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Existence and non-existence of a positive solution for (p,q)-Laplacian with singular weights

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abstract: We use the Hardy-Sobolev inequality to study existence and non-existence results for a positive solution of the quasilinear elliptic problem

$$-\triangle_p u - \mu\triangle_q u = \lambda[m_p(x)|u|^{p-2}u + \mu m_q(x)|u|^{q-2}u] \text{ in } \Omega$$

driven by nonhomogeneous operator (p,q)-Laplacian with singular weights under the Dirichlet boundary condition. We also prove that in the case where $\mu > 0$ and with $1 < q < p < \infty$ the results are completely different from those for the usual eigenvalue problem for the p-Laplacian with singular weight under the Dirichlet boundary condition, which is retrieved when $\mu = 0$. Precisely, we show that when $\mu > 0$ there exists an interval of eigenvalues for our eigenvalue problem.

Key Words: Nonlinear eigenvalue problem; (p,q)-Laplacian; singular weight; Indefinite weight; Hardy-Sobolev inequality; Harnack inequality .

Contents

1. Introduction

Consider the (p,q)-Laplacian eigenvalue problem

$$(P_{\lambda,\mu})\begin{cases} \text{To find } (u,\lambda) & \in & (W_0^{1,p}(\Omega)\setminus\{0\})\times\mathbb{R} & \text{such that}\\ -\triangle_p u - \mu\triangle_q u & = & \lambda[m_p(x)|u|^{p-2}u + \mu m_q(x)|u|^{q-2}u] & \text{in } \Omega,\\ u & = & 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with piecewise C^1 boundary $\partial\Omega$, λ, $\mu\in\mathbb{R}^+$ and $1 < q < p < \infty$. For $r = p,q$, $\triangle_r u = div(|\nabla u|^{r-2}\nabla u)$ indicate the r-Laplacian and the weight m_r may be unbounded and change sign. As in [14], we assume for $r = p,q$ that $m_r\delta^\tau \in L^a(\Omega)$ with $\delta(x) = dist(x,\partial\Omega)$ and $m_r^+ \not\equiv 0$, where a, r and τ satisfy one of the following conditions:

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(H1): $\partial\Omega$ is piecewise C^1, $0 < \tau < 1$, $\frac{r}{1-\tau} \leq a$ and $a \leq \frac{Nr}{N-\tau r}$ if $N > \tau r$;

(H2): $\partial\Omega$ is piecewise C^1, $0 < \tau < 1$, $r < \frac{N}{1-\tau} \leq a$;

(H3): $\partial\Omega$ is piecewise C^1, $\tau = 1$ and $a = \infty$;

(H4): Ω is any bounded domain, $\tau = 0$ and $a = \infty$.

The problem $(P_{\lambda,\mu})$ comes, for example, from a general reaction diffusion system

$$u_t = div(D(u)\nabla u) + c(x, u), \tag{1.1}$$

where $D(u) = (|\nabla u|^{p-2} + \mu|\nabla u|^{q-2})$. This system has a wide range of applications in physics and related sciences like chemical reaction design [2], biophysics [8] and plasma physics [18]. In such applications, the function u describes a concentration, the first term on the right-hand side of (1.1) corresponds to the diffusion with a diffusion coefficient $D(u)$; whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x; u)$ has a polynomial form with respect to the concentration.

Our problem was addressed in [15] for domains with boundary C^2 and bounded weights, when only the condition (H4) holds true. These work proved that in the case where $\mu > 0$, there exists an interval of eigenvalues. The authors proved the existence of positive solutions in resonant cases. A non-existence result is also given. Here we will assume that the boundary $\partial\Omega$ is a piecewise C^1 and singular weights m_r $(r = p, q)$ which satisfy one of the conditions (H1), (H2), (H3) or (H4). Our work represent developments of the study performed in [15] because we prove all results of this paper by considering others conditions that represent the singularity of the domain and the weights. Our main tool is the Hardy-Sobolev inequality, see Lemma 2.2 in preliminary section.
Many authors have studied the nonhomogeneous operator (p, q)-Laplacian (see [12, 16,21,22]). However, there are few results one the eigenvalue problems for the (p, q)-Laplacian. In [4,5], the authors established the existence of the principal eigenvalue and of a continuous family of eigenvalues for problem

$$-\triangle_p u - \triangle_q u = \lambda g(x)|u|^{p-2}u \text{ in } \mathbb{R}^N.$$

where g is a bounded positive weight. Eigenvalue problem for a $(p, 2)$-Laplacian was studied in [3]. The existence of non trivial solution for the following Dirichlet equation is proved in [6]

$$-\triangle_p u - \mu\triangle u = \lambda|u|^{p-2}u + g(u) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

in the case where $p > 2$, $g \in C^1$ and $\lambda \notin \sigma(-\Delta_p)$, where $\sigma(-\Delta_p)$ is the spectrum of $(-\Delta_p)$. Under the Neumann boundary condition, [13] determined the set of eigenvalues for the equation

$$-\triangle_p u - \triangle u = \lambda u \text{ in } \Omega,$$

where $p > 2$. In [19], M. Tanaka has completely described the generalized eigenvalue λ for which the following equation

$$-\triangle_r u - \mu \triangle u_{r*} = \lambda m_r |u|^{r-2} u \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega.$$

has a positive solution, where $1 < r \neq r* < \infty$ and $\mu > 0$.

We recall that a value $\lambda \in \mathbb{R}$ is an eigenvalue of problem $(P_{\lambda,\mu})$ if and only if there exists $u \in W_0^{1,p}(\Omega)\backslash\{0\}$ such that

$$\int_\Omega (|\nabla u|^{p-2} + \mu |\nabla u|^{q-2}) \nabla u \nabla \varphi dx = \lambda \left[\int_\Omega (m_p(x)|u|^{p-2} + \mu m_q(x)|u|^{q-2}) u \varphi dx \right] \tag{1.2}$$

for all $\varphi \in W_0^{1,p}(\Omega)$. u is then called an eigenfunction of λ.

Letting $\mu \to 0^+$, our problem $(P_{\lambda,\mu})$ turns into the $(p-1)$-homogeneous problem known as the usual weighted eigenvalue problem for the p-Laplacian with singular weight m_p:

$$(P_{\lambda,m_p}) \begin{cases} -\triangle_p u &= \lambda m_p(x)|u|^{p-2}u \quad \text{in } \Omega, \\ \quad u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

Moreover, after multiplying our equation $(P_{\lambda,\mu})$ by $1/\mu$ and then letting $\mu \to +\infty$, we obtain the $(q-1)$-homogeneous equation:

$$(P_{\lambda,m_q}) \begin{cases} -\triangle_q u &= \lambda m_q(x)|u|^{q-2}u \quad \text{in } \Omega, \\ \quad u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

Nonlinear eigenvalue problem (P_{λ,m_r}), where $r = p, q$ and with bounded weight have been studied by several authors, for example (see [1,7,9,11,17,18]). These works proved that there exists a first eigenvalue $\lambda_1(r, m_r) > 0$, where

$$\lambda_1(r, m_r) := \inf \left\{ \frac{1}{r} \int_\Omega |\nabla u|^r dx; u \in W_0^{1,r}(\Omega) \text{ and } \frac{1}{r} \int_\Omega m_r(x)|u|^r dx = 1 \right\}, \tag{1.3}$$

which is simple in the sense that two eigenfunctions corresponding to it are proportional. Moreover, the corresponding first eigenfunction $\phi_1(r, m_r)$ can be assumed to be positive. It was also shown (see [1]) that $\lambda_1(r, m_r)$ is simple and isolated. Recently, the problem (P_{λ,m_r}) with singular weight m_r satisfying the conditions (H1), (H2), (H3) or (H4), was studied in [14]. The authors use the Hardy-Sobolev inequality to characterize the first eigenvalue. In some cases it is shown that $\lambda_1(r, m_r) > 0$ is positive simple, isolated and has a nonnegative corresponding eigenfunction $\phi_1(r, m_r) \in L^\infty(\Omega)$. Higher eigenvalues, in particular the second one, are also determined.

The plan of this paper is the following. In Section 2, which has a preliminary character, we collect some results concerning the first eigenvalue $\lambda_1(r, m_r)$ of problem (P_{λ,m_r}), where $r = p, q$. In Section 3, we study Rayleigh quotient for our problem $(P_{\lambda,\mu})$. In contrast to homogeneous case, we prove that if $\lambda_1(p, m_p) \neq \lambda_1(q, m_q)$ or $\phi_1(p, m_p) \neq k\phi_1(q, m_q)$ for every $k > 0$, then the infimum in Rayleigh quotient is not attained. We also show nonexistence results for positive solutions of the eigenvalue problem $(P_{\lambda,\mu})$ formulated as Theorem 3.5. Our existence results for positive solutions of the eigenvalue problem $(P_{\lambda,\mu})$ are presented in Section 4. After studying the non-resonant cases (Theorem 4.1) which prove that when $\mu > 0$ there exists an interval of positive eigenvalues for the problem $(P_{\lambda,\mu})$, we present the resonant cases in Theorem 4.8.

2. Preliminaries

Throughout this paper Ω will be a bounded domain of $\mathbb{R}^N$ with piecewise C^1 boundary,
$1 < q < p < \infty$ and $r = p$ or q. We will always assume for $r = p, q$ that $m_r\delta^\tau \in L^a(\Omega)$ with $\delta(x) = dist(x, \partial\Omega)$ and $m_r^+ \not\equiv 0$, where a, r and τ satisfy one of the conditions (H1), (H2), (H3) or (H4).

Remark 2.1. Condition (H4) implies $m_r\delta^\tau = m_r \in L^\infty(\Omega)$, including results of the previously cited paper [15]. Here $\partial\Omega$ is piecewise C^1 except for (H4).

We will write $\|u\|_r := \left(\int_\Omega |u|^r dx\right)^{1/r}$ for the $L^r(\Omega)-$norm and $W_0^{1,r}(\Omega)$ will denote the usual Sobolev space with usual norm $\|\nabla u\|_r$.

In the sequel, we collects some results relative to the first eigenvalue $\lambda_1(r, m_r)$ defined by (1.3) and its corresponding normalized eigenfunction $\phi_1(r, m_r)$. The following lemma concerns the Hardy-Sobolev inequality proved in [10], which characterize the first eigenvalue $\lambda_1(r, m_r)$ of problem (P_{λ,m_r}). This inequality is our main tool in this paper.

Lemma 2.2. *[10] Let $0 \leq \tau \leq 1$ and s such that $\frac{1}{s} = \frac{1}{r} - \frac{1-\tau}{N}$ for $r < N$ and $\frac{1}{s} = \frac{\tau}{r}$ for $r \geq N$. If $\partial\Omega$ is piecewise C^1, then $\left\|\frac{u}{\delta^\tau}\right\|_{L^s(\Omega)} \leq C\|\nabla u\|_{L^r(\Omega)}$ for all $u \in W_0^{1,r}(\Omega)$, where $\delta(x) = dist(x, \partial\Omega)$ and $C = C(N, r, \tau) > 0$ is a constant. In the case $s = r = p, q$, no regularity on $\partial\Omega$ is needed.*

We give now an example of the weight m_r such that $m_r\delta^\tau \in L^a(\Omega)$ with $m_r^+ \not\equiv 0$, where a, τ and r satisfying the condition (H2).

Example 2.3. The weight $m_r(x) = \delta(x)^{-\beta} = (1-|x|)^{-\beta}$ is admissible in the open unit ball of $\mathbb{R}^N$ (i.e. $\Omega = B_1(0)$). For $1/2 < \beta < 25/42$, $p = 3/2$, $N = 3$, $\tau = 1/2$ and $a = 21/2$, we have $m_r \notin L^{N/r}(\Omega) = L^2(\Omega)$, but $m_r\delta^\tau \in L^a(\Omega) = L^{21/2}(\Omega)$.

To use Harnack inequality as in [14] and [20], we make now the following definitions involving locally integrable weights. Let $\epsilon(\rho)$ be a smooth function defined for $\rho > 0$ such that

$$\lim_{\rho\to 0^+} \epsilon(\rho) = 0 \text{ and } \int_0^{\rho*} \frac{\epsilon(\rho)}{\rho} d\rho < \infty, \tag{2.1}$$

for some $\rho* > 0$. We denote by $K_{x_0}(\rho)$ an N-dimensional cube contained in Ω whose edges are of length ρ and are parallel to the coordinate axes. We define

$$\begin{aligned} L^t_{\epsilon(\rho)}(\Omega) &= \{u \in L^t(\Omega) : \|u\|_{t,\epsilon(\rho),\Omega} < \infty\}, \text{ where} \\ \|u\|_{t,\epsilon(\rho),\Omega} &= \sup_{x_0\in\Omega,\rho>0} \frac{\|u\|_{L^t(K_{x_0}(\rho)\cap\Omega)}}{\epsilon(\rho)}. \end{aligned} \tag{2.2}$$

Remark 2.4. The weight m_r in Example 2.3 is such that $m_r \in L^{N/r}_{\epsilon(\rho)}(\Omega)$, but $m_r \notin L^s(\Omega)$ for $s > N/r$ if $1 < r \leq N$.

The following theorem guarantees the simplicity and isolation of $\lambda_1(r, m_r)$, where $r = p, q$. This result is proved by M. Montenegro and S. Lorca in [14]. To ensure positiveness of $\phi_1(r, m_r)$, the authors apply the Harnack inequality of [20].

Theorem 2.5. *[14] If one supposes $\partial\Omega$ is piecewise C^1 and $m_r\delta^\tau \in L^a(\Omega)$ with $m_r^+ \not\equiv 0$, where a, τ and r satisfy (H1), (H2), (H3) or (H4), then the number $\lambda_1(r, m_r)$ is attained by some $\phi_1(r, m_r) \in W^{1,p}_0(\Omega)$, where we may assume that $\phi_1(r, m_r) \geq 0$ a.e. in Ω, $\phi_1(r, m_r)^+ \not\equiv 0$. Moreover $\lambda_1(r, m_r)$ is positive and isolated.*
If in addition one assumes $m_r \in L^1(\Omega)$ for $r > N$ or $m_r \in L^{N/p}_{\epsilon(\rho)}(\Omega)$ for $1 < r \leq N$, then the first eigenvalue $\lambda_1(r, m_r)$ is simple and any positive eigenvalue other than $\lambda_1(r, m_r)$ has no positive eigenfunctions.

3. Rayleigh quotient and non-existence results

3.1. Rayleigh quotient for the problem $(P_{\lambda,\mu})$

This subsection concerns the Rayleigh quotient for our problem $(P_{\lambda,\mu})$.

Remark 3.1. We start by pointing out to find a solution for the problem $(P_{\lambda,\mu})$ is equivalent to seek a solution in the case $\mu = 1$, that is to solve the problem $(P_{\lambda,1})$. Indeed, if u is a solution of $(P_{\lambda,1})$, then multiplying equation $(P_{\lambda,1})$ by s^{p-1} for $s > 0$ we deduce that $v = su$ is a solution for problem $(P_{\lambda,\mu=s^{p-q}})$.
Conversely, let u be a solution of problem $(P_{\lambda,\mu})$. Then it follows that $v = \mu^{1/q-p}u$ is a solution of $(P_{\lambda,1})$.

We introduce now the functionals A and B on $W^{1,p}_0(\Omega)$ by

$$A(u) := \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{q}\int_\Omega |\nabla u|^q dx \tag{3.1}$$

$$B(u) := \frac{1}{p}\int_\Omega m_p(x)|u|^p dx + \frac{1}{q}\int_\Omega m_q(x)|u|^q dx \tag{3.2}$$

for all $u \in W^{1,p}_0(\Omega)$.

Proposition 3.2. *(i) The functional A is well defined and sequently weakly lower semi-continuous.*
(ii) If $m_r\delta^\tau \in L^a(\Omega)$ and $m_r^+ \not\equiv 0$ $(r = p, q)$, where a, r and τ satisfy one of the conditions (H1), (H2), (H3) or (H4), then the functional B are also well defined and weakly continuous.

Proof: (i) The functional A is well defined. indeed, since Ω bounded and $q < p$, we have $W_0^{1,p}(\Omega) \subset W_0^{1,q}(\Omega)$. Then for all $u \in W_0^{1,p}(\Omega)$, $\frac{1}{p}\int_\Omega |\nabla u|^p dx < \infty$ and $\frac{1}{q}\int_\Omega |\nabla u|^q dx < \infty$. It follows that $A(u) < \infty$. It is clear that A is sequently weakly lower semi-continuous.
(ii) The functional B is also well defined. Indeed, for $u \in W_0^{1,p}(\Omega)$, by Hölder's inequality with $\frac{1}{a} + \frac{1}{b} + \frac{r-1}{r} = 1$, where $r = p, q$ and $b = b(r) = \frac{ar}{a-r}$ if $a < \infty$ and $b = b(r) = r$ if $a = \infty$, we obtain

$$\begin{aligned}\frac{1}{r}\int_\Omega m_r(x)|u|^r dx &\leq \frac{1}{r}\int_\Omega m_r\delta^\tau \frac{|u|}{\delta^\tau}|u|^{r-1}dx\\ &\leq \frac{1}{r}\|m_r\delta^\tau\|_a \left\|\frac{u}{\delta^\tau}\right\|_b \||u|^{r-1}\|_{\frac{r}{r-1}}.\end{aligned}$$

Under assumption (H1) and Lemma 2.2, we have

$$\left\|\frac{u}{\delta^\tau}\right\|_b \| \leq C\|\nabla u\|_{\tau b} < \infty, \text{ because } \tau b \leq r.$$

Condition (H2) and Lemma 2.2 imply

$$\left\|\frac{u}{\delta^\tau}\right\|_b \| \leq C\|\nabla u\|_{\frac{bN}{N+b(1-\tau)}} < \infty, \text{ because } bN < r(N + b(1-\tau)).$$

By virtue Lemma 2.2 and (H3) or (H4),

$$\left\|\frac{u}{\delta^\tau}\right\|_b \| \leq C\|\nabla u\|_r < \infty.$$

Finally, in each case $B(u) < \infty$ and $C = C(r, N, a, \tau) > 0$ is a constant that may differ in each case except if $a = \infty$, $C = C(N, r) > 0$.
Let us now show that B is weakly continuous. If $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$, up to a subsequence, $u_n \to u$ strongly in $L^r(\Omega)$ and $|u_n|^{r-1} \to |u|^{r-1}$ strongly in $L^{r/r-1}(\Omega)$ with $r = p, q$, because $\|\nabla u_n\|_p$ is bounded and the embedding $W_0^{1,p}(\Omega) \subset L^r(\Omega)$ is compact. Hence by Hölder's inequality, we have

$$\begin{aligned}|B(u_n) - B(u)| &\leq \frac{1}{p}\left|\int_\Omega m_p(x)(|u_n|^p - |u|^p)dx\right| + \frac{1}{q}\left|\int_\Omega m_q(x)(|u_n|^q - |u|^q)dx\right|\\ &\leq C_p\|m_p\delta^\tau\|_a \left\|\frac{|u_n| + |u|}{\delta^\tau}\right\|_{b(p)} \left\||u_n|^{p-1} - |u|^{p-1}\right\|_{p/p-1}\\ &+ C_q\|m_q\delta^\tau\|_a \left\|\frac{|u_n| + |u|}{\delta^\tau}\right\|_{b(q)} \left\||u_n|^{q-1} - |u|^{q-1}\right\|_{q/q-1}\\ &\to 0.\end{aligned}$$

because $\left\| |u_n|^{r-1} - |u|^{r-1} \right\|_{r/r-1} \to 0$ and under (H1), (H2), (H3) or (H4) the norms $\|m_r \delta^\tau\|_a$ and $\left\| \frac{|u_n|+|u|}{\delta^\tau} \right\|_{b(r)}$ are bounded. The constant $C(r) = C/r > 0$, where C comes from the inequality $|\alpha^r - \beta^r| \le C(\alpha+\beta)|\alpha^{r-1} - \beta^{r-1}|$ for positive numbers α and β. To be precise, $C = 1$ if $r \ge 2$ and $C > r/(r-1)$ if $1 < r < 2$. Thus B is weakly continuous. □

Define now the Rayleigh quotient

$$\lambda^* = \inf \left\{ \frac{A(u)}{B(u)}; u \in W_0^{1,p}(\Omega), B(u) > 0 \right\}. \tag{3.3}$$

Proposition 3.3. *One assumes the same conditions as for Theorem 2.5. If $\lambda_1(p, m_p) \ne \lambda_1(q, m_q)$ or $\phi_1(p, m_p) \ne k\phi_1(q, m_q)$, for every $k > 0$. Then the infimum in 3.3 is not attained.*

For the proof of Proposition 3.3, we will need to use the following lemma.

Lemma 3.4. *The infimum in 3.3 verifies*

$$\lambda^* = \min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\}$$

Proof: For sufficiently large $k > 0$, using (3.1) et (3.2), we have

$$B(k\phi_1(p, m_p)) = k^q \left(k^{p-q} + \frac{1}{q} \int_\Omega m_q(x) \phi_1^q(p, m_p) dx \right) > 0.$$

and

$$A(k\phi_1(p, m_p)) = k^p \left(\lambda_1(p, m_p) + \frac{1}{q} k^{q-p} \int_\Omega |\nabla \phi_1(p, m_p)|^q dx \right).$$

By (3.3), we find

$$\begin{aligned} \lambda^* &\le \frac{A(k\phi_1(p, m_p))}{B(k\phi_1(p, m_p))} \\ &= \frac{\lambda_1(p, m_p) + \frac{1}{q} k^{q-p} \int_\Omega |\nabla \phi_1(p, m_p)|^q dx}{1 + \frac{1}{q} k^{q-p} \int_\Omega m_q(x) \phi_1^q(p, m_p) dx} \\ &\to \lambda_1(p, m_p) \text{ as } k \to +\infty, \text{ because } q < p. \end{aligned}$$

It follows that $\lambda^* \le \lambda_1(p, m_p)$. On the other hand, we also have

$$\begin{aligned} \lambda^* &\le \frac{A(k\phi_1(q, m_q))}{B(k\phi_1(q, m_q))} \\ &= \frac{\lambda_1(q, m_q) + \frac{1}{p} k^{p-q} \int_\Omega |\nabla \phi_1(q, m_q)|^p dx}{1 + \frac{1}{p} k^{p-q} \int_\Omega m_p(x) \phi_1^p(q, m_q) dx} \\ &\to \lambda_1(q, m_q) \text{ as } k \to 0^+, \text{ because } q < p. \end{aligned}$$

Thus, we obtain $\lambda^* \le \lambda_1(q, m_q)$, which implies that

$$\lambda^* \le \min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\}$$

Conversely, suppose by contradiction that $\lambda^* < \min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\}$. Then, by (3.3), there exists $u \in W_0^{1,p}(\Omega)$ such that $B(u) > 0$ and

$$\frac{A(u)}{B(u)} < \min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\}.$$

We distinguish three cases.
Case (i): Suppose that $\int_\Omega m_p|u|^p dx > 0$ and $\int_\Omega m_q|u|^q dx \le 0$. There hold $pB(u) \le \int_\Omega m_p|u|^p dx$ and $pA(u) \ge \|\nabla u\|_p^p$. Using the definition of $\lambda_1(p, m_p)$, we arrive at the contradiction.

$$\min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\} > \frac{A(u)}{B(u)} \ge \frac{\|\nabla u\|_p^p}{\int_\Omega m_p|u|^p dx} \ge \lambda_1(p, m_p). \tag{3.4}$$

Case (ii): Suppose that $\int_\Omega m_p|u|^p dx \le 0$ and $\int_\Omega m_q|u|^q dx > 0$. Using the definition of $\lambda_1(q, m_q)$, we also arrive at contradiction

$$\min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\} > \frac{A(u)}{B(u)} \ge \frac{\|\nabla u\|_q^q}{\int_\Omega m_q|u|^q dx} \ge \lambda_1(q, m_q). \tag{3.5}$$

Case (iii): Suppose now that $\int_\Omega m_p|u|^p dx > 0$ and $\int_\Omega m_q|u|^q dx > 0$. It follows from the definition of $\lambda_1(r, m_r)$, where $r = p, q$ that

$$\|\nabla u\|_r^r \ge \lambda_1(r, m_r) \int_\Omega m_r|u|^r dx.$$

Hence we get

$$\begin{aligned} A(u) &\ge \frac{\lambda_1(p, m_p)}{p} \int_\Omega m_p|u|^p dx + \frac{\lambda_1(q, m_q)}{q} \int_\Omega m_q|u|^q dx \\ &\ge \min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\} B(u). \end{aligned} \tag{3.6}$$

Against the assumption in our reasoning by contradiction.

□

Proof: [Proof of Proposition 3.3.] By contradiction, we suppose that there exists $u \in W_0^{1,p}(\Omega)$ such that $B(u) > 0$ and $\frac{A(u)}{B(u)} = \lambda^*$. Using Lemma 3.4, we give

$$\frac{A(u)}{B(u)} = \lambda^* = \min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\}. \tag{3.7}$$

We argue by considering the three cases in the proof of Lemma 3.4.
Case (i): By (3.4), (3.7) and $\int_\Omega m_q|u|^q dx \le 0$, we have

$$\lambda^* = \frac{A(u)}{B(u)} \ge \frac{\|\nabla u\|_p^p + \frac{p}{q}\|\nabla u\|_q^q}{\int_\Omega m_p|u|^p dx} \ge \frac{\|\nabla u\|_p^p}{\int_\Omega m_p|u|^p dx} \ge \lambda_1(p, m_p) \ge \lambda^*.$$

We deduce that

$$\|\nabla u\|_p^p = \lambda_1(p, m_p) \int_\Omega m_p |u|^p dx \text{ and } \|\nabla u\|_q = 0.$$

This contradicts the fact that $u \neq 0$.
Case (ii): similarly, By (3.5), (3.7) and $\int_\Omega m_p |u|^p dx \leq 0$, we get

$$\|\nabla u\|_q^q = \lambda_1(q, m_q) \int_\Omega m_q |u|^q dx \text{ and } \|\nabla u\|_p = 0.$$

Which contradicts $u \neq 0$.
Case (iii): In this case, using (3.6) and (3.7), we find

$$A(u) = \lambda^* B(u) = \frac{\lambda_1(p, m_p)}{p} \int_\Omega m_p |u|^p dx + \frac{\lambda_1(q, m_q)}{q} \int_\Omega m_q |u|^q dx.$$

It follows

$$[\lambda_1(p, m_p) - \lambda^*] \int_\Omega m_p |u|^p dx + [\lambda_1(q, m_q) - \lambda^*] \int_\Omega m_q |u|^q dx = 0.$$

Since $\int_\Omega m_p |u|^p dx > 0$, $\int_\Omega m_q |u|^q dx > 0$ and $\lambda^* = \min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\}$, we have

$$\lambda^* = \lambda_1(p, m_p) = \lambda_1(q, m_q).$$

We deduce that

$$\frac{\|\nabla u\|_p^p}{\int_\Omega m_p |u|^p dx} = \lambda_1(p, m_p) = \lambda_1(q, m_q) = \frac{\|\nabla u\|_q^q}{\int_\Omega m_q |u|^q dx}.$$

Hence, the simplicity of eigenvalue $\lambda_1(r, m_r)$ (for $r = p, q$), given by Theorem 2.5, guarantees that $u = t\phi_1(p, m_p) = s\phi_1(q, m_q)$ for some $t \neq 0$ and $s \neq 0$. The hypothesis of proposition is thus contradicted. □

3.2. Non-existence results

This subsection studies a non-existence results for the problem $(P_{\lambda,1})$, so for the problem $(P_{\lambda,\mu})$. This work is inspired from [15]. The following theorem is the main result of this section.

Theorem 3.5. *One assumes the same conditions as for Theorem 2.5.*

1. *If it holds $0 < \lambda < \lambda^*$, then the problem $(P_{\lambda,1})$ has no non-trivial solutions.*
2. *Moreover, if one of the following conditions holds*
   (i) $\lambda_1(p, m_p) \neq \lambda_1(q, m_q)$;
   (ii) $\phi_1(p, m_p) \neq k\phi_1(q, m_q)$, for every $k > 0$,

   then the problem $(P_{\lambda,1})$, with $\lambda = \lambda^*$ has no non-trivial solutions.

Remark 3.6. It is easy to see that if $\lambda_1(p, m_p) = \lambda_1(q, m_q)$ and $\phi_1(p, m_p) = k\phi_1(q, m_q)$, for some $k > 0$, then $\phi_1(p, m_p)$ and $\phi_1(q, m_q)$ are positive solutions of problem $(P_{\lambda,1})$, with $\lambda = \lambda_1(p, m_p) = \lambda_1(q, m_q)$.

Proof: [Proof of Theorem 3.5.] Assume by contradiction that there exists a non-trivial solution u of problem $(P_{\lambda,1})$. Then, for every $s > 0$, we have that $v = su$ is a non-trivial solution of problem $(P_{\lambda,s^{p-q}})$ (see Remark 3.1). Choose $s^{p-q} = p/q$ and then act with su as test function on the problem $(P_{\lambda,s^{p-q}})$. We arrive at

$$0 < pA(su) = p\lambda B(su). \tag{3.8}$$

From the estimate (3.8) and according to Lemma 3.4, we obtain

$$\lambda = \frac{A(su)}{B(su)} \geq \lambda^* = \min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\}.$$

This contradiction yields the first assertion of the theorem.
The second part of the Theorem 3.5 follows by Proposition 3.3. □

4. Existence results

4.1. Non-resonant cases

The following theorem is our main existence result for problem $(P_{\lambda,1})$ (or $(P_{\lambda,\mu})$) in the non-resonant cases. This result prove that there exists an interval of positive eigenvalues for the problem $(P_{\lambda,1})$ (or $(P_{\lambda,\mu})$, with $\mu > 0$).

Theorem 4.1. *In addition to the hypotheses of Theorem 2.5 one supposes that* $\lambda_1(p, m_p) \neq \lambda_1(q, m_q)$. *If*

$$\min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\} < \lambda < \max\{\lambda_1(p, m_p), \lambda_1(q, m_q)\},$$

then the problem $(P_{\lambda,1})$ *has at least one positive solution.*

Remark 4.2. The proof of Theorem 4.1 reduces to provide a non-trivial critical point of the functional I_{λ,m_p,m_q} defined for all $u \in W_0^{1,p}(\Omega)$ by

$$I_{\lambda,m_p,m_q}(u) := A(u) - \lambda B(u^+),$$

where $u^+ = \max\{u, 0\}$ and A, B are the functionals defined by (3.1) and (3.2). This non-trivial critical point u of I_{λ,m_p,m_q} is a non-negative solution of the problem $(P_{\lambda,1})$. We can check that $u \in L^\infty(\Omega)$ (see Remark 1.7 in [14]). Then the Harnack inequality of [20] can be applied to ensure positiveness of u.

The argument will be separately developed in two cases:
(a) $\lambda_1(q, m_q) < \lambda < \lambda_1(p, m_p)$.
(b)$\lambda_1(p, m_p) < \lambda < \lambda_1(q, m_q)$.
In case (a), we apply the minimum principle and in case (b), we use the mountain pass theorem.

Proof of case (a). By Proposition 3.2, A is sequently weakly lower semi-continuous and B is weakly continuous. It follows that I_{λ,m_p,m_q} is sequently weakly lower semi-continuous. It is remains to show that I_{λ,m_p,m_q} is coercive and bounded from below.
We distinguish two cases:
(i) For $u \in W_0^{1,p}(\Omega)$ such that $\int_\Omega m_p(u^+)^p dx \le 0$. A calculation similar to that in the proof of Proposition 3.2 for $r = q$ gives

$$\begin{aligned} I_{\lambda,m_p,m_q}(u) &\ge \frac{1}{p}\|\nabla u\|_p^p - \frac{\lambda}{q}\|m_q\delta^\tau\|_a \left\|\frac{u^+}{\delta^\tau}\right\|_{b(q)} \left\|(u^+)^{q-1}\right\|_{q/q-1} \\ &\ge \frac{1}{p}\|\nabla u\|_p^p - \frac{C\lambda}{q}\|m_q\delta^\tau\|_a\|\nabla u\|_q\|u\|_q^{q-1} \\ &\ge \frac{1}{p}\|\nabla u\|_p^p - \frac{CC'\lambda}{q}\|m_q\delta^\tau\|_a\|\nabla u\|_q^q \\ &\ge \frac{1}{p}\|\nabla u\|_p^p - \frac{CC'C''\lambda}{q}\|m_q\delta^\tau\|_a\|\nabla u\|_p^q, \end{aligned} \tag{4.1}$$

where $C, C', C'' > 0$ are the constants given respectively by the Hardy-Sobolev inequality (see Lemma 2.2), the compact embedding $W_0^{1,q}(\Omega) \subset L^q(\Omega)$ and the continuous embedding $W_0^{1,p}(\Omega) \subset W_0^{1,q}(\Omega)$.
(ii) For $u \in W_0^{1,p}(\Omega)$ such that $\int_\Omega m_p(u^+)^p dx > 0$. Fix $\epsilon > 0$ such that

$$(1-\epsilon)\lambda_1(p, m_p) > \lambda, \tag{4.2}$$

which is possible due to the assumption in case (a). By the definition of $\lambda_1(p, m_p)$ we have

$$\|\nabla u^+\|_p^p \ge \lambda_1(p, m_p) \int_\Omega m_p(u^+)^p dx.$$

Then taking into account (4.2), we derive

$$\begin{aligned} I_{\lambda,m_p,m_q}(u) &\ge \frac{\epsilon}{p}\|\nabla u\|_p^p + \frac{(1-\epsilon)\lambda_1(p,m_p) - \lambda}{p}\int_\Omega m_p(u^+)^p dx \\ &\quad - \frac{C\lambda}{q}\|m_q\delta^\tau\|_a\|\nabla u\|_q\|u\|_q^{q-1} \\ &\ge \frac{\epsilon}{p}\|\nabla u\|_p^p - \frac{CC'C''\lambda}{q}\|m_q\delta^\tau\|_a\|\nabla u\|_p^q. \end{aligned} \tag{4.3}$$

Since $q < p$, it follows from (4.1) and (4.3) that the functional I_{λ,m_p,m_q} is coercive and bounded from below. Consequently, by minimum principle, there exists a global minimizer u_0 of I_{λ,m_p,m_q}. Finally, $u_0 \ne 0$, indeed it suffices to prove that $I_{\lambda,m_p,m_q}(u_0) = \min_{W_0^{1,p}(\Omega)} I_{\lambda,m_p,m_q} < 0$. For sufficiently small $k > 0$, we have

$$I_{\lambda,m_p,m_q}(k\phi_1(q,m_q)) = k^q\left(\frac{k^{p-q}}{p}\|\nabla\phi_1(q,m_q)\|_p^p - \frac{\lambda k^{p-q}}{p}\int_\Omega m_p\phi_1^p(q,m_p)dx \right. \\ \left. + \frac{\lambda_1(q,m_q)-\lambda}{q}\right).$$

Then $I_{\lambda,m_p,m_q}(k\phi_1(q,m_q)) < 0$, because $\lambda_1(q,m_q) < \lambda$, which completes the proof of case (a).

Proof of case (b). We organize the proof of this case in several lemmas. In the sequel, we design by $o(1)$ a quantity tending to 0 as $n \to \infty$.

Lemma 4.3. *Suppose that $m_r\delta^\tau \in L^a(\Omega)$ and $m_r^+ \not\equiv 0$ $(r = p,q)$, where a, r and τ satisfy one of the conditions (H1), (H2), (H3) or (H4). In addition, we assume that $m_r \in L^1(\Omega)$ for $r > N$ or $m_r \in L^{N/p}_{\epsilon(\rho)}(\Omega)$ for $1 < r \leq N$. If $\lambda \neq \lambda_1(p,m_p)$, then the functional I_{λ,m_p,m_q} satisfies the Palais-Smale condition on $W_0^{1,p}(\Omega)$.*

Proof: Let $(u_n) \subset W_0^{1,p}(\Omega)$ be a sequence such that

$$I_{\lambda,m_p,m_q}(u_n) \to c \text{ for } c \in \mathbb{R} \text{ and } I'_{\lambda,m_p,m_q}(u_n) \to 0 \text{ in } (W_0^{1,p}(\Omega))^* \text{ as } n \to \infty.$$

Let us first show that the sequence u_n is bounded. It is sufficient only to prove the boundedness of $\|u_n\|_p$ because

$$\|\nabla u_n\|_p^p \leq pc + o(1) + C_p\|\nabla u_n\|_p\|u_n\|_p^{p-1} + \frac{p\alpha\beta C_q}{q}\|\nabla u_n\|_p\|u_n\|_p^{q-1}, \tag{4.4}$$

where α, β and C are respectively the constants in inequalities $\|u\|_q \leq \alpha\|u\|_p$, $\|\nabla u\|_q \leq \beta\|\nabla u\|_p$(since Ω bounded and $q < p$) and $\left\|\frac{u}{\delta^\tau}\right\|_{b(r)} \leq C\|\nabla u\|_r$ (in each case (H1), (H2), (H3) or (H4), see the proof of Proposition 3.2) and $C_r = \lambda C\|m_r\delta^\tau\|_a$ $(r = p,q)$. Suppose by contradiction that $\|u_n\|_p \to \infty$ and let $v_n := \frac{u_n}{\|u_n\|_p}$. The sequence v_n is bounded in $W_0^{1,p}(\Omega)$. Indeed, dividing (4.4) by $\|u_n\|_p^p$, we have

$$\begin{aligned}\|\nabla v_n\|_p^p &\leq o(1) + C_p\|\nabla v_n\|_p + \frac{p\alpha\beta}{q\|u_n\|_p^{p-q}}C_q\|\nabla v_n\|_p \\ &= o(1) + (C_p + o(1))\|\nabla v_n\|_p.\end{aligned} \tag{4.5}$$

Since $p > 1$, the inequality (4.5) implies the boundedness of v_n in $W_0^{1,p}(\Omega)$. For a subsequence, $v_n \to v$ weakly in $W_0^{1,p}(\Omega)$. By the compact embedding $W_0^{1,p}(\Omega) \subset L^r(\Omega)$ $(r = p,q)$, we have $v_n \to v$ strongly in $L^r(\Omega)$ $(r = p,q)$. First we, observe that $v^- \equiv 0$ in Ω. In fact, acting with $-u_n^-$ as test function, we have

$$o(1)\|\nabla(u_n^-)\|_p = \langle I'_{\lambda,m_p,m_q}(u_n), -u_n^-\rangle = \|\nabla(u_n^-)\|_p^p + \|\nabla(u_n^-)\|_q^q \geq \|\nabla(u_n^-)\|_p^p. \tag{4.6}$$

Because $p > 1$, the inequality (4.6) guarantees the boundedness of $\|\nabla(u_n^-)\|_p$ and so $\|\nabla v_n^-\|_p = \|\nabla(u_n^-)\|_p/\|u_n\|_p \to 0$. Thus $v^- \equiv 0$ holds, hence $v \geq 0$ in Ω.

Now, by taking $(v_n - v)/\|u_n\|_p^{p-1}$ as test function, we have

$$\begin{aligned} o(1) &= \left\langle I'_{\lambda,m_p,m_q}(u_n), \frac{v_n - v}{\|u_n\|_p^{p-1}} \right\rangle \\ &= \int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)dx + \frac{1}{\|u_n\|_p^{p-q}} \int_\Omega |\nabla v_n|^{q-2}\nabla v_n \nabla(v_n - v)dx \\ &- \lambda \int_\Omega m_p|v_n|^{p-2}v_n(v_n - v)dx - \frac{\lambda}{\|u_n\|_p^{p-q}} \int_\Omega m_q|v_n|^{q-2}v_n(v_n - v)dx \\ &= \int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)dx + o(1), \end{aligned} \tag{4.7}$$

because $q < p$, $\|u_n\|_p \to +\infty$, v_n is bounded in $W_0^{1,p}(\Omega)$ and converges to v strongly in $L^r(\Omega)$ (r=p,q). Thus by (4.7) and (S_+) property of $(-\Delta_p)$ on $W_0^{1,p}(\Omega)$, we deduce that $v_n \to v$ strongly in $W_0^{1,p}(\Omega)$. For any $\varphi \in W_0^{1,p}(\Omega)$, by taking $\varphi/\|u_n\|_p^{p-1}$ as test function, we obtain

$$\begin{aligned} o(1) &= \left\langle I'_{\lambda,m_p,m_q}(u_n), \frac{\varphi}{\|u_n\|_p^{p-1}} \right\rangle \\ &= \int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla\varphi dx + \frac{1}{\|u_n\|_p^{p-q}} \int_\Omega |\nabla v_n|^{q-2}\nabla v_n \nabla\varphi dx \\ &- \lambda \int_\Omega m_p|v_n|^{p-2}v_n\varphi dx - \frac{\lambda}{\|u_n\|_p^{p-q}} \int_\Omega m_q|v_n|^{q-2}v_n\varphi dx. \end{aligned} \tag{4.8}$$

Passing to the limit in (4.8), we see that v is a non-negative and non-trivial solution of problem (P_{λ,m_p}) (note $v \geq 0$ and $\|\nabla v\|_p = 1$). According to the Harnack inequality (see Remark 1.7 in [14]), we have $v > 0$ in Ω. This implies that $\lambda = \lambda_1(p, m_p)$ because any positive eigenvalue other than $\lambda_1(p, m_p)$ has no positive eigenfunctions (see Theorem 2.5). Therefore, we obtain a contradiction since we assumed $\lambda \neq \lambda_1(p, m_p)$. Hence u_n is bounded in $W_0^{1,p}(\Omega)$. For a subsequence, $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$ and $u_n \to u$ strongly in $L^r(\Omega)$ $(r = p, q)$.

We claim now that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$. It suffices to prove that $\|\nabla u_n\|_p \to \|\nabla u\|_p$, because $W_0^{1,p}(\Omega)$ is uniformly convex. It is clear that

$$\begin{aligned} o(1) =&\langle I'_{\lambda,m_p,m_q}(u_n) - I'_{\lambda,m_p,m_q}(u), u_n - u\rangle \\ &= \int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n - u)dx \\ &+ \int_\Omega (|\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u)\nabla(u_n - u)dx + o(1). \end{aligned} \tag{4.9}$$

Using Hölder inequality and for $r = p, q$, we have

$$\begin{aligned}
&\int_\Omega (|\nabla u_n|^{r-2}\nabla u_n - |\nabla u|^{r-2}\nabla u)\nabla(u_n - u)dx \\
&= \int_\Omega |\nabla u_n|^r dx + \int_\Omega |\nabla u|^r dx - \int_\Omega |\nabla u_n|^{r-2}\nabla u_n \nabla u dx - \int_\Omega |\nabla u|^{r-2}\nabla u \nabla u_n dx \\
&\geq \int_\Omega |\nabla u_n|^r dx + \int_\Omega |\nabla u|^r dx - \left(\int_\Omega |\nabla u_n|^r dx\right)^{(r-1)/r} \left(\int_\Omega |\nabla u|^r dx\right)^{1/r} \\
&- \left(\int_\Omega |\nabla u_n|^r dx\right)^{1/r} \left(\int_\Omega |\nabla u|^r dx\right)^{(r-1)/r} \\
&= (\|\nabla u_n\|_r^{r-1} - \|\nabla u\|_r^{r-1})(\|\nabla u_n\|_r - \|\nabla u\|_r) \\
&\geq 0.
\end{aligned} \tag{4.10}$$

Moreover, (4.9) and (4.10) imply that $\|\nabla v_n\|_r \to \|\nabla v\|_r$ (for $r = p, q$). Thus $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$.

□

Lemma 4.4. *Suppose $m_r\delta^\tau \in L^a(\Omega)$ and $m_r^+ \not\equiv 0$ $(r = p, q)$, where a, r and τ satisfy one of the conditions (H1), (H2), (H3) or (H4) If $\lambda < \lambda_1(q, m_q)$, then there exist $\alpha > 0$ and $\beta > 0$ such that*

$$I_{\lambda,m_p,m_q}(u) \geq \alpha \text{ whenever } \|u\|_q = \beta. \tag{4.11}$$

To prove the Lemma 4.4, we need the following lemma.

Lemma 4.5. *Suppose τ, a and p as in Lemma 4.4 and let b be such that $b = \frac{ap}{a-p}$ if $a < \infty$ and $b = p$ if $a = \infty$. Set $X(d) := \{u \in W_0^{1,p}(\Omega); \|\nabla u\|_p^p \leq d\|\frac{u}{\delta^\tau}\|_b \|u\|_p^{p-1}\}$, for $d > 0$. Then there exists $\alpha(d) > 0$ such that $\|\nabla u\|_p \leq \alpha(d)\|u\|_q$, for all $u \in X(d)$.*

Remark 4.6. Conditions (H4) implies that the Lemma 4.5 includes the Lemma 11 of the previously cited paper [15].

Proof: Suppose, by contradiction, that

$$(\forall n \in \mathbb{N}^*)\ (\exists u_n \in X(d)) : \ \frac{1}{n}\|\nabla u_n\|_p > \|u_n\|_q. \tag{4.12}$$

Because of $\|\nabla u_n\|_p \neq 0$, we set $v_n := \frac{u_n}{\|\nabla u_n\|_p}$. Thus, by (4.12), $v_n \to 0$ strongly in $L^q(\Omega)$. Since $\|\nabla v_n\|_p = 1$, the sequence v_n is bounded in $W_0^{1,p}(\Omega)$. For a subsequence, $v_n \to v$ weakly in $W_0^{1,p}(\Omega)$. By the compact embedding $W_0^{1,p}(\Omega) \subset L^r(\Omega)$ $(r = p, q)$, we have $v_n \to v$ strongly in $L^r(\Omega)$ $(r = p, q)$. Hence, $\|v_n\|_q \to \|v\|_q$ and $\|v_n\|_p \to \|v\|_q$. By uniqueness of limit, we deduce that $\|v\|_r = 0$. It follows that $v = 0$. As $u_n \in X(d)$, we obtain

$$\frac{1}{d} \leq \frac{\|\frac{u_n}{\delta^\tau}\|_b}{\|\nabla u_n\|_p}\|v_n\|_p^{p-1}. \tag{4.13}$$

By Lemma 2.2 and under one of the properties (H1), (H2), (H3) or (H4), the sequence $\frac{\|\frac{u_n}{\delta^\tau}\|_b}{\|\nabla u_n\|_p}$ is bounded (see the proof of Proposition 3.2). Passing to the limit in (4.13), we get $\frac{1}{d} \leq 0$. This contradiction completes the proof of Lemma 4.5. □

Proof: [Proof of Lemma 4.4.] According to Lemma 4.5, choose

$$d > \max\{1, \lambda\|m_p\delta^\tau\|_a, C\lambda\|m_p\delta^\tau\|_a\}, \tag{4.14}$$

where $C > 0$ is a constant that may differ in each case (H1), (H2), (H3) or (H4) (see the proof of Proposition 3.2).
For any $u \in X(d)$ satisfying $\int_\Omega m_q(u^+)^q dx \leq 0$, there exists $\alpha(d) > 0$ such that $\|\nabla u\|_p \leq \alpha(d)\|u\|_q$ and we have

$$\begin{aligned} I_{\lambda,m_p,m_q}(u) &\geq \frac{1-d}{p}\|\nabla u\|_p^p + \frac{\lambda_1(q,1)}{q}\|u\|_q^q + \frac{d}{p}\|\nabla u\|_p^p - \frac{C\lambda\|m_p\delta^\tau\|_a}{p}\|\nabla u\|_p^p \\ &\geq \frac{1-d}{p}\|\nabla u\|_p^p + \frac{\lambda_1(q,1)}{q}\|u\|_q^q + \frac{1}{p}\|\nabla u\|_p^p\,(d - C\lambda\|m_p\delta^\tau\|_a) \\ &\geq \frac{(1-d)[\alpha(d)]^p}{p}\|u\|_q^p + \frac{\lambda_1(q,1)}{q}\|u\|_q^q. \end{aligned} \tag{4.15}$$

For any $u \notin X(d)$ satisfying $\int_\Omega m_q(u^+)^q dx \leq 0$, we have $\|\nabla u\|_p^p > d\|\frac{u}{\delta^\tau}\|_b\|u\|_p^{p-1}$. It follows that

$$\begin{aligned} I_{\lambda,m_p,m_q}(u) &\geq \left(\frac{d - \lambda\|m_p\delta^\tau\|_a}{p}\right)\left\|\frac{u}{\delta^\tau}\right\|_b \|u\|_p^{p-1} + \frac{\lambda_1(q,1)}{q}\|u\|_q^q \\ &\geq \frac{\lambda_1(q,1)}{q}\|u\|_q^q. \end{aligned} \tag{4.16}$$

If $u \in W_0^{1,p}(\Omega)$ satisfying $\int_\Omega m_q(u^+)^q dx > 0$, by the definition of $\lambda_1(q, m_q)$, we get

$$\|\nabla u\|_q^q \geq \|\nabla u^+\|_q^q \geq \lambda_1(q, m_q)\int_\Omega m_q(u^+)^q dx. \tag{4.17}$$

Our assumption in λ enables us to fix $0 < \epsilon < 1$ with

$$(1-\epsilon)\lambda_1(q, m_q) > \lambda. \tag{4.18}$$

If in addition $u \notin X(d)$, then due to (4.14), (4.17) and (4.18) we have

$$\begin{aligned} I_{\lambda,m_p,m_q}(u) \quad &\geq \quad \left(\frac{d - \lambda\|m_p\delta^\tau\|_a}{p}\right)\left\|\frac{u}{\delta^\tau}\right\|_b \|u\|_p^{p-1} \\ &\quad + \frac{\epsilon}{q}\|\nabla u\|_q^q + \frac{1}{q}[(1-\epsilon)\lambda_1(q, m_q) - \lambda]\int_\Omega m_q(u^+)^q dx \\ &\geq \quad \frac{\lambda_1(q,1)}{q}\|u\|_q^q. \end{aligned} \tag{4.19}$$

Finally, if $u \in X(d)$ and $\int_\Omega m_q(u^+)^q dx > 0$, then (4.14), (4.17) and (4.18) imply

$$\begin{aligned} I_{\lambda,m_p,m_q}(u) &\geq \frac{1-d}{p}\|\nabla u\|_p^p + \frac{\epsilon}{q}\|\nabla u\|_q^q + \frac{1}{q}[(1-\epsilon)\lambda_1(q,m_q) - \lambda]\int_\Omega m_q(u^+)^q dx \\ &\quad + \frac{d}{p}\|\nabla u\|_p^p - \frac{\lambda\|m_p\delta^\tau\|_a}{p}\left\|\frac{u}{\delta^\tau}\right\|_b \|u\|_p^{p-1} \\ &\geq \frac{(1-d)[\alpha(p)]^p}{p}\|u\|_q^p + \frac{\epsilon\lambda_1(q,1)}{q}\|u\|_q^q + \frac{d - C\lambda\|m_p\delta^\tau\|_a}{p}\|\nabla u\|_p^p \\ &\geq \frac{(1-d)[\alpha(p)]^p}{p}\|u\|_q^p + \frac{\epsilon\lambda_1(q,1)}{q}\|u\|_q^q. \end{aligned} \tag{4.20}$$

Using that $q < p$, the claim (4.11) in Lemma 4.4 follows from (4.15), (4.16), (4.19) and (4.20).

□

Lemma 4.7. *Suppose $m_r\delta^\tau \in L^a(\Omega)$ and $m_r^+ \not\equiv 0$ $(r = p, q)$, where a, r and τ satisfy one of the conditions (H1), (H2), (H3) or (H4). If $\lambda_1(p, m_p) < \lambda$, then there is $R > 0$ such that*

$$\|R\phi_1(p, m_p)\|_q > \beta \text{ and } I_{\lambda,m_p,m_q}(R\phi_1(p, m_p)) < 0, \tag{4.21}$$

where $\beta > 0$ is the constant in (4.11).

Proof: For sufficiently large $R > 0$, taking into account that $q < p$ and $\lambda_1(p, m_p) < \lambda$, we have

$$\begin{aligned} \frac{I_{\lambda,m_p,m_q}(R\phi_1(p, m_p))}{R^\beta} &= \frac{\lambda_1(p, m_p) - \lambda}{p} \\ &\quad + \frac{1}{qR^{p-q}}\left(\|\phi_1(p, m_p)\|_q^q - \int_\Omega m_q\phi_1^q(p, m_p)dx\right) < 0. \end{aligned}$$

□

Recalling that the functional I_{λ,m_p,m_q} satisfies the Palais-Smale condition by virtue of Lemma 4.3, the properties (4.11) of Lemma 4.4 and (4.21) of Lemma 4.7 allow us to apply the mountain pass theorem, which guarantees the existence of a critical value $c \geq \alpha$ of I_{λ,m_p,m_q}, with $\alpha > 0$ in (4.11), namely $c := \inf_{\gamma\in\Sigma} \max_{t\in[0,1]} I_{\lambda,m_p,m_q}(\gamma(t))$, where $\Sigma := \{\gamma \in C([0, 1], W_0^{1,p}(\Omega)); \gamma(0) = 0, \gamma(1) = R\phi_1(p, m_p)\}$. This completes the proof of Theorem 4.1.

4.2. Resonant cases

In this section, we study the existence result for problem $(P_{\lambda,1})$ (or $(P_{\lambda,\mu})$) in the resonant cases. The following theorem is our main result in this section.

Theorem 4.8. *One assumes the same conditions as for Theorem 2.5. If one of the following assertions holds*

(i) $\lambda = \lambda_1(p, m_p) > \lambda_1(q, m_q)$ *and* $\int_\Omega |\nabla \phi_1(p, m_p)|^q dx - \lambda \int_\Omega m_q |\phi_1(p, m_p)|^q dx > 0$*;*

(ii) $\lambda = \lambda_1(q, m_q) > \lambda_1(p, m_p)$ *and* $\int_\Omega |\nabla \phi_1(q, m_q)|^p dx - \lambda \int_\Omega m_p |\phi_1(q, m_q)|^p dx > 0$,

then the problem $(P_{\lambda,1})$ *(or* $(P_{\lambda,\mu})$*) has at least one positive solution.*

Proof: Proof of case (i): Since $m_p^+ \not\equiv 0$, the Lebesgue mesure of $\{x \in \Omega; m_p(x) > 0\}$ is positive. Thus there exists an $n_0 \in \mathbb{N}$ such that $(m_p - 1/n_0) \not\equiv 0$. For $n \geq n_0$, we define, as in [15], the functional I_n on $W_0^{1,p}(\Omega)$ by

$$I_n(u) = I_{\lambda,m_p,m_q}(u) + \frac{\lambda}{pn}\|u^+\|_p^p = I_{\lambda,m_p-1/n,m_q}(u).$$

Using the strict monotonicity of the first eigenvalue of problem (P_{λ,m_p}) (see [14]), we obtain $\lambda_1(p, m_p - 1/n) > \lambda_1(p, m_p) = \lambda$. Thus we are able to apply Theorem 3.5 obtaining a positive solution u_n of the problem

$$\begin{cases} -\triangle_p u - \triangle_q u = \lambda[(m_p(x) - 1/n)|u|^{p-2}u + m_q(x)|u|^{q-2}u] & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We may assume that u_n is a global minimizer of I_n and $I_n(u_n) < 0$ (see the case (a) in the proof of Theorem 3.5). In addition, observing that $I_n \leq I_{n_0}$ provided $n \geq n_0$, we infer that for all $n \geq n_0$,

$$I_n(u_n) = \min_{W_0^{1,p}(\Omega)} I_n \leq I_n(u_{n_0}) \leq I_{n_0}(u_{n_0}) < 0.$$

We claim that if u_n is bounded in $W_0^{1,p}(\Omega)$, then u_n is a bounded Palais-Smale sequence of I_{λ,m_p,m_q}. Indeed, there exists $c \in \mathbb{R}$ such that $I_{\lambda,m_p,m_q}(u_n) \to c$ because $I_n(u_n)$ is a convergent sequence and $\|\nabla u_n\|_p$ is bounded. On the other hand, since $I_n'(u_n) = 0$, we have

$$\begin{aligned} \|I'_{\lambda,m_p,m_q}(u_n)\|_{(W_0^{1,p}(\Omega))^*} &= \|I'_{\lambda,m_p,m_q}(u_n) - I_n'(u_n)\|_{(W_0^{1,p}(\Omega))^*} \\ &\leq \frac{\lambda[\lambda_1(p,1)]^{-1/p}}{n}\|\nabla(u_n)^+\|_p^{p-1}. \end{aligned}$$

As $\|\nabla u_n\|_p$ is bounded, we obtain $\|I_\lambda'(u_n\|_{(W_0^{1,p}(\Omega))^*} \to 0$. This completes the proof of our claim.

We prove now the boundedness of u_n in $W_0^{1,p}(\Omega)$ by way of contradiction. Suppose that $\|\nabla u_n\|_p \to \infty$ and let $v_n := u_n/\|\nabla u_n\|_p$. For a subsequence, $v_n \to v$ weakly in $W_0^{1,p}(\Omega)$ and $v_n \to v$ strongly in $L^p(\Omega)$. Following the same steps of the proof of

Lemma 4.3, we show that v is a positive solution of problem (P_{λ,m_p}). This entails v is a positive eigenfunction corresponding to $\lambda_1(p,m_p)$. Thus the simplicity of $\lambda_1(p,m_p)$ implies that $v=\phi_1(p,m_p)$. The facts that $I_n(u_n)<0$ for all $n\geq n_0$ and u_n is a critical point of I_n result in

$$0>\frac{I_n(u_n)}{\|\nabla u_n\|_p^q}=\left(\frac{1}{q}-\frac{1}{p}\right)\left(\|\nabla v_n\|_q^q-\lambda\int_\Omega m_q v_n^q dx\right).$$

Passing to the limit, we obtain $\int_\Omega |\nabla\phi_1(p,m_p)|^q dx-\lambda\int_\Omega m_q|\phi_1(p,m_p)|^q dx\leq 0$ which contradicts second point of assertion (i). Thus u_n is a bounded Palais-Smale sequence of I_{λ,m_p,m_q}. Since $\lambda\neq\lambda_1(p,m_p)$, I_{λ,m_p,m_q} satisfies Pais-Smale condition (see Lemma 4.3). It follows that u_n has a subsequence converging to some critical point u_0 of I_{λ,m_p,m_q}.
We note that $u_0\neq 0$ because $I_{\lambda,m_p,m_q}(u_0)=\lim_n I_n(u_n)\leq I_{n_0}(u_0)<0$. Therefore u_0 is a positive solution of problem $(P_{\lambda,1})$ (see Remark 4.2).

Proof of case (ii): As in the proof of case (i), we can choose $n_0\in\mathbb{N}$ such that $(m_q-1/n_0)\neq 0$. For $n\geq n_0$, we define the functional J_n on $W_0^{1,p}(\Omega)$ by

$$J_n(u)=I_{\lambda,m_p,m_q}(u)+\frac{\lambda}{pn}\|u^+\|_q^q=I_{\lambda,m_p,m_q-1/n}(u).$$

Using the strict monotonicity of the first eigenvalue of problem (P_{λ,m_q}), we obtain $\lambda_1(q,m_q-1/n)>\lambda_1(q,m_q)=\lambda$, for any $n\geq n_0$. Thus we may apply Theorem 3.5 obtaining a positive solution u_n of the problem

$$\begin{cases} -\triangle_p u-\triangle_q u &= \lambda[m_p(x)|u|^{p-2}u+(m_q(x)-1/n)|u|^{q-2}u] \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega.\end{cases}$$

Following the pattern of the proof of case (i), this time proceeding as in case (b) in the proof of Theorem 3.5, we deduce that $J_n(u_n)>0$ for all $n\geq n_0$. For the boundedness of u_n in $W_0^{1,p}(\Omega)$, proceeding as in the proof of case (i) and the contradiction follows from the condition $\lambda=\lambda_1(q,m_q)>\lambda_1(p,m_p)$. The bounded sequence u_n is a Palais-Smale sequence for the functional I_{λ,m_p,m_q} as can be seen from the estimate

$$\|I'_{\lambda,m_p,m_q}(u_n)\|_{(W_0^{1,p}(\Omega))^*}=\|I'_{\lambda,m_p,m_q}(u_n)-J'_n(u_n)\|_{(W_0^{1,p}(\Omega))^*}\leq\frac{c}{n}\|u_n\|_q^{q-1},$$

where c is a positive constant independent of n. It follows that u_n has a subsequence converging to some critical point u_0 of I_{λ,m_p,m_q}. In order to complete the proof, due to the Harnack inequality, it suffices to justify that $u_0\neq 0$. We assume, by contradiction, that $u_n\to 0$ strongly in $W_0^{1,p}(\Omega)$. Set $v_n:=\frac{u_n}{\|\nabla u_n\|_p}$. Then, for a subsequence, $v_n\to v$ weakly in $W_0^{1,p}(\Omega)$, weakly in $W_0^{1,q}(\Omega)$ and strongly in $L^p(\Omega)$. It is easy to see that $v\geq 0$ in Ω. Using $\frac{v_n-v}{\|\nabla u_n\|_p^{q-1}}$ as test function, we obtain

$$0=\left\langle J'_n(u_n),\frac{v_n-v}{\|u_n\|^{q-1}}\right\rangle=\int_\Omega|\nabla v_n|^{q-2}\nabla v_n\nabla(v_n-v)dx+o(1).$$

By (S_+) property for $-\Delta_q$ on $W_0^{1,q}(\Omega)$, we have $v_n \to v$ strongly in $W_0^{1,q}(\Omega)$. We claim that $v \neq 0$ in Ω. Using u_n as test function, we may write

$$\begin{aligned}0 &= \langle J_n'(u_n), u_n\rangle \\ &= \|\nabla u_n\|_p^p - \lambda \int_\Omega m_p u_n^p dx + \|\nabla u_n\|_q^q - \lambda \int_\Omega (m_q - \frac{1}{n}) u_n^q dx \\ &\geq \|\nabla u_n\|_p^p - \lambda \int_\Omega m_p u_n^p dx,\end{aligned} \tag{4.22}$$

because $\lambda = \lambda_1(q, m_q)$ and by the definition of $\lambda_1(q, m_q)$, we have $\|\nabla u_n\|_q^q - \lambda \int_\Omega m_q u_n^q dx \geq 0$. Thus according (H1), (H2), (H3) or (H4), we have $\|\nabla u\|_p^p \leq \lambda \|m_p \delta^\tau\|_a \|\frac{u}{\delta^\tau}\|_b \|u\|_p^{p-1}$, where τ, a and b as in Lemma 4.5. Whence $u_n \in X(d)$ with $d = \lambda \|m_p \delta^\tau\|_a$. Therefore, Lemma 4.5 guarantees the existence of a constant $\alpha(d) > 0$ such that

$$\|\nabla u\|_p \leq \alpha(d) \|u_n\|_q \leq \alpha(d) [\lambda_1(q, 1)]^{-1/q} \|\nabla u_n\|_q, \text{ for all } n \geq n_0.$$

Consequently, recalling that $v_n \to v$ strongly in $W_0^{1,q}(\Omega)$, we have

$$\|\nabla v\|_q = \lim \|\nabla v_n\|_q = \lim \frac{\|\nabla u_n\|_q}{\|\nabla u_n\|_p} \geq \frac{1}{\alpha(d)[\lambda_1(q,1)]^{-1/q}} > 0,$$

thus proving our claim.
For any $\varphi \in W^{1,p}(\Omega)$, by using $\frac{\varphi}{\|\nabla u_n\|_p^{q-1}}$ as test function we show, as in proof of case (i), that v is a nonnegative nontrivial solution of (P_{λ, m_q}). The simplicity of $\lambda_1(q, m_q)$ guarantees that $v = \phi_1(q, m_q)$ is a positive eigenfunction corresponding to $\lambda_1(q, m_q)$.
Using that $J_n(u_n) > 0$ for all $n \geq n_0$ in conjunction with (4.22), we have

$$0 \leq \frac{J_n(u_n)}{\|\nabla u_n\|_p^p} = \left(\frac{1}{p} - \frac{1}{q}\right) \left(\|\nabla v_n\|_p^p - \lambda \int_\Omega m_p v_n^p dx\right).$$

Since $q < p$, it follows that

$$\|\nabla v_n\|_p^p - \lambda \int_\Omega m_p v_n^p dx \leq 0.$$

By passing to the limit inferior we obtain

$$\|\nabla \phi_1(q, m_q)\|_p^p - \lambda \int_\Omega m_p [\phi_1(q, m_q)]^p dx \leq 0.$$

By this contradiction the proof of Theorem 4.8 is achieved.

□

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