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On Homological Properties of Some Module Derivations on Banach Algebras

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ABSTRACT: In recent years, lots of papers have been published on module amenability. In this paper, our main aim is to study the homological properties of various module derivations and prove some results about module amenability. So this paper continous a line investigation in [3], [4] for Banach algebras.

Key Words: Banach modules, bimodules, module derivation, homological functor, inverse semigroup.

Contents

1. Introduction

In noncommutative rings, the notion of a derivation was extended to a (σ, τ)-derivation, a right (left) derivation, a Jordan derivation, a Lie derivation, and a central derivation etc. The properties of various derivations were discussed in many papers with respect to the ring structures. Later, these derivations were generalized by M. Brešar [5] and A. Nakajima [14]. Furthermore, some properties of derivations and generalized derivations, such as homological or categorical properties, have been obtained on several algebras, especially on Banach algebras.

In recent years, lots of papers have been published on amenability and certain kinds of amenability of Banach algebras, related with the concept of derivation. A Banach algebra A is called *module amenable* (as a U-module where U is a Banach algebra) if for any commutative Banach A-U-module X, each module derivation $D : A \to X^*$ is inner, equivalently if $H(A, X^*) = \{0\}$ where $H(A, X^*)$ is the

2000 *Mathematics Subject Classification:* Primary 46H25; Secondary 20M18 first relative (to U) cohomology group of A with coefficients in X^*. The module cohomology of certain Banach algebras is studied in [15] (see for details; [1,7,8,11, 12]). Our main objective is to fill the gap between the homological properties of various module derivations and module amenability.

This paper is closely connected to the earlier works [3] and [4]. Building upon ideas from [14] and [9], we shall give similar results and leave some open questions.

The paper is organized as follows.

In section 2, we give the main definitions about module derivations and basic facts on amenability and introduce the notations needed later.

In section 3, we give a necessary and sufficient condition for $Z^B(A,X)$ to be isomorphic to $Z^G(A,X)$. In the following two sections, we discuss the functorial and opposite properties of module derivations.

Finally, we extend the previous results to module (σ,τ)-derivations and give some open questions. In the cases where the proofs are similar to the classical case, the results are stated without proof.

2. Preliminaries

2.1. Some Structures

Let U be a Banach algebra. A Banach space X which is also an U-bimodule is called a *Banach U-bimodule* if there exists a constant $K > 0$ such that

$$\|\alpha \cdot x\| \leq K\|\alpha\|\|x\| \text{ and } \|x \cdot \alpha\| \leq K\|\alpha\|\|x\|$$

for each $\alpha \in U$ and $x \in X$.

Throughout this paper, as in [2], A and U are Banach algebras such that A is a Banach U-bimodule with the compatible actions, as follows:

$$\alpha \cdot (ab) = (\alpha \cdot a)b \quad , \quad (ab) \cdot \alpha = a(b \cdot \alpha) \quad (\alpha \in U \ , \quad a, b \in A)$$

Let X be a Banach A-bimodule and a Banach U-bimodule with the compatible actions, that is;

$$\alpha{\cdot}(a{\cdot}x) = (\alpha{\cdot}a){\cdot}x \ , \ (\alpha{\cdot}x){\cdot}a = \alpha(x{\cdot}a) \ , \ (a{\cdot}\alpha){\cdot}x = a{\cdot}(\alpha{\cdot}x) \quad (\alpha \in U \ , \ \ a \in A, \ \ x \in X)$$

and similarly for the right or two-sided actions. Then, we say that X is a *Banach A-U-module*. If moreover $\alpha \cdot x = x \cdot \alpha$ ($\alpha \in U, \ x \in X$), then X is called a *commutative Banach A-U-module*. Furthermore if $a \cdot x = x \cdot a$ for all $x \in X$ and $a \in A$, then X is called a *bi-commutative Banach A-U-module*. Throughout this paper, by a A-U-module, we shall always mean a commutative Banach A-U-module.

Note that in general, A is not a Banach A-U-module because A does not satisfy the compatibility condition $a(\alpha \cdot b) = (a \cdot \alpha)b$ for $\alpha \in U$ and $a, b \in A$. But when A is a commutative U-bimodule and acts on itself by algebra multiplication from both sides, then it is also a Banach A-U-module.

If X is a (commutative) Banach A-U-module, then so is X^*, where the actions of A and U on X^* are defined as follows:

$$< f\cdot\alpha, x >=< f, \alpha\cdot x > , \quad < f\cdot a, x >=< f, a\cdot x >,$$
$$< \alpha\cdot f, x >=< f, x\cdot\alpha >, \text{ and } < a\cdot f, x >=< f, x\cdot a >$$

for each $a \in A, \alpha \in U, x \in X$ and $f \in X^*$.

Let $X\widehat{\otimes}Y$ denote the projective tensor product of two Banach spaces X and Y. We consider the module projective tensor product $A\widehat{\otimes}_U A$ which is a Banach A-U-module with canonical actions. Let I be the closed linear span of

$$\{(a\cdot\alpha)\otimes b - a\otimes(\alpha\cdot b) \mid \alpha\in U, a, b\in A\}$$

in A. Also consider the bounded linear map $w : A\widehat{\otimes}A \to A$ defined by $w(a\otimes b) = ab$ and the closed ideal J of A generated by $w(I)$.

It follows immediately that I and J are both A-submodules and U-submodules of $A\widehat{\otimes}A$ and A, respectively, so the module projective tensor product $A\widehat{\otimes}_U A \cong (A\widehat{\otimes}A)/I$ and the quotient Banach algebra A/J are both Banach A-modules and Banach U-modules. Also A/J is a Banach A-U-module with the compatible actions when A acts on A/J canonically.

The following proposition, which is proved in [16, Proposition 4.1], characterizes the $A\widehat{\otimes}_U A^{op}$-$U$-module homomorphisms where A^{op} is the opposite of A. By a *left essential* A-module X we mean a left Banach A-module X such that the linear span of $A\cdot X = \{a\cdot x \mid a\in A, x\in X\}$ is dense in X. Right essential A-modules and two-sided essential A-bimodules are defined similarly.

Proposition 2.1. *Let A be a commutative Banach U-module, and let X, Y be A-U-modules.*

(i) *If $\phi : X \to Y$ is a bounded A-U-module homomorphism, then it is a left $A\widehat{\otimes}_U A^{op}$-$U$-module homomorphism;*

(ii) *If Y is an essential A-bimodule and $\varphi : X^* \to Y^*$ is a bounded right $A\widehat{\otimes}_U A^{op}$-$U$-module homomorphism, then φ is an A-U-module homomorphism.*

2.2. Some Maps

Let A and B be Banach algebras and Banach U-bimodules with compatible actions, a *U-module map* is a bounded map $h : A \to B$ with

$$h(a\pm b) = h(a)\pm h(b),\ h(\alpha\cdot a) = \alpha\cdot h(a),\ h(a\cdot\alpha) = h(a)\cdot\alpha \quad (\alpha\in U, a, b\in A).$$

Note that h is bounded if there exists $M > 0$ such that $\|h(a)\| \le M\|a\|$, for each $a \in A$. Here h is not necessarily linear, so it is not necessarily a U-module homomorphism.

h is called *multiplicative U-module map* (or called *U-module morphism*) if $h(ab) = h(a)h(b)$ $(a, b \in A)$. We denote by $Hom_U(A, B)$ the metric space of all multiplicative U-module maps from A into B, with the metric derived from the usual linear operator norm $\|.\|$ on $L_U(A, B)$; the set of all bounded linear operators from A into B, and denote $Hom_U(A, A)$ by $Hom(A)$.

Let A and U be as above and X be a Banach A-U-module. A *module derivation* $D : A \to X$ is a U-module map such that $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in A$. The set of all module derivations from A to X is denoted by $Z^U(A, X)$ (abb. $Z(A, X)$). Note that D is not necessarily linear, but still its boundedness implies its norm continuity since D preserves subtraction.

Let f and g be module derivations from A to X and $\alpha \in U$, so are $f+g$ and αf. Since X is a Banach U-bimodule, we have that $Z(A, X)$ is a Banach U-bimodule.

For $x \in X$, define a map by $D_x : A \to X$, $a \longmapsto a \cdot x - x \cdot a$, $a \in A$. When X is a A-U-module, it is clear that D_x is a module derivation. Module derivations of this kind are called *inner* and denoted by $InnZ(A, X)$. (In the amenability papers, generally, the notation $B(A, X)$ is used for the set of inner module derivations from A to X, instead of $InnZ(A, X)$).

We consider the quotient space $H(A, X) = Z(A, X)/InnZ(A, X)$ which call the first relative (to U) cohomology group of A with coefficients in X. Hence A is *module amenable* if and only if $H(A, X^*) = \{0\}$, for each A-U-module X.

A *Jordan module derivation* $D : A \to X$ is a U-module map such that $D(a^2) = D(a) \cdot a + a \cdot D(a)$ for all $a \in A$. The set of all Jordan module derivations from A to X is denoted by $JZ(A, X)$.

The U-module map $f : A \to X$ is said to be *Lie module derivation* if the identity

$$f([a, b]) = [f(a), b] + [a, f(b)]$$

holds for all $a, b \in A$. The set of all Lie module derivations is denoted by $LieZ(A,X)$. Here $[a, b] = ab - ba$.

By a *Brešar generalized module derivation* (f, D), we mean $f : A \to X$ is a U-module map such that $f(ab) = f(a) \cdot b + a \cdot D(b)$ for all $a, b \in A$, where D is a module derivation on A. We denote by $Z^B(A, X)$ is the set of Brešar generalized module derivation from A to X.

If (f_1, D_1) and (f_2, D_2) are Brešar generalized module derivations and $\alpha \in U$, then $(f_1 + f_2, D_1 + D_2)$ and $(\alpha f_1, \alpha D_1)$ are also Brešar generalized module derivations and hence, $Z^B(A, X)$ is a Banach U-bimodule.

For $x, y \in X$, a U-module map satisfies the identity

$$f_{x,y} : A \ni a \mapsto (x \cdot a + a \cdot y) \in X$$

for all $a \in A$ is called a *Brešar generalized inner module derivation.*

We also consider the quotient space $H^B(A, X) = Z^B(A, X)/InnZ^B(A, X)$, called the first generalized cohomology group from A into X (as in [13]). Then, A is said to be *generalized module amenable* (resp. *weakly generalized module amenable*) if $H^B(A, X^*) = \{0\}$ (resp. $H^B(A, A^*) = \{0\}$), for each A-U-module X.

For a U-module map $f : A \to X$ is called a *Brešar generalized Jordan module derivation* if

$$f(a^2) = f(a) \cdot a + a \cdot D(a)$$

for all $a \in A$. Here D is a Jordan module derivation. We denote the set of Brešar generalized Jordan module derivations from A to X by $JZ^B(A, X)$.

The U-module map $f : A \to X$ is said to be *Brešar generalized Lie module derivation* if the identity

$$f([a, b]) = [f(a), b] + [a, D(b)]$$

holds for all $a, b \in A$. Here D is a Lie module derivation. We denote the set by $LieZ^B(A, X)$.

For a U-module map $f : A \to X$ and an element $x \in X$, a pair (f, x) is called a *generalized module derivation* in the sense of *Nakajima*, if

$$f(ab) = f(a) \cdot b + a \cdot f(b) + a \cdot x \cdot b$$

for all $a, b \in A$. We denote the set of this type of generalized module derivations by $Z^G(A, X)$. This is also a Banach U-bimodule for a Banach A-U-module X.

For $x, y \in X$, a U-module map $f_{x,y} : A \to X$ is called *generalized inner module derivation* if

$$f_{x,y}(ab) = f_{x,y}(a) \cdot b + a \cdot f_{x,y}(b) + a \cdot (-x - y) \cdot b$$

for all $a, b \in A$. We denote this derivation by $(f_{x,y}, -x - y)$.

We also denote the first Nakajima generalized cohomology group of A with coefficients in X by the quotient space $H^G(A, X) = Z^G(A, X)/InnZ^G(A, X)$. Similar to other definitions of amenability of Banach algebras we say A is a *Nakajima generalized module amenable* (resp. *Nakajima weakly generalized module amenable*) if $H^G(A, X^*) = \{0\}$ (resp. $H^G(A, A^*) = \{0\}$), for every A-U-module X.

A pair (f, x) is called a *generalized Jordan module derivation* if

$$f(a^2) = f(a) \cdot a + a \cdot f(a) + a \cdot x \cdot a$$

for all $a \in A$. We denote the set of generalized Jordan module derivations from A to X by $JZ^G(A, X)$.

The pair (f, x) is called a *generalized Lie module derivation* if the relation

$$f([a, b]) = [f(a), b] + [a, f(b)] + a \cdot x \cdot b - b \cdot x \cdot a$$

holds for all $a, b \in A$ and the set of generalized Lie module derivations from A to X can be denoted by $LieZ^G(A, X)$.

If $x = 0$, then these definitions lead to the conventional notions of generalized Jordan and Lie module derivations.

Throughout this paper we use the following notations for the above sets:

$Z(A, X)$, the set of module derivations,

$InnZ(A, X)$, the set of inner module derivations,

$JZ(A, X)$, the set of Jordan module derivations,

$LieZ(A, X)$, the set of Lie module derivations,

$Z^B(A, X)$, the set of Breŝar generalized module derivations,

$InnZ^B(A, X)$, the set of Breŝar generalized inner module derivations,

$JZ^B(A, X)$, the set of Breŝar generalized Jordan module derivations,

$LieZ^B(A, X)$, the set of Breŝar generalized Lie module derivations,

$Z^G(A, X)$, the set of generalized module derivations,

$InnZ^G(A, X)$, the set of generalized inner module derivations,

$JZ^G(A, X)$, the set of generalized Jordan module derivations,

$LieZ^G(A, X)$, the set of generalized Lie module derivations.

Furthermore, we need some extra new derivation sets:

Let σ and τ be arbitrary elements of $Hom_U(A, A)$ (abb. $Hom(A)$). A U-module map $D : A \to X$ is called a *module* (σ, τ)*-derivation* if

$$D(ab) = D(a) \cdot \tau(b) + \sigma(a) \cdot D(b)$$

for all $a, b \in A$.

When X is a A-U-module, each $x \in X$ defines a module (σ, τ)-derivation $D^x_{(\sigma,\tau)} : A \to X$ by $D^x_{(\sigma,\tau)}(a) = x \cdot \sigma(a) - \tau(a) \cdot x$ $(a \in A)$. These are called *inner module* (σ, τ)*-derivations*.

We consider the quotient space $H_{(\sigma,\tau)}(A, X) = Z_{(\sigma,\tau)}(A, X)/InnZ_{(\sigma,\tau)}(A, X)$, called the first (σ, τ)-cohomology group of A with coefficients in X (see [6]). Hence A is *module* (σ, τ)*-amenable* if and only if $H_{(\sigma,\tau)}(A, X^*) = \{0\}$, for all A-U-module X.

Recall that a *Jordan module* (σ, τ)*-derivation* is a U-module map $D : A \to X$ satisfying the identity

$$D(a^2) = D(a) \cdot \tau(a) + \sigma(a) \cdot D(a)$$

for all $a \in A$.

The U-module map $f : A \to X$ is called a *Lie module* (σ, τ)*-derivation* if

$$f([a, b]) = [f(a), b]_{\sigma,\tau} - [f(b), a]_{\sigma,\tau}$$

for all $a, b \in A$. Here $[a, b]_{\sigma,\tau} = a\tau(b) - \sigma(b)a$ for all $a, b \in A$.

A U-module map $f : A \to X$ is called a *Brešar generalized module* (σ, τ)-*derivation*, denoted by (f, D), if there is a module (σ, τ)-derivation $D : A \to X$ such that

$$f(ab) = f(a) \cdot \tau(b) + \sigma(a) \cdot D(b)$$

for all $a, b \in A$. We use the notion $Z^B_{(\sigma,\tau)}(A, X)$ for all Brešar generalized module (σ, τ)- *derivations* from A to X. If f satisfies the relation

$$f(a^2) = f(a) \cdot \tau(a) + \sigma(a) \cdot D(a)$$

for all $a \in A$, then it is called a *Brešar generalized Jordan module* (σ, τ)-*derivation.*

For $x, y \in X$, the U-module map $f_{x,y} : A \to X$ is called *Brešar generalized inner module* (σ, τ)-*derivation* if $f_{x,y}(a) = x \cdot \tau(a) + \sigma(a) \cdot y$ holds for all $a \in A$.

We use notation $H^B_{(\sigma,\tau)}(A, X)$ for the quotient space $Z^B_{(\sigma,\tau)}(A, X)/InnZ^B_{(\sigma,\tau)}(A, X)$ which call the first generalized (σ, τ)-cohomology group of A with coefficients in X. Then, A is called *generalized module* (σ, τ)-*amenable* if $H^B_{(\sigma,\tau)}(A, X^*) = \{0\}$, for all A-U-module X.

A U-module map $f : A \to X$ is called a *Brešar generalized Lie module* (σ, τ)-*derivation* if there exists a Lie module (σ, τ)-derivation L from A to X such that

$$f([a, b]) = [L(a), b]_{\sigma,\tau} - [f(b), a]_{\sigma,\tau}$$

for all $a, b \in A$.

Finally, we introduce the similar notations in the sense of *Nakajima*:

A U-module map $f : A \to X$ is called a *generalized module* (σ, τ)-*derivation* if the identity

$$f(ab) = f(a) \cdot \tau(b) + \sigma(a) \cdot f(b) + \sigma(a) \cdot x \cdot \tau(b)$$

holds for all $a, b \in A$ and some $x \in X$. This module derivation is denoted by $(f, x)_{\sigma,\tau}$ (abb. (f, x)).

We say that (f, x) is a *generalized Jordan module* (σ, τ)-*derivation* if

$$f(a^2) = f(a) \cdot \tau(a) + \sigma(a) \cdot f(a) + \sigma(a) \cdot x \cdot \tau(a)$$

for all $a \in A$.

The pair (f, x) is called a *generalized Lie module* (σ, τ)-*derivation* if

$$f([a, b]) = [f(a), b]_{\sigma,\tau} - [f(b), a]_{\sigma,\tau} + \tau(a) \cdot x \cdot \sigma(b) - \tau(b) \cdot x \cdot \sigma(a)$$

for all $a, b \in A$.

For $x, y \in X$, a U-module map $f_{x,y} : A \to X$ is called a *generalized inner module* (σ, τ)-*derivation* if

$$f_{x,y}(ab) = f_{x,y}(a) \cdot \tau(b) + \sigma(a) \cdot f_{x,y}(b) + \sigma(a) \cdot (-x - y) \cdot \tau(b)$$

for all $a, b \in A$ and we denote this module derivation by $(f_{x,y}, -x - y)$.

Also we denote the first Nakajima generalized (σ, τ)-cohomology group from A into X by the quotient space $H^G_{(\sigma,\tau)}(A, X) = Z^G_{(\sigma,\tau)}(A, X)/InnZ^G_{(\sigma,\tau)}(A, X)$. Then, A is called *Nakajima generalized module* (σ, τ)*-amenable* if $H^G_{(\sigma,\tau)}(A, X^*) = \{0\}$, for all A-U-module X.

Throughout this paper we also use the following notations for some (σ, τ)-derivation sets:

$Z_{(\sigma,\tau)}(A, X)$, the set of module (σ, τ)-derivations,

$InnZ_{(\sigma,\tau)}(A, X)$, the set of inner module (σ, τ)-derivations,

$JZ_{(\sigma,\tau)}(A, X)$, the set of Jordan module (σ, τ)-derivations,

$LieZ_{(\sigma,\tau)}(A, X)$, the set of Lie module (σ, τ)-derivations,

$Z^B_{(\sigma,\tau)}(A, X)$, the set of Brešar generalized module (σ, τ)-derivations,

$InnZ^B_{(\sigma,\tau)}(A, X)$, the set of Brešar generalized inner module (σ, τ)-derivations,

$JZ^B_{(\sigma,\tau)}(A, X)$, the set of Brešar generalized Jordan module (σ, τ)-derivations,

$LieZ^B_{(\sigma,\tau)}(A, X)$, the set of Brešar generalized Lie module (σ, τ)-derivations,

$Z^G_{(\sigma,\tau)}(A, X)$, the set of generalized module (σ, τ)-derivations,

$InnZ^G_{(\sigma,\tau)}(A, X)$, the set of generalized inner module (σ, τ)-derivations,

$JZ^G_{(\sigma,\tau)}(A, X)$, the set of generalized Jordan module (σ, τ)-derivations,

$LieZ^G_{(\sigma,\tau)}(A, X)$, the set of generalized Lie module (σ, τ)-derivations.

Moreover, we can say that the all above derivation sets are Banach U-bimodule.

Now we want to define the following special sets:

A U-module map $f : A \to X$ is said to be *left module multiplier* if $f(ab) = f(a) \cdot b$ for all $a, b \in A$. We denote by $Mull^U(A, X)$ (abb. $Mull(A, X)$) the set of all left module multipliers from A to X. Especially if $f(a^2) = f(a) \cdot a$ for all $a \in A$, then f is called *Jordan left module multiplier* and we denote the set of these maps by $JMull(A, X)$. Furthermore, we can define the set

$$LieMull(A, X) = \{f \mid f : A \to X, \text{ module map and } f[a, b] = [-f(b), a] \text{ for all } a, b \in A\}$$

which is called the set of *Lie left module multipliers*.

Similarly, the definitions of the above special sets can be modified to the *left module* (σ, τ)*-multipliers*:

$$Mull_{\tau}(A,X) = \{f \mid f : A \to X, \text{ module map and } f(ab) = f(a)\cdot\tau(b) \text{ for all } a,b \in A\}$$

and these maps are called *left module (σ,τ)-multipliers*.

If we just take only $a \in A$ in the last set, f is called *Jordan left module (σ,τ)-multiplier*. The set of these maps is denoted by $JMull_{\tau}(A,X)$.

A U-module map $f : A \to X$ is said to be *Lie left module (σ,τ)-multiplier* if $f[a,b] = [-f(b),a]_{\sigma,\tau}$ for all $a,b \in A$. We denote by $LieMull_{\sigma,\tau}(A,X)$ the set of all Lie left module (σ,τ)-multipliers from A to X.

If f and g are left module multipliers in all types and $\alpha \in U$, then $f+g$ and αf are also left module multipliers in all types, hence all the above special sets are Banach U-bimodules.

At the end of this section, we want to give a well-known lemma which will be used several times in the next sections in our paper:

Lemma 2.2. *[10, Lemma 2.2 (Five Lemma)] Let*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\alpha_1"] & A_2 \arrow[r] \arrow[d, "\alpha_2"] & A_3 \arrow[r] \arrow[d, "\alpha_3"] & A_4 \arrow[r] \arrow[d, "\alpha_4"] & A_5 \arrow[d, "\alpha_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

be a commutative diagram of R-modules and R-module homomorphisms, with exact rows, the followings hold:

(i) If α_1 is an epimorphism and α_2, α_4 are monomorphisms, then α_3 is a monomorphism;

(ii) If α_5 is a monomorphism and α_2, α_4 are epimorphisms, then α_3 is a monomorphism

3. Homological Properties of Generalized Module Derivations and Generalized Module (σ,τ)-Derivations

In this section, we first discuss the relation between the U-bimodules $Z^B(A,X)$ and $Z^G(A,X)$. Now, we give some elementary lemmas which show the relation between module derivations and our generalized module derivations.

Lemma 3.1. ***(i)*** *If $(f,x) : A \to X$ is a generalized module derivation, then there exists a module derivation $d = f + l_x : A \to X$, where $l_x : A \to X$ is a left multiplication, i.e., $l_x(a) = xa$, such that $f(ab) = f(a)b + ad(b)$ for all $a,b \in A$. Moreover if $\{x \in X \mid Ax = 0\} = 0$, then d is uniquely determined by f.*

(ii) *If $D : A \to X$ is a module derivation, then for any nonzero element $x \in X$, $(f = D + l_x, -x) : A \to X$ is a generalized module derivation such that $f \neq D$ and D associates to f.*

(iii) *If $(f, x) : A \to X$ is a generalized module derivation, then $(f, f + l_x) : A \to X$ is a Brešar generalized module derivation.*

(iv) *If A contains a unit element and $(f, D) : A \to X$ is a Brešar generalized module derivation, then $(f, -f(1)) : A \to X$ is a generalized module derivation. It means that the notions of generalized module derivations of Nakajima and Brešar coincide when A contains an identity element.*

Proof: We only need to check the boundedness of the map $d = f + l_x : A \to X$. Since f is a U-module map and X is a A-U-bimodule, then we get

$$\|(f + l_x)(a)\| \leq \|f(a)\| + \|x \cdot a\| \leq M\|a\| + K\|x\|\|a\| = (M + K\|x\|)\|a\|$$

for each $a \in A$. This means that the map $f + l_x$ is bounded. The other parts of the proof can be done easily. $\square$

Remark: Throughout this paper, the most important thing which we have to check is the boundedness of the maps (for U-module maps). In the next parts of the paper, we have omitted the boundedness of the maps (because all of them are done similarly).

Corollary 3.2. *The following sequence of U-modules $Z^G(A, X)$ and $Z(A, X)$ is exact and splitting:*

$$0 \to X \xrightarrow{\varphi_1} Z^G(A, X) \xrightarrow{\varphi_2} Z(A, X) \to 0$$

where $\varphi_1(x) = (l_x, -x)$ and $\varphi_2((f, x)) = f + l_x$ are U-module maps. Hence, we get $Z^G(A, X) \cong X \oplus Z(A, X)$.

Our aim is to give necessary and sufficient condition for $Z^B(A, X)$ to be isomorphic to $Z^G(A, X)$ as a Banach U-bimodule when A *does not have a unit element.*

Theorem 3.3. *Suppose that $\Phi : Z^G(A, X) \to Z^B(A, X)$ and $\psi : X \to \mathrm{Mull(A, X)}$ are U-module morphisms such that $\Phi((f, x)) = (f, f + l_x)$ and $\psi(x) = l_x$. Then Φ is a U-module isomorphism if and only if ψ is a U-module isomorphism.*

Proof: We have the following split exact sequence of Banach U-bimodules:

$$0 \to Mull(A, X) \xrightarrow{\psi_1} Z^B(A, X) \xrightarrow{\psi_2} Z(A, X) \to 0$$

where $\psi_1(g) = (g, 0)$ and $\psi_2((f, D)) = D$.

Define a map $\psi_2' : Z(A, X) \to Z^B(A, X)$ by $\psi_2'(D) = (D, D)$. Then $\psi_2\psi_2' = id_{Z(A,X)}$, and thus is split exact. This gives the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \stackrel{\varphi_1}{\longrightarrow} & Z^G(A, X) & \stackrel{\varphi_2}{\longrightarrow} & Z(A, X) & \longrightarrow & 0 \\
 & & \downarrow \psi & & \downarrow \Phi & & \downarrow id & & \\
0 & \longrightarrow & Mull(A, X) & \stackrel{\psi_1}{\longrightarrow} & Z^B(A, X) & \stackrel{\psi_2}{\longrightarrow} & Z(A, X) & \longrightarrow & 0
\end{array}$$

Hence we complete the proof of the theorem by using Five Lemma. □

Corollary 3.4. *The following sequence of Banach U-bimodules $JZ(A, X)$ and $JZ^G(A, X)$, is exact and splitting:*

$$0 \to X \stackrel{\psi_X}{\longrightarrow} JZ^G(A, X) \stackrel{\phi_X}{\longrightarrow} JZ(A, X) \to 0$$

where $\psi_X(x) = (l_x, -x)$ and $\phi_X((f, x)) = f + l_x$.

Corollary 3.5. *Suppose that $\Phi : JZ^G(A, X) \to JZ^B(A, X)$ and $\psi : X \to$ JMull(A, X) are U-module morphisms such that $\Phi((f, x)) = (f, f + l_x)$ and $\psi(x) = l_x$. Then Φ is a U-module isomorphism if and only if ψ is a U-module isomorphism.*

Corollary 3.6. *The following sequence of Banach U-bimodules $LieZ(A, X)$ and $LieZ^G(A, X)$, is exact and splitting:*

$$0 \to X \stackrel{\psi_X}{\longrightarrow} LieZ^G(A, X) \stackrel{\phi_X}{\longrightarrow} LieZ(A, X) \to 0$$

where $\psi_X(x) = (l_x, -x)$ and $\phi_X((f, x)) = f + l_x$.

Corollary 3.7. *Suppose that $\Phi : LieZ^G(A, X) \to LieZ^B(A, X)$ and $\psi : X \to$ LieMull(A, X) are U-module morphisms such that $\Phi((f, x)) = (f, f + l_x)$ and $\psi(x) = l_x$. Let us define the set*

$$\mathcal{X}(A) = \{x \in X \mid [x, a] = 0 \text{ for all } a \in A\}.$$

If $\mathcal{X}(A) = X$ (If X is a bi-commutative Banach A-U-module), then Φ is a U-module isomorphism if and only if ψ is a U-module isomorphism.

Corollary 3.8. *Let X be a A-U-module, then the following diagram is commutative and the rows are split exact:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\psi_1} & InnZ^G(A,X) & \xrightarrow{\psi_2} & InnZ(A,X) & \longrightarrow & 0 \\ & & \downarrow i_0 & & \downarrow i & & \downarrow i_1 & & \\ 0 & \longrightarrow & X & \xrightarrow{\varphi_1} & Z^G(A,X) & \xrightarrow{\varphi_2} & Z(A,X) & \longrightarrow & 0 \end{array}$$

where i_0, i_1, i are the canonical module injections and $\psi_1(x) = (f_{x,0}, -x)$, $\psi_2(f_{x,y}, -x-y) = f_{x,y} + l_{(-x-y)}$.

Proof: All maps in the above diagram are U-module maps, and the commutativity of the diagram is easily seen. If $\psi_2(f_{x,y}, -x-y) = 0$, then we see that $f_{x+y,0} = f_{x,y}$. Thus $Ker\psi_2 = Im\psi_1$. The other part is clear by Corollary 3.2 using the definitions of φ_1 and φ_2. □

Afterwards, we generalize above theorems and corollaries to the module (σ, τ)-derivations. All results in this part are similarly proved to the corresponding results in the previous parts, so we omit the proofs.

Lemma 3.9. ***(i)*** *If $(f, x) : A \to X$ is a generalized module (σ, τ)-derivation, then there is a module (σ, τ)-derivation $d : A \to X$ such that $d = f + l_x\tau$. Moreover if $\{x \in X \mid Ax = 0\} = 0$ and the map $\sigma : A \to A$ is surjective, then d is determined by f uniquely.*

(ii) *If $D : A \to X$ is a module (σ, τ)-derivation, then for any nonzero element $x \in X$, the map $(f = D + l_x\tau, -x) : A \to X$ is a generalized module (σ, τ)-derivation such that $f \neq D$ and D associates to f.*

(iii) *If A contains a unit element and $(f, D) : A \to X$ is a Brešar generalized module derivation, then $(f, -f(1)) : A \to X$ is a generalized module (σ, τ)-derivation.*

Theorem 3.10. *Suppose that $\{x \in X \mid Ax = 0\} = 0$ and σ is surjective. For the Banach U-bimodules $Z_{(\sigma,\tau)}(A, X)$ and $Z^G_{(\sigma,\tau)}(A, X)$, the following sequence is split exact:*

$$0 \to X \xrightarrow{\varphi_1} Z^G_{(\sigma,\tau)}(A, X) \xrightarrow{\varphi_2} Z_{(\sigma,\tau)}(A, X) \to 0$$

where $\varphi_1(x) = (l_x\tau, -x)$ and $\varphi_2((f, x)) = f + l_x\tau$ are U-module maps. Hence, we get $Z^G_{(\sigma,\tau)}(A, X) \cong X \oplus Z_{(\sigma,\tau)}(A, X)$.

Theorem 3.11. *Suppose that $\Phi : Z^G_{(\sigma,\tau)}(A, X) \to Z^B_{(\sigma,\tau)}(A, X)$ and $\psi : X \to Mull_\tau(A, X)$ are U-module morphisms such that $\Phi((f, x)) = (f, f + l_x\tau)$ and $\psi(x) = l_x\tau$. Then Φ is a U-module isomorphism if and only if ψ is a U-module isomorphism.*

Proof: Here we use the following split exact sequence of Banach U-bimodules:

$$0 \to Mull_{\tau}(A,X) \xrightarrow{\varphi_1} Z^B_{(\sigma,\tau)}(A,X) \xrightarrow{\varphi_1} Z(A,X)_{(\sigma,\tau)} \to 0$$

where $\varphi_1(g) = (g,0)$ and $\varphi_2((f,D)) = D$. □

Corollary 3.12. *Suppose that $\{x \in X \mid Ax = 0\} = 0$ and σ is surjective. Then the following sequence of Banach U-bimodules $JZ_{(\sigma,\tau)}(A,X)$ and $JZ^G_{(\sigma,\tau)}(A,X)$, is exact and splitting:*

$$0 \to X \xrightarrow{\psi_X} JZ^G_{(\sigma,\tau)}(A,X) \xrightarrow{\phi_X} JZ_{(\sigma,\tau)}(A,X) \to 0$$

where $\psi_X(x) = (l_x\tau, -x)$ and $\phi_X((f,x)) = f + l_x\tau$.

Corollary 3.13. *Suppose that $\Phi : JZ^G_{(\sigma,\tau)}(A,X) \to JZ^B_{(\sigma,\tau)}(A,X)$ and $\psi : X \to JMull_{\tau}(A,X)$ are U-module morphisms such that $\Phi((f,x)) = (f, f + l_x\tau)$ and $\psi(x) = l_x\tau$. Then Φ is a U-module isomorphism if and only if ψ is a U-module isomorphism.*

Corollary 3.14. *Suppose that $\{x \in X \mid Ax = 0\} = 0$ and σ is surjective. Then the following sequence of Banach U-bimodules $LieZ_{(\sigma,\tau)}(A,X)$ and $LieZ^G_{(\sigma,\tau)}(A,X)$, is exact and splitting:*

$$0 \to X \xrightarrow{\psi_X} LieZ^G_{(\sigma,\tau)}(A,X) \xrightarrow{\phi_X} LieZ_{(\sigma,\tau)}(A,X) \to 0$$

where $\psi_X(x) = (l_x\tau, -x)$ and $\phi_X((f,x)) = f + l_x\tau$.

Corollary 3.15. *Suppose that $\Phi : LieZ^G_{(\sigma,\tau)}(A,X) \to LieZ^B_{(\sigma,\tau)}(A,X)$ and $\psi : X \to LieMull_{(\sigma,\tau)}(A,X)$ are U-module morphisms such that $\Phi((f,x)) = (f, f + l_x\tau)$ and $\psi(x) = l_x\tau$. Let us define the set*

$$X_{(\sigma,\tau)}(A) = \{x \in X \mid [x,a]_{\sigma,\tau} = 0 \text{ for all } a \in A\}.$$

If $X_{(\sigma,\tau)}(A) = X$, then Φ is U-module isomorphism if and only if ψ is U-module isomorphism.

Theorem 3.16. *The following diagram is commutative and the rows are split exact:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\psi_1} & InnZ^G_{(\sigma,\tau)}(A,X) & \xrightarrow{\psi_2} & InnZ_{(\sigma,\tau)}(A,X) & \longrightarrow & 0 \\
 & & \downarrow i_0 & & \downarrow i & & \downarrow i_1 & & \\
0 & \longrightarrow & X & \xrightarrow{\varphi_1} & Z^G_{(\sigma,\tau)}(A,X) & \xrightarrow{\varphi_2} & Z_{(\sigma,\tau)}(A,X) & \longrightarrow & 0
\end{array}$$

where i_0, i_1, i *are the canonical module injections and* $\psi_1(x) = (f_{x,0}, -x)$, $\psi_2(f_{x,y}, -x-y) = f_{x,y} + l_{(-x-y)}\tau$.

4. Functorial Relations and Opposite Properties

4.1. Functorial Relations

Firstly, we give a functorial relation between $Z(A,-)$ and $Z^G(A,-)$ as follows:

Theorem 4.1. *Let* X_1 *and* X_2 *be Banach A-U-modules and* $\gamma : X_1 \to X_2$ *be a U-module morphism. Then* γ *induces a U-module map*

$$\gamma' : Z^G(A, X_1) \to Z^G(A, X_2)$$

such that $\gamma'((f, x_1)) = (\gamma f, \gamma(x_1))$ *and* $Z^G(A,-)$ *is a covariant functor from the category of Banach A-U-modules to the category of Banach U-bimodules.*

Proof: The map

$$\gamma' : Z^G(A, X_1) \to Z^G(A, X_2), \quad (f, x_1) \mapsto (\gamma f, \gamma(x_1))$$

is a U-module map.

Let $\mathcal{X}$ be a category of Banach A-U-modules and $\mathcal{M}$ be a category of Banach U-bimodules. Define the functor as follows:

$$Z^G(A,-) : \mathcal{X} \to \mathcal{M}, \quad X \mapsto \mathcal{X}(X) = Z^G(A, X)$$

If $\gamma : X_1 \to X_2, \quad \gamma_* : X_2 \to X_3$ are U-module morphisms, then for the following map, the first condition is satisfied:

$$Z^G(A,-)(\gamma_* \circ \gamma) : Z^G(A, X_1) \to Z^G(A, X_3), \quad (f, x_1) \mapsto ((\gamma_* \circ \gamma) \circ f, (\gamma_* \circ \gamma)(x_1))$$

On the other hand,

$$\begin{aligned}
Z^G(A,-)(\gamma_*)((Z^G(A,-)(\gamma))(f, x_1)) &= Z^G(A,-)(\gamma_*)(\gamma f, \gamma(x_1)) \\
&= (\gamma_* \circ (\gamma f), \gamma_*(\gamma(x_1)).
\end{aligned}$$

Thus $Z^G(A,-)(\gamma_*) \circ Z^G(A,-)(\gamma) = Z^G(A,-)(\gamma_* \circ \gamma)$.

For the second condition, we use the map,

$$Z^G(A,-)(1_X): Z^G(A,X) \to Z^G(A,X), \quad (f,x_1) \mapsto (1_X f, 1_X(x_1)) = (f,x_1)$$

Therefore, $Z^G(A,-)(1_X) = 1_{Z^G(A,-)(X)}$. □

Theorem 4.2. *Let $\Phi : Z^G(A,-) \to Z(A,-) \oplus F$ be a map of functors where F is the forgetful functor from the category of Banach A-U-modules to the category of Banach U-bimodules. Then Φ assigns to each Banach A-U-module X of $\mathcal{X}$, a U-module isomorphism $\Phi_X : Z^G(A,X) \to Z(A,-) \oplus X$ of $\mathcal{M}$ such that $\Phi_X((f,x)) = (f + l_x, x)$ where $x \in X$ and $X = X_1, X_2$; in such a way that for every U-module morphism of Banach A-U-modules $\gamma : X_1 \to X_2$ of $\mathcal{X}$, the diagram*

$$\begin{array}{ccc} Z^G(A,X_1) & \xrightarrow{\alpha_*} & Z^G(A,X_2) \\ \downarrow \Phi_{X_1} & & \downarrow \Phi_{X_2} \\ Z(A,X_1) \oplus X_1 & \xrightarrow{\overline{\alpha_*}} & Z(A,X_2) \oplus X_2 \end{array}$$

in $\mathcal{M}$ is commutative, where $\Phi_X(d,x) = (\gamma d, d(x))$ and $\overline{\alpha_*}(f,x_1) = (\gamma f, f(x_1))$. Hence we can say that Φ is a natural transformation of functors.

Since Φ_X is an equivalence for every A-U-module X_1 in $\mathcal{X}$ by Theorem 4.2, we have the following corollary:

Corollary 4.3. *The functors $Z^G(A,-)$ and $Z^G(A,-) \oplus F$ from the category of Banach A-U-modules to the category of Banach U-bimodules are naturally equivalent.*

Now, we give a similar functorial relation between $Z_{(\sigma,\tau)}(A,-)$ and $Z^G_{(\sigma,\tau)}(A,-)$ as follows:

Theorem 4.4. *Let X_1 and X_2 be Banach A-U-modules and $\gamma : X_1 \to X_2$ be a U-module morphism. Then γ induces a U-module map*

$$\gamma' : Z^G_{(\sigma,\tau)}(A,X_1) \to Z^G_{(\sigma,\tau)}(A,X_2)$$

such that $\gamma'((f,x_1)) = (\gamma f, \gamma(x_1))$ and $Z^G_{(\sigma,\tau)}(A,-)$ is a covariant functor from the category of Banach A-U-modules to the category of Banach U-bimodules.

Theorem 4.5. *Let $\Phi : Z^G_{(\sigma,\tau)}(A,-) \to Z_{(\sigma,\tau)}(A,-) \oplus F$ be a map of functors where F is the forgetful functor from the category of Banach A-U-modules to the category of Banach U-bimodules. Then Φ assigns to each Banach A-U-module X of $\mathcal{X}$, a U-module isomorphism $\Phi_X : Z^G_{(\sigma,\tau)}(A,X) \to Z_{(\sigma,\tau)}(A,-) \oplus X$ of $\mathcal{M}$ such that $\Phi_X((f,x)) = (f + l_x\tau, x)$ where $x \in X$ and $X = X_1, X_2$; in such a way that for every U-module morphism of Banach A-U-modules $\gamma : X_1 \to X_2$ of $\mathcal{X}$, the diagram*

$$\begin{array}{ccc} Z^G_{(\sigma,\tau)}(A,X_1) & \xrightarrow{\alpha_*} & Z^G_{(\sigma,\tau)}(A,X_2) \\ \downarrow \Phi_{X_1} & & \downarrow \Phi_{X_2} \\ Z_{(\sigma,\tau)}(A,X_1)\oplus X_1 & \xrightarrow{\overline{\alpha_*}} & Z_{(\sigma,\tau)}(A,X_2)\oplus X_2 \end{array}$$

in $\mathcal{M}$ is commutative. Hence we can say that Φ is a natural transformation of functors.

Corollary 4.6. *The functors $Z^G_{(\sigma,\tau)}(A,-)$ and $Z^G_{(\sigma,\tau)}(A,-)\oplus F$ from the category of Banach A-U-modules to the category of Banach U-bimodules are naturally equivalent.*

4.2. Opposite Properties

In this section, we first consider the universal problem for generalized module derivations.

Let A be a Banach algebra. Define a new algebra A^{op} with underlying set consisting of elements $\{a^o \mid a \in A\}$. Addition in A^{op} coincides with addition in A and multiplication in A^{op} is the map

$$o : A \to A^{op}, \qquad a \mapsto a^o$$

such that $a^o.b^o = (ba)^o$ where ba is the product in A. Moreover, this map is linear. Endow A^{op} with the Banach space structure $\|a^o\| = \|a\|$. Then A^{op} is again a Banach algebra, called the *opposite* of A. If A is a commutative Banach U-bimodule, then it can be easily verified that A^{op} is also a commutative Banach U-bimodule.

Furthermore, $A\widehat{\otimes}_U A^{op}$ is a Banach algebra with the following product rule:

$$(a\otimes b^o)(c\otimes d^o) := (ac)\otimes(db)^o \qquad (a,b,c,d\in A).$$

Let A be a Banach algebra and a Banach U-bimodule, and let X be a Banach A-U-module. Then X is a left Banach $A\widehat{\otimes}_U A^{op}$- module by the action defined by $(a\otimes b^o)x := a\cdot x\cdot b, \quad (a,b\in A, x\in X)$. Then the map

$$\mu : A\widehat{\otimes}_U A^{op} \ni a\otimes b^o \mapsto ab \in A$$

is a left $A\widehat{\otimes}_U A^{op}$-module map and its kernel I is a left Banach $A\widehat{\otimes}_U A^{op}$-module.

If A has an identity element 1, then the map

$$\delta_A : A \ni a \mapsto a\otimes 1^o - 1\otimes a^o \in I$$

is a module derivation and δ_A has the following universal mapping property:

For every left Banach $A\widehat{\otimes}_U A^{op}$-module X and every module derivation $d : A \to X$, there exists a unique $A\widehat{\otimes}_U A^{op}$-module map $h : I \to X$ such that $d = h\delta_A$.

Using Corollary 4.3, we get the following theorem.

Theorem 4.7. *Let A be a Banach algebra and a Banach U-bimodule with identity element 1, and let X be a Banach A-U-module and $(f,x) : A \to X$ be a generalized module derivation. Then the map $g\delta_A : A \to I \oplus (A\widehat{\otimes}_U A^{op})$ defined by $g\delta_A(a) = (\delta_A(a), -1 \otimes 1^\circ)$ gives a generalized module derivation $(g\delta_A, (0, 1 \otimes 1^\circ))$ and there exists a unique left $A\widehat{\otimes}_U A^{op}$-module map $h_f : I \oplus (A\widehat{\otimes}_U A^{op}) \to X$ such that $f = h_f(g\delta_A)$ and $h_f((0, 1 \otimes 1^\circ) = x$. This induces the map*

$$\xi_x : Hom_{A\widehat{\otimes}_U A^{op}}(I \oplus (A\widehat{\otimes}_U A^{op}, X) \ni \gamma \mapsto (\gamma(g_{\delta_A}), \gamma(0, 1 \otimes 1^o)) \in Z^G(A, X)$$

which is a U-module isomorphism.

Proof: Since X is a left Banach $A\widehat{\otimes}_U A^{op}$-module, by Corollary 4.3 and the above universal mapping property, we have a chain of U-module isomorphisms of Banach U-bimodules

$$\begin{aligned} Z^G(A,X) &\cong Z(A,X) \oplus X \\ &\cong Hom_{A\widehat{\otimes}_U A^{op}}(I,X) \oplus Hom_{A\widehat{\otimes}_U A^{op}}(A\widehat{\otimes}_U A^{op}, X) \\ &\cong Hom_{A\widehat{\otimes}_U A^{op}}(I \oplus (A\widehat{\otimes}_U A^{op}), X). \end{aligned}$$

By these U-module isomorphisms, a generalized module derivation (f,x) corresponds to the map

$$h_f : I \oplus (A\widehat{\otimes}_U A^{op}) \ni (a \otimes 1^o - 1 \otimes a^o, b \otimes c^o) \mapsto (f + l_x)(a) + bxc \in X.$$

Define a map $g\delta_A : A \to I \oplus (A\widehat{\otimes}_U A^{op})$ by $g\delta_A(a) = (\delta_A(a), -1 \otimes a^o)$.

Then $(g\delta_A, (0, 1\otimes 1^o))$ is a generalized module derivation, $h_f(0, 1\otimes 1^o) = x\, h_f(g\delta_A)(a)$ $= f(a) + xa - xa = f(a)$. Moreover, since $\{g\delta_A(a) \mid a \in A\}$ generates $I \oplus (A\widehat{\otimes}_U A^{op})$ as a left Banach $A\widehat{\otimes}_U A^{op}$-module, h_f is uniquely determined as a $A\widehat{\otimes}_U A^{op}$-module map and thus $(g\delta_A, (0, 1 \otimes 1^o)$ is as required, completing the proof. □

Corollary 4.8. *Let A be a commutative Banach algebra with identity, X a left bi-commutative Banach A-U-module and $(f,x) : A \to X$ a generalized module derivation. Let $d_A : A \to I/I^2$ $d_A(a) = \overline{a \otimes 1 - 1 \otimes a}$ be the canonical module derivation. Then the map*

$$\delta_A : A \to (I/I^2) \oplus A,\ a \mapsto (\overline{\delta_A(a)}, -a)$$

gives a generalized module derivation $(g\delta_A, (\overline{0}, 1))$ and there exists a unique U-module map $h_f : I/I^2 \oplus A \to X$such that $f = h_f(g\delta_A)$ and $h_f((\overline{0}, 1)) = x$. This induces the module map

$$\xi_x : Hom_A I/I^2 \oplus (A, X) \to Z^G(A, X),\ \alpha \mapsto (\alpha g\delta_A, \alpha(\overline{0}, 1))$$

which is a U-module isomorphism.

Now, we consider the universal problem for generalized module (σ,τ)-derivations.

Let A be a Banach algebra and a Banach U-bimodule, and let X be a Banach A-U-module. Then X is a left Banach $A\widehat{\otimes}_U A^{op}$- module. Also the map

$$\mu : A\widehat{\otimes}_U A^{op} \ni a \otimes b^o \mapsto \sigma(a)\tau(b) \in A$$

is a left $A\widehat{\otimes}_U A^{op}$-module map and its kernel I is a left Banach $A\widehat{\otimes}_U A^{op}$-module.

If A has an identity element 1, then the following map, whose boundedness can be shown by properties of tensor norm,

$$\delta_A : A \ni a \mapsto \sigma(a) \otimes 1^o - 1 \otimes (\tau(a))^o \in I$$

is a module (σ, τ)-derivation and δ_A has the following universal mapping property:

For every left Banach $A\widehat{\otimes}_U A^{op}$-module X and every module (σ, τ)-derivation $d : A \to X$, there exists a unique $A\widehat{\otimes}_U A^{op}$-module map $h : I \to X$ such that $d = h\delta_A$.

Theorem 4.9. *Let A be a Banach algebra and a Banach U-bimodule with identity element 1, and let X be a Banach A-U-module and $(f, x) : A \to X$ be a generalized module (σ, τ)-derivation. Then the map $g\delta_A : A \to I \oplus (A\widehat{\otimes}_U A^{op})$ defined by $g\delta_A(a) = (\delta_A(a), -1 \otimes a^\circ)$ gives a generalized module (σ, τ)-derivation $(g\delta_A, (0, 1 \otimes 1^\circ))$ and there exists a unique left $A\widehat{\otimes}_U A^{op}$-module map $h_f : I \oplus (A\widehat{\otimes}_U A^{op}) \to X$ such that $f = h_f(g\delta_A)$ and $h_f((0, 1 \otimes 1^\circ) = x$. This induces the map*

$$\xi_x : Hom_{A\widehat{\otimes}_U A^{op}}(I \oplus (A\widehat{\otimes}_U A^{op}, X) \ni \gamma \mapsto (\gamma(g_{\delta_A}), \gamma(0, 1 \otimes 1^o)) \in Z^G(A, X)$$

which is a U-module isomorphism.

Proof: We define the map

$$h_f : I \oplus (A\widehat{\otimes}_U A^{op}) \ni (\sigma(a) \otimes 1^o - 1 \otimes (\tau(a))^o, b \otimes c^o) \mapsto (f + l_x\tau)(a) + b \cdot x \cdot \tau(c) \in X.$$

such that $h_f(g\delta_A) = f$ and $h_f((0, 1 \otimes 1^\circ)) = x$. The rest is clear. □

Corollary 4.10. *Let A be a commutative Banach algebra with identity, X a left A-U-module and $(f, x) : A \to X$ is generalized module (σ, τ)-derivation. Let $d_A : A \to I/I^2$ $d_A(a) = \overline{a \otimes 1 - 1 \otimes a}$ be the canonical (σ, τ)-module derivation. Then the map*

$$\delta_A : A \to (I/I^2) \oplus A,\ a \mapsto (\overline{\delta_A(a)}, -a)$$

gives the generalized module (σ, τ)-derivation $(g\delta_A, (\overline{0}, 1))$ and there exists a unique A-U-module map such that $f = h_f(g\delta_A)$ and $h_f((\overline{0}, 1)) = x$. This induces the module map

$$\xi_x : Hom_A(I/I^2) \oplus A, X) \to Z^G(A, X),\ f \mapsto (fg\delta_A, f(0,1))$$

which is a U-module isomorphism.

Open Questions:

1. Can the right (left), central etc. module derivations be defined and can one modify the above results to these types of module derivations?

2. What are the necessary and sufficient conditions which give us the generalized module amenability (or weak generalized module amenability) relation between the generalized cohomology groups $H^B(A, X^*)$ (or $H^B(A, A^*)$) and $H^G(A, X^*)$ (or $H^G(A, A^*)$) for each A-U-module X?

3. What are the necessary and sufficient conditions which give us the generalized module (σ, τ)-amenability relation between the generalized (σ, τ)-cohomology groups $H^B_{(\sigma,\tau)}(A, X^*)$ and $H^G_{(\sigma,\tau)}(A, X^*)$ for each A-U-module X?

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