–138.
- 8. Kuratowski K., Topologies I, Warszawa, (1933).
- Jyothis T. and Sunil J., μ-Compactness in Generalized Topological Spaces, Journal of Advanced Studies in Topology. 3, (2012), no. 3,18–22.
- Mashhour A. S., Abd El-Monsef M.E., I.A. Hasanein and T. Noiri, Strongly compact spaces, Delta J. Sci. 8 (1) (1984), 30–46.
- Mustafa J. M., μ-Semi Compcatness and μ-semi Lindelofness in generalized topological sppaces, International Journal of Pure and Applied Mathematics. 78, No. 4, (2012), 535–241.

- 12. Nasef A. A. and Noiri T., On $\alpha\text{-compact modulo an ideal, Far East J. Math. Sci. 6, No. 6, (1998), 857–865.$
- Nasef A. A., Some classes of compactness in terms of ideals, Soochow Journal of Mathematics. 27, No. 3, (2001), 343–352.
- 14. Newcomb R. L., *Topologies Which Are Compact Modulo an Ideal*, Ph. D. Dissertation, Univ. of Cal. at Santa Barbara, (1967).
- Rajamani M., Inthumathi V. and Ramesh R., Some new generalized topologies via hereditary classes, Bol. Soc. Parana. Mat. 30 (2012), no. 2, 71–77.
- Xum GE. and Ying GE., µ-seaparations in generalized topological spaces, Appl Math. J Chinese Univ. 25 (2010), 243–252
- Zahram A. M., El-Saady K. and Ghareeb A., Modification of weak structures via hereditary clases, Appl. Math. Letters. 25 (2012), 869–872.

C. Carpintero Departamento de Matemáticas Universidad de Oriente Núcleo De Sucre Cumaná, VENEZUELA. E-mail address: carpintero.carlos@gmail.com

and

E. Rosas Departamento de Matemáticas Universidad de Oriente Núcleo De Sucre Cumaná, VENEZUELA. E-mail address: ennisrafael@gmail.com

and

M. Salas-Brown Departamento de Matemáticas Universidad De Oriente Núcleo De Sucre Cumaná, VENEZUELA. E-mail address: salasbrown@gmail.com

and

J. Sanabria Departamento de Matemáticas Universidad De Oriente Núcleo De Sucre Cumaná, VENEZUELA. E-mail address: jesanabri@gmail.com