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On Zweier Sequence Spaces and de la Vallée-Poussin mean of order α and some geometric properties

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abstract: The main purpose of this paper is to study some geometrical properties such as order continuous, the Fatou property and the Banach-Saks property of the new space $[\mathcal{Z}_\lambda^\alpha]_\infty(p)$.

Key Words: Zweier operator; order continuous; Fatou property; Banach-Saks property.

Contents

1. Introduction

We denote w, ℓ_∞, c and c_0, the spaces of all, bounded, convergent, null sequences, respectively. Also, by ℓ_1 and ℓ_p, we denote the spaces of all absolutely summable and p-absolutely summable series, respectively. Recall that a sequence $(x(i))_{i=1}^\infty$ in a Banach space X is called *Schauder* (or *basis*) of X if for each $x \in X$ there exists a unique sequence $(a(i))_{i=1}^\infty$ of scalars such that $x = \sum_{i=1}^\infty a(i)x(i)$, i.e. $\lim_{n\to\infty} \sum_{i=1}^n a(i)x(i) = x$. A sequence space X with a linear topology is called a *K-space* if each of the projection maps $P_i : X \to \mathbb{C}$ defined by $P_i(x) = x(i)$ for $x = (x(i))_{i=1}^\infty \in X$ is continuous for each natural i. A *Fréchet space* is a complete metric linear space and the metric is generated by a *F-norm* and a Fréchet space which is a *K-space* is called an *FK-space* i.e. a *K-space* X is called an *FK-space* if X is a complete linear metric space. In other words, X is an *FK-space* if X is a Fréchet space with continuous coordinatewise projections. All the sequence spaces mentioned above are FK-space except the space c_{00} which is the space of real sequences which have only a finite number of non-zero coordinates. An FK-space X which contains the space c_{00} is said to have the *property* AK if for every sequence $(x(i))_{i=1}^\infty \in X, x = \sum_{i=1}^\infty x(i)e(i)$ where $e(i) = (0, 0, ...1^{i^{th}place}, 0, 0, ...)$.

A Banach space X is said to be a *Köthe sequence space* if X is a subspace of w such that

(a) if $x \in w, y \in X$ and $|x(i)| \leq |y(i)|$ for all $i \in \mathbb{N}$, then $x \in X$ and $||x|| \leq ||y||$

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(b) there exists an element $x \in X$ such that $x(i) > 0$ for all $i \in \mathbb{N}$.

We say that $x \in X$ is *order continuous* if for any sequence $(x_n) \in X$ such that $x_n(i) \leq |x(i)|$ for all $i \in \mathbb{N}$ and $x_n(i) \to 0$ as $n \to \infty$ we have $||x_n|| \to 0$ holds.

A Köthe sequence space X is said to be *order continuous* if all sequences in X are order continuous. It is easy to see that $x \in X$ order continuous if and only if $||(0, 0, ..., 0, x(n+1), x(n+2), ...)|| \to 0$ as $n \to \infty$.

A Köthe sequence space X is said to be the *Fatou property* if for any real sequence x and (x_n) in X such that $x_n \uparrow x$ coordinatewisely and $\sup_n ||x_n|| < \infty$, we have that $x \in X$ and $||x_n|| \to ||x||$.

A Banach space X is said to have the *Banach-Saks property* if every bounded sequence (x_n) in X admits a subsequence (z_n) such that the sequence $(t_k(z))$ is convergent in X with respect to the norm, where

$$t_k(z) = \frac{z_1 + z_2 + ... + z_k}{k} \text{ for all } k \in \mathbb{N}.$$

Some of works on geometric properties of sequence space can be found in [4,5,9,10,16,20].

Şengönül [22] defined the sequence $y = (y_k)$ which is frequently used as the Z^i-transformation of the sequence $x = (x_k)$ i.e.

$$y_k = ix_k + (1-i)x_{k-1}$$

where $x_{-1} = 0, k \neq 0, 1 < k < \infty$ and Z^i denotes the matrix $Z^i = (z_{nk})$ defined by

$$z_{nk} = \begin{cases} i, & \text{if } n = k; \\ 1-i, & \text{if } n-1 = k; \\ 0, & \text{otherwise.} \end{cases}$$

Şengönül [22] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows

$$\mathcal{Z} = \{x = (x_k) \in w : Z^i x \in c\}$$

and

$$\mathcal{Z}_0 = \{x = (x_k) \in w : Z^i x \in c_0\}.$$

For details on Zweier sequence spaces we refer to [6,12,13,15,17,18].

Let $\lambda = (\lambda_r)$ be an increasing sequence of positive real numbers tending to ∞ such that $\lambda_r \leq \lambda_r + 1, \lambda_1 = 1$. The generalized de la Vallée-Poussin mean is defined by $t_r(x) = \frac{1}{\lambda_r} \sum_{k \in I_r} x_k$ where $I_r = [r - \lambda_r + 1, r]$ for $r = 1, 2, 3, ...$ A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L if $t_r(x) \to L$ as $r \to \infty$ (see [19]). If $\lambda_r = r$, then (V, λ)-summability is reduced to Cesáro summability.

We denote Λ the set of all increasing sequences of positive real numbers tending to ∞ such that $\lambda_r \leq \lambda_r + 1, \lambda_1 = 1$. For details on (V, λ)-summability we refer to [1,2,3,7,8,11,14,21].

2. New set of sequences of order α

In this section let $\alpha \in (0,1]$ be any real number, let $\lambda = (\lambda_r)$ be an increasing sequence of positive real numbers tending to ∞ such that $\lambda_r \leq \lambda_r + 1, \lambda_1 = 1$, and p be a positive real number such that $1 \leq p < \infty$. Now we define the following sequence space.

$$[\mathcal{Z}_\lambda^\alpha]_\infty(p) = \left\{ x \in w : \sup_r \frac{1}{\lambda_r^\alpha} \sum_{k \in I_r} |\left(Z^i x\right)_k|^p < \infty \right\}.$$

Special cases:

(a) For $p = 1$ we have $[\mathcal{Z}_\lambda^\alpha]_\infty(p) = [\mathcal{Z}_\lambda^\alpha]_\infty$.

(b) For $\alpha = 1$ and $p = 1$ we have $[\mathcal{Z}_\lambda^\alpha]_\infty(p) = [\mathcal{Z}_\lambda]_\infty$.

Theorem 2.1. *Let $\alpha \in (0,1]$ and p be a positive real number such that $1 \leq p < \infty$. Then the sequence space $[\mathcal{Z}_\lambda^\alpha]_\infty(p)$ is a BK-space normed by*

$$||x||_\alpha = \sup_r \frac{1}{\lambda_r^\alpha} \left(\sum_{k \in I_r} |\left(Z^i x\right)_k|^p \right)^{\frac{1}{p}}.$$

Proof: The proof of the result is straightforward, so omitted. □

Theorem 2.2. *Let $\alpha \in (0,1]$ and p be a positive real number such that $1 \leq p < \infty$. Then $[\mathcal{Z}_\lambda^\alpha]_\infty \subset [\mathcal{Z}_\lambda^\alpha]_\infty(p)$.*

Proof: The proof of the result is straightforward, so omitted. □

Theorem 2.3. *Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and p be a positive real number such that $1 \leq p < \infty$. Then $[\mathcal{Z}_\lambda^\alpha]_\infty(p) \subset [\mathcal{Z}_\lambda^\beta]_\infty(p)$.*

Proof: The proof of the result is straightforward, so omitted. □

Theorem 2.4. *Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and p be a positive real number such that $1 \leq p < \infty$. For any two sequences $\lambda = (\lambda_r)$ and $\mu = (\mu_r)$ for all r, then $[\mathcal{Z}_\lambda^\alpha]_\infty(p) \subset [\mathcal{Z}_\mu^\beta]_\infty(p)$ if and only if $\sup_r \left(\frac{\lambda_r^\alpha}{\mu_r^\beta}\right) < \infty$.*

Proof: Let $x=(x_k)\in[\mathcal{Z}^{\alpha}_{\lambda}]_{\infty}(p)$ and $\sup_r\left(\frac{\lambda^{\alpha}_r}{\mu^{\beta}_r}\right)<\infty$. Then

$$\sup_r \frac{1}{\lambda^{\alpha}_r}\sum_{k\in I_r}|\left(Z^i x\right)_k|^p<\infty$$

and there exists a positive number K such that $\lambda^{\alpha}_r\le K\mu^{\beta}_r$ and so that $\frac{1}{\mu^{\beta}_r}\le\frac{K}{\lambda^{\alpha}_r}$ for all r. Therefore, we have

$$\frac{1}{\mu^{\beta}_r}\sum_{k\in I_r}|\left(Z^i x\right)_k|^p\le\frac{K}{\lambda^{\alpha}_r}\sum_{k\in I_r}|\left(Z^i x\right)_k|^p.$$

Now taking supremum over r, we get

$$\sup_r\frac{1}{\mu^{\beta}_r}\sum_{k\in I_r}|\left(Z^i x\right)_k|^p\le\sup_r\frac{K}{\lambda^{\alpha}_r}\sum_{k\in I_r}|\left(Z^i x\right)_k|^p$$

and hence $x\in[\mathcal{Z}^{\beta}_{\mu}]_{\infty}(p)$.

Next suppose that $[\mathcal{Z}^{\alpha}_{\lambda}]_{\infty}(p)\subset[\mathcal{Z}^{\beta}_{\mu}]_{\infty}(p)$ and $\sup_r\left(\frac{\lambda^{\alpha}_r}{\mu^{\beta}_r}\right)=\infty$. Then there exists an incresing sequence (r_i) of natural numbers such that $\lim_i\left(\frac{\lambda^{\alpha}_{r_i}}{\mu^{\beta}_{r_i}}\right)=\infty$. Let L be a positive real number, then there exists $i_0\in\mathbb{N}$ such that $\frac{\lambda^{\alpha}_{r_i}}{\mu^{\beta}_{r_i}}>L$ for all $r_i\ge i_0$. Then $\lambda^{\alpha}_{r_i}>L\mu^{\beta}_{r_i}$ and so $\frac{1}{\mu^{\beta}_{r_i}}>\frac{L}{\lambda^{\alpha}_{r_i}}$. Therefore we can write

$$\frac{1}{\mu^{\beta}_{r_i}}\sum_{k\in I_r}|\left(Z^i x\right)_k|^p>\frac{L}{\lambda^{\alpha}_{r_i}}\sum_{k\in I_r}|\left(Z^i x\right)_k|^p \text{ for all } r_i\ge i_0.$$

Now taking supremum over $r_i\ge i_0$ then we get

$$\sup_{r_i\ge i_0}\frac{1}{\mu^{\beta}_{r_i}}\sum_{k\in I_{r_i}}|\left(Z^i x\right)_k|^p>\sup_{r_i\ge i_0}\frac{L}{\lambda^{\alpha}_{r_i}}\sum_{k\in I_{r_i}}|\left(Z^i x\right)_k|^p. \tag{2.1}$$

Since the relation (2.1) holds for all $L\in\mathbb{R}^+$ (we may take the number L sufficienlty large), we have

$$\sup_{r_i\ge i_0}\frac{1}{\mu^{\beta}_{r_i}}\sum_{k\in I_{r_i}}|\left(Z^i x\right)_k|^p=\infty$$

but $x=(x_k)\in[\mathcal{Z}^{\alpha}_{\lambda}]_{\infty}(p)$ with

$$\sup_r\left(\frac{\lambda^{\alpha}_r}{\mu^{\beta}_r}\right)<\infty.$$

Therefore $x\notin[\mathcal{Z}^{\beta}_{\mu}]_{\infty}(p)$ which contradicts that $[\mathcal{Z}^{\alpha}_{\lambda}]_{\infty}(p)\subset[\mathcal{Z}^{\beta}_{\mu}]_{\infty}(p)$. Hence $\sup_{r\ge1}\left(\frac{\lambda^{\alpha}_r}{\mu^{\beta}_r}\right)<\infty$. □

Corollary 2.5. *Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and p be a positive real number such that $1 \leq p < \infty$. For any two sequences $\lambda = (\lambda_r)$ and $\mu = (\mu_r)$ in Λ for all $r \geq 1$, then*

(a) $[\mathcal{Z}_\lambda^\alpha]_\infty(p) = [\mathcal{Z}_\mu^\beta]_\infty(p)$ *if and only if* $0 < \inf_r \left(\frac{\lambda_r^\alpha}{\mu_r^\beta}\right) < \sup_r \left(\frac{\lambda_r^\alpha}{\mu_r^\beta}\right) < \infty$.

(b) $[\mathcal{Z}_\lambda^\alpha]_\infty(p) = [\mathcal{Z}_\mu^\alpha]_\infty(p)$ *if and only if* $0 < \inf_r \left(\frac{\lambda_r^\alpha}{\mu_r^\alpha}\right) < \sup_r \left(\frac{\lambda_r^\alpha}{\mu_r^\alpha}\right) < \infty$.

(c) $[\mathcal{Z}_\lambda^\alpha]_\infty(p) = [\mathcal{Z}_\lambda^\beta]_\infty(p)$ *if and only if* $0 < \inf_r \left(\frac{\lambda_r^\alpha}{\lambda_r^\beta}\right) < \sup_r \left(\frac{\lambda_r^\alpha}{\lambda_r^\beta}\right) < \infty$.

Theorem 2.6. $\ell_p \subset [\mathcal{Z}_\lambda^\alpha]_\infty(p) \subset \ell_\infty$.

Proof: The proof of the result is straightforward, so omit it. □

Theorem 2.7. *If $0 < p < q$, then $[\mathcal{Z}_\lambda^\alpha]_\infty(p) \subset [\mathcal{Z}_\lambda^\alpha]_\infty(q)$.*

Proof: The proof of the result is straightforward, so omit it. □

3. Some geometric properties of the new space

In this section we study some of the geometric properties like order continuous, the Fatou property and the Banach-Saks property in this new sequence space.

Theorem 3.1. *The space $[\mathcal{Z}_\lambda^\alpha]_\infty(p)$ is order continuous.*

Proof: We have to show that the space $[\mathcal{Z}_\lambda^\alpha]_\infty(p)$ is an AK-space. It is easy to see that $[\mathcal{Z}_\lambda^\alpha]_\infty(p)$ contains c_{00} which is the space of real sequences which have only a finite number of non-zero coordinates. By using the definition of AK-properties, we have that $x = (x(i)) \in [\mathcal{Z}_\lambda^\alpha]_\infty(p)$ has a unique representation $x = \sum_{i=1}^\infty x(i)e(i)$ i.e. $||x - x^{[j]}||_\alpha = ||(0, 0, .., x(j), x(j+1), ...)||_\alpha \to 0$ as $j \to \infty$, which means that $[\mathcal{Z}_\lambda^\alpha]_\infty(p)$ has AK. Therefore BK-space $[\mathcal{Z}_\lambda^\alpha]_\infty(p)$ containing c_{00} has AK-property, hence the space $[\mathcal{Z}_\lambda^\alpha]_\infty(p)$ is order continuous. □

Theorem 3.2. *The space $[\mathcal{Z}_\lambda^\alpha]_\infty(p)$ has the Fatou property.*

Proof: Let x be a real sequence and (x_j) be any nondecreasing sequence of non-negative elements form $[\mathcal{Z}_\lambda^\alpha]_\infty(p)$ such that $x_j(i) \to x(i)$ as $j \to \infty$ coordinatewisely and $\sup_j ||x_j||_\alpha < \infty$.

Let us denote $T = \sup_j ||x_j||_\alpha$. Since the supremum is homogeneous, then we have

$$\frac{1}{T} \sup_r \frac{1}{\lambda_r^\alpha} \left(\sum_{k \in I_r} | \left(Z^i x_j(i)\right)_k |^p \right)^{\frac{1}{p}}$$

$$\leq \sup_r \frac{1}{\lambda_r^\alpha} \left(\sum_{k \in I_r} \left| \frac{\left(Z^i x_j(i)\right)_k}{||x_n||_\alpha} \right|^p \right)^{\frac{1}{p}}$$

$$= \frac{1}{||x_n||_\alpha} ||x_n||_\alpha = 1.$$

Also by the assumptions that (x_j) is non-dreceasing and convergent to x coordinatewisely and by the Beppo-Levi theorem, we have

$$\frac{1}{T} \lim_{j \to \infty} \sup_r \frac{1}{\lambda_r^\alpha} \left(\sum_{k \in I_r} | \left(Z^i x_j(i)\right)_k |^p \right)^{\frac{1}{p}}$$

$$= \sup_r \frac{1}{\lambda_r^\alpha} \left(\sum_{k \in I_r} \left| \frac{\left(Z^i x(i)\right)_k}{T} \right|^p \right)^{\frac{1}{p}} \leq 1,$$

whence

$$||x||_\alpha \leq T = \sup_j ||x_j||_\alpha = \lim_{j \to \infty} ||x_j||_\alpha < \infty.$$

Therefore $x \in [\mathcal{Z}_\lambda^\alpha]_\infty(p)$. On the other hand, since $0 \leq x$ for any natural number j and the sequence (x_j) is non-decreasing, we obtain that the sequence $(||x_j||_\alpha)$ is bounded form above by $||x||_\alpha$. Therefore $\lim_{j\to\infty} ||x_j||_\alpha \leq ||x||_\alpha$ which contadicts the above inequality proved already, yields that $||x||_\alpha = \lim_{j\to\infty} ||x_j||_\alpha$. □

Theorem 3.3. *The space $[\mathcal{Z}_\lambda^\alpha]_\infty(p)$ has the Banach-Saks property.*

Proof: The proof of the result follows from the standard technique. □

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