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The PCD Method on Composite Grid

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abstract: We introduce a discretization method of boundary value problems (BVP) in the case of the two dimensional diffusion equation on a rectangular mesh with refined zones. The method consists in representing the unknown distribution and its derivatives by piecewise constant distributions (PCD) on distinct meshes together with an appropriate approximate variational formulation of the exact BVP on this piecewise constant distributions space. This method, named the PCD method, has the advantage of producing the most compact possible discrete stencil. Here, we analyze and prove the convergence of the PCD method and determine upper bounds on its convergence rate.

Key Words: Boundary value problem, PCD method, local mesh refinement, most compact discrete scheme, interface boundary, $O(h)$-convergence rate.

Contents

1. Introduction

We propose a discretization technique of boundary value problems (BVP) in which the unknown distribution and its derivatives are all represented by piecewise constant distributions (PCD) but on distinct meshes. The only difficulty of the method lies in the appropriate choice of these meshes. Once done, it becomes

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rather straightforward to introduce an appropriate approximate variational formulation of the exact BVP on this piecewise constant distribution space. We apply and analyze the method, named the PCD method, in the case of the diffusion equation on a rectangular mesh with refined zones. It has the advantage, compared with other discretization methods, of producing the most compact discrete stencil. Essentially, the PCD method does not make use of the so-called slave nodes that appear in some finite element discretizations with local mesh refinement. Particularly, the graph of the discrete matrix turns out to be the grid itself of the mesh used for the unknown distribution.

The interest for piecewise constant approximations has been stressed since the early days of discretization techniques for partial differential equations. This motivated the earlier analysis by Aubin [2,3,4], Cea [10], Temam [18], Girault [14], Bank and Rose [5] and Weiser and Wheeler [21]. Numerous contributions by Cai, Mandel and McCormick [8], Cai and McCormick [9], Ewing, Lazarov and Vassilevski [12] and Vassilevski, Petrova and Lazarov [19], to quote a few of them, clearly demonstrate a persistent interest for these approximations.

Our approach cannot however rely on the results obtained so far because we use piecewise constant approximations not only for the unknown distribution itself but also for its derivatives. This feature makes its mathematical analysis more difficult and requires further developments. To keep them as simple as possible, we restrict the present contribution to the analysis of the two dimensional diffusion equation on rectangular mesh with local mesh refinement.

More specifically, we consider solving the following partial differential equation on a rectangular domain Ω :

$$- \operatorname{div} (p(x) \nabla u(x)) + q(x)u(x) = s(x) \qquad x \in \Omega \tag{1.1}$$

$$u(x) = 0 \qquad x \in \Gamma_0 \tag{1.2}$$

$$n \cdot \nabla u(x) + \omega(x)u(x) = 0 \qquad x \in \Gamma_1 \tag{1.3}$$

where n denotes the unit normal to $\Gamma = \partial\Omega$ and $\Gamma = \Gamma_0 \cup \Gamma_1$.
By a rectangular domain we understand any connected subset of the plane with orthogonal sides. It may be L-shaped and need only to be simply connected. We assume that $p(x)$ is bounded and strictly positive on $\overline{\Omega}$ and that $q(x)$ and $\omega(x)$ are bounded and nonnegative on Ω and Γ_1 respectively, and we assume that $s(x)$ is in $L^2(\Omega)$. In the case where $q(x)$ and $\omega(x)$ would be identically zero, we require that Γ_0 has positive measure. Otherwise, one would extend the results developed below by requiring that $s(x)$ and $u(x)$ be orthogonal to unity in the $L^2(\Omega)$ scalar product, as usually done in such analysis. We use homogeneous boundary conditions for simplicity. The extension of the theory to general boundary conditions does not raise new difficulties.

The discrete version of this problem will be based on its variational formulation:

$$\text{find} \quad u \in H \quad \text{such that} \quad \forall\, v \in H \quad a(\,u,\, v\,) = (s,\, v)_\Omega \tag{1.4}$$

where $H = H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma_0\}$,

$$a(u,v) = \int_\Omega p(x)\, \nabla u(x) \cdot \nabla v(x)\, dx + \int_\Omega q(x) u(x)\, v(x)\, dx$$
$$+ \int_{\Gamma_1} p(x)\omega(x) u(x)\, v(x)\, ds\,, \tag{1.5}$$

and $(s,\, v)_\Omega$ denotes the $L^2(\Omega)$ scalar product. We use here the standard notation for Sobolev spaces as defined in Adams [1]:

$$H^m = H^m(\Omega) = \{\, u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega),\, |\alpha| \leq m\}, \quad m \geq 0.$$

The norm in $H^m(\Omega)$ is denoted $\|\,.\,\|_{m,\Omega}$ or simply $\|\,.\,\|_m$ and the semi-norm is denoted $|\,.\,|_{m,\Omega}$ or simply $|\,.\,|_m$. The norm in $L^2(\Omega)$ is denoted $\|\,.\,\|_\Omega$ or simply $\|\,.\,\|$.

The precise meaning of Equation (1.1) with boundary conditions (1.2)-(1.3) and $s(x) \in L^2(\Omega)$ is to define an equation

$$A\, u = s \tag{1.6}$$

where A is a linear operator with domain $\mathcal{D}(A) \subset L^2(\Omega)$ and range $\mathcal{R}(A) = L^2(\Omega)$ (distributions belonging to $L^2(\Omega)$ space acting on $L^2(\Omega)$ distributions). So that Equation (1.6) is actually equivalent to stating that:

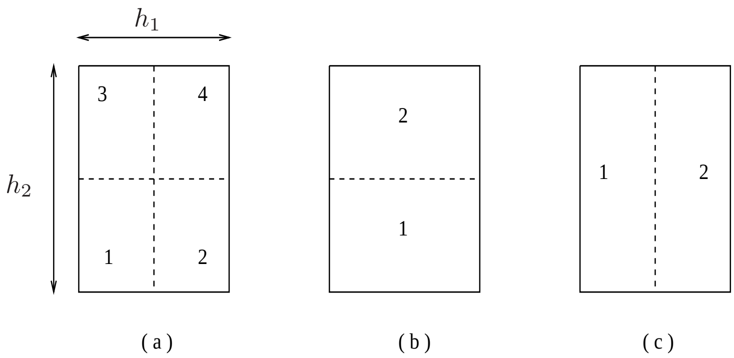
$$\forall\, v \in L^2(\Omega) \ : < Au, v > = < s, v > \tag{1.7}$$

where we have used the bilinear notation $<\,,\,>$ rather than the usual scalar product notation in order to stress that $A\,u$, s and v are distributions. This holds in particular when $v \in H$ and integration by parts is then generally used to get Equation (1.4) which is at the root of the study of the properties of the operator A.

Integration by parts can however be done more generally, for example when v has only piecewise square summable derivatives, with possible jumps at the inner boundaries of the subdomains of Ω on which it has square summable first derivatives (and provided that $p(x)$ is not discontinuous across these boundaries). See for example [7, pp. 94-95]. It turns out that the issue may again be written:

$$a(u,v) = (s,v)_\Omega\,, \tag{1.8}$$

although the roles of u and v are no more symmetrical in that case, since ∇v may present Dirac behaviors across some inner boundaries on which $p\,\nabla u$ is required to be continuous.



Submeshes used to represent v_h (a), $\partial_{h1}\, v_h$ (b) and $\partial_{h2}\, v_h$ (c) on Ω_ℓ

Figure 1: Regular (rectangular) element

2. The PCD discretization

2.1. Space discretization :

The discretization technique proposed here splits the open domain Ω under investigation into elements $\Omega_\ell\,(\ell \in J)$ open subsets of Ω such that

$$\overline{\Omega} = \bigcup_{\ell \in J} \overline{\Omega}_\ell \qquad \Omega_k \cap \Omega_\ell = \emptyset \qquad \text{if } k \neq \ell \ .$$

We denote by h the mesh size defined by: $h = \max(h_\ell)\,(\ell \in J)$ where $h_\ell = \mathrm{diam}(\Omega_\ell)$ $(\ell \in J)$ and we denote by $h_{\ell 1}$ and $h_{\ell 2}$, the width and the height of the element Ω_ℓ. We define several submeshes on each element Ω_ℓ for the representation of $v \in H^1(\Omega)$ and its derivatives $\partial_i v$ $(i = 1, 2)$. These representations, denoted v_h and $\partial_{hi}\, v_h$ $(i = 1, 2)$ respectively, are piecewise constant on each of these submeshes (a specific one for each) with the additional requirement for v_h that it must be continuous across the element boundaries (i.e. along the normal to the element boundary). In the case of the rectangular meshes considered in the present work, the operators ∂_{hi} $(i = 1, 2)$ will be finite difference quotients taken along the element edges. They would need to be appropriately adapted for other elements.

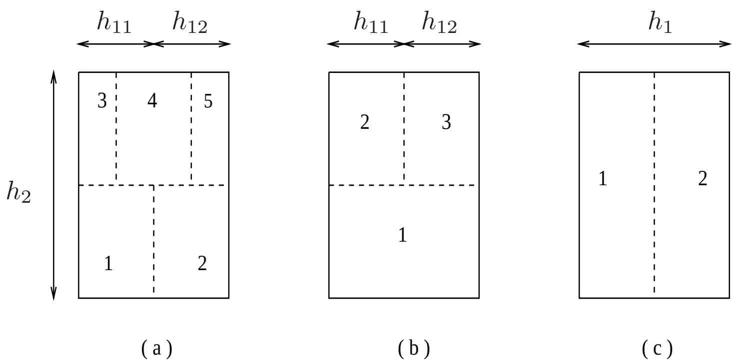
Figure 1 gives an example of submeshes used to define $v_h|_{\Omega_\ell}$, $\partial_{h1}\, v_h|_{\Omega_\ell}$ and $\partial_{h2}\, v_h|_{\Omega_\ell}$ on a regular rectangular element Ω_ℓ, i.e. an element from a regular rectangular mesh.
$v_h|_{\Omega_\ell}$ is the piecewise constant distribution with 4 values v_{hi} on the regions denoted i with $i = 1, ..., 4$ on Fig. 1 (a) .
$\partial_{h1}\, v_h|_{\Omega_\ell}$ is the piecewise constant distribution with constant values:

$$(\partial_{h1}\, v_h)_1 = \frac{v_{h2} - v_{h1}}{h_1} \quad , \quad (\partial_{h1}\, v_h)_2 = \frac{v_{h4} - v_{h3}}{h_1}$$

on the regions denoted 1, 2 on Fig. 1 (b) and $\partial_{h2} v_h|_{\Omega_\ell}$ is similarly the piecewise constant distribution with constant values:

$$(\partial_{h2} v_h)_1 = \frac{v_{h3} - v_{h1}}{h_2} \quad , \quad (\partial_{h2} v_h)_2 = \frac{v_{h4} - v_{h2}}{h_2}$$

on the regions denoted 1, 2 on Fig. 1 (c). In addition, v_h must be continuous across element boundaries. Thus for example if the bottom boundary of Ω_ℓ is common with the top boundary of Ω_k, one must have that $v_{h1}(\Omega_\ell) = v_{h3}(\Omega_k)$ and $v_{h2}(\Omega_\ell) = v_{h4}(\Omega_k)$.



Submeshes used to represent v_h (a), $\partial_{h1} v_h$ (b) and $\partial_{h2} v_h$ (c) on Ω_ℓ

Figure 2: Irregular (rectangular) element

Figure 2 provides another example of submeshes used to define $v_h|_{\Omega_\ell}$, $\partial_{h1} v_h|_{\Omega_\ell}$ and $\partial_{h2} v_h|_{\Omega_\ell}$ now in the case of an irregular element, i.e. an element located along the bottom boundary of a refined zone. In this case $v_h|_{\Omega_\ell}$ assumes 5 values v_{hi} , $i = 1, ..., 5$, $\partial_{h1} v_h|_{\Omega_\ell}$ 3 values:

$$(\partial_{h1} v_h)_1 = \frac{v_{h2} - v_{h1}}{h_1} \quad , \quad (\partial_{h1} v_h)_2 = \frac{v_{h4} - v_{h3}}{h_{11}} \quad , \quad (\partial_{h1} v_h)_3 = \frac{v_{h5} - v_{h4}}{h_{12}}$$

where $h_1 = h_{11} + h_{12}$, and $\partial_{h2} v_h|_{\Omega_\ell}$ 2 values:

$$(\partial_{h2} v_h)_1 = \frac{v_{h3} - v_{h1}}{h_2} \quad , \quad (\partial_{h2} v_h)_2 = \frac{v_{h5} - v_{h2}}{h_2}$$

Remember that v_h must be continuous across element boundaries. Thus if the top boundary of Ω_ℓ is common with the bottom boundaries of the 2 cells Ω_{k1} and Ω_{k2} of widths h_{11} and h_{12} we have that:
$v_{h3}(\Omega_\ell) = v_{h1}(\Omega_{k1})$, $v_{h4}(\Omega_\ell) = v_{h2}(\Omega_{k1}) = v_{h1}(\Omega_{k2})$ and $v_{h5}(\Omega_\ell) = v_{h2}(\Omega_{k2})$.

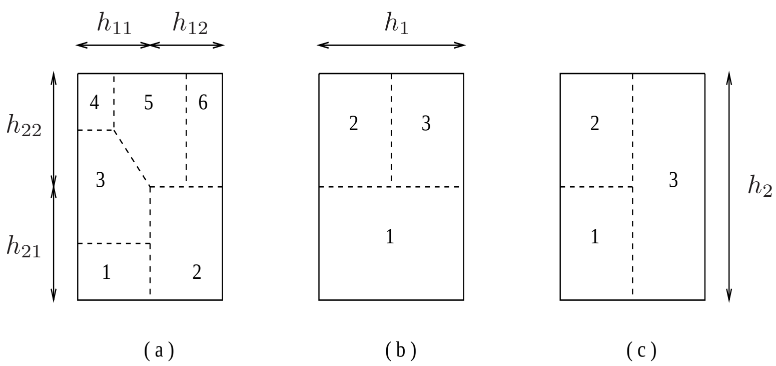
For simplicity, we consider a mesh refinement by a factor 2. Higher order ratio can of course be handled in a similar way, see Tahiri [15].

Figure 3 provides a particular case of submeshes used to define $v_h|_{\Omega_\ell}$, $\partial_{h1} v_h|_{\Omega_\ell}$ and $\partial_{h2} v_h|_{\Omega_\ell}$, when the boundary of a refined zone covers two different sides of Ω_ℓ. In this case $v_h|_{\Omega_\ell}$ assumes 6 values v_{hi}, $i = 1, ..., 6$, $\partial_{h1} v_h|_{\Omega_\ell}$ 3 values, where $h_1 = h_{11} + h_{12}$

$$(\partial_{h1} v_h)_1 = \frac{v_{h2} - v_{h1}}{h_1}\ ,\ (\partial_{h1} v_h)_2 = \frac{v_{h5} - v_{h4}}{h_{11}}\ ,\ (\partial_{h1} v_h)_3 = \frac{v_{h6} - v_{h5}}{h_{12}}$$

and $\partial_{h2} v_h|_{\Omega_\ell}$ 3 values, where $h_2 = h_{21} + h_{22}$

$$(\partial_{h2} v_h)_1 = \frac{v_{h3} - v_{h1}}{h_{21}}\ ,\ (\partial_{h2} v_h)_2 = \frac{v_{h4} - v_{h3}}{h_{22}}\ ,\ (\partial_{h2} v_h)_3 = \frac{v_{h6} - v_{h2}}{h_2}$$



Submeshes used to represent v_h (a), $\partial_{h1} v_h$ (b) and $\partial_{h2} v_h$ (c) on Ω_ℓ

Figure 3: Particular case

The further handling and analysis of our discretization method require appropriate notation for the various spaces involved, which we introduce now.

By $\mathbf{X}$ we denote $(L^2(\Omega))^3$ with norm $\|(u, v, w)\|^2_{\mathbf{X}} = \|u\|^2 + \|v\|^2 + \|w\|^2$.
By $\mathbf{Y}$ we denote the subspace of $\mathbf{X}$ of the elements of the form $(v, \partial_1 v, \partial_2 v)$.
By H_{h0} and H_{hi} $(i = 1, 2)$ we denote the spaces of piecewise constant distributions used to define v_h and $\partial_{hi} v_h$ $(i = 1, 2)$, equipped with the $L^2(\Omega)$ scalar product.
$\mathbf{X}_h = H_{h0} \times H_{h1} \times H_{h2}$ with norm $\|(u_h, v_h, w_h)\|^2_{\mathbf{X}} = \|u_h\|^2 + \|v_h\|^2 + \|w_h\|^2$ and $\mathbf{Y}_h$ is the subspace of $\mathbf{X}_h$ of the elements of the form $(v_h, \partial_{h1} v_h, \partial_{h2} v_h)$.

We further denote by H_h the space H_{h0} equipped with the inner product:

$$(v_h, w_h)_h = (v_h, w_h)_\Omega + (\partial_{h1} v_h, \partial_{h1} w_h)_\Omega + (\partial_{h2} v_h, \partial_{h2} w_h)_\Omega, \quad (2.1)$$

and its associate norm is denoted $\|\,.\,\|_h$.
Clearly H and H_h are isomorphic to $\mathbf{Y}$ and $\mathbf{Y}_h$ respectively and we let f and f_h denote the bijections of H and H_h into $\mathbf{X}$ and $\mathbf{X}_h$ ($\mathbf{Y} = f(H)$ and $\mathbf{Y}_h = f_h(H_h)$).

By r_{hi} we denote the L^2-orthogonal projection from $L^2(\Omega)$ onto H_{hi} $(i = 0, 1, 2)$ and we let $r_h v$, for any $v \in H$, denotes the element of H_h determined by $r_{h0} v \in H_{h0}$. We precise, $r_{h0} v \in H_{h0}$ and $f_h(r_h v) = (r_{h0} v,\, \partial_{h1}(r_{h0} v),\, \partial_{h2}(r_{h0} v)) \in \mathbf{Y}_h$. We call r_h the L^2-orthogonal projection on H_h. Finally, by p_h we denote the canonical injection of $\mathbf{X}_h$ into $\mathbf{X}$.

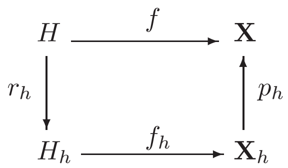


Figure 4: Structure of the discretization analysis

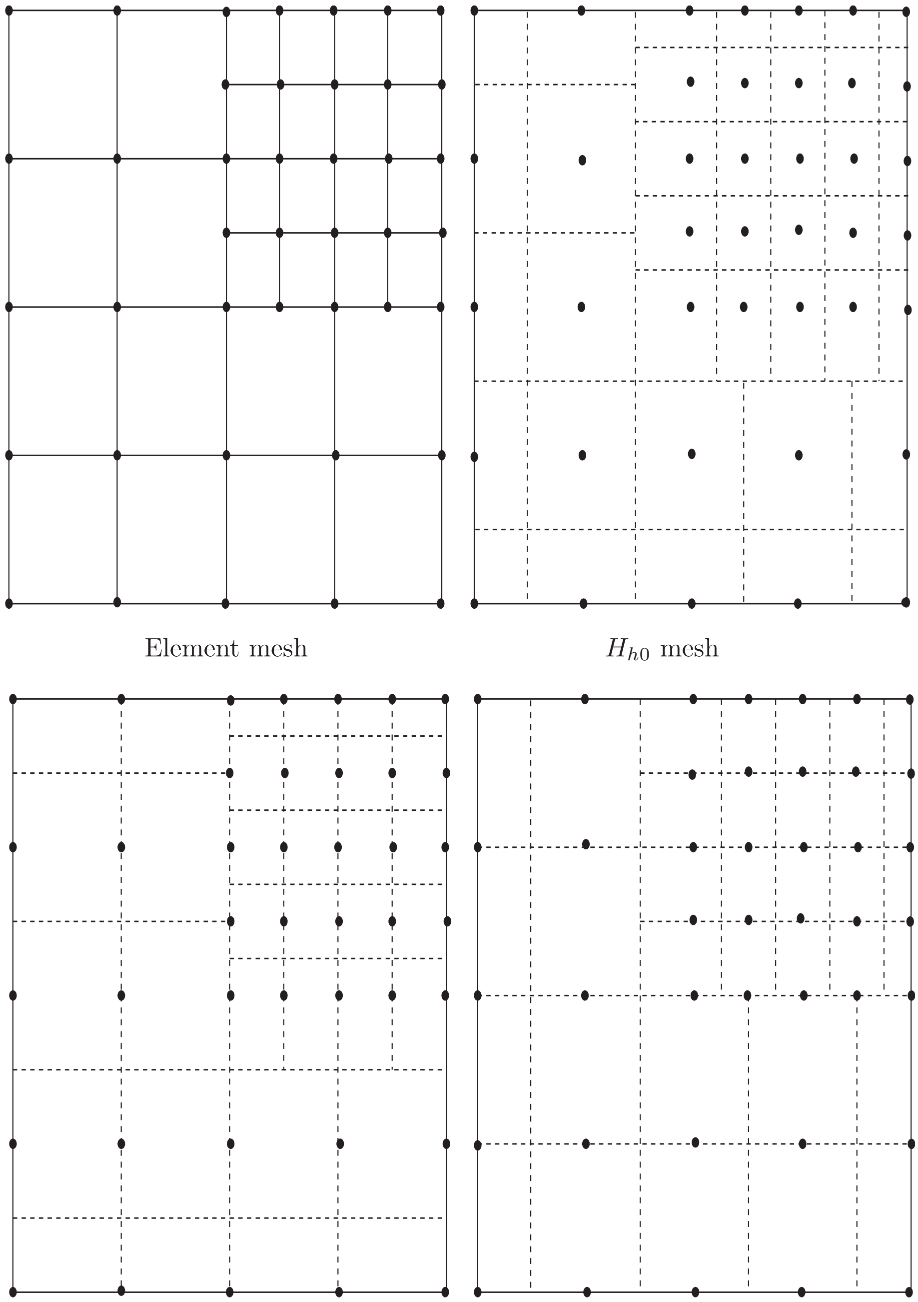
On Figure 4 we represent the structure of this discretization and the used operators. The motivation for using this space structure is that, while we cannot directly compare the elements of H and H_h, we can use the norm of $\mathbf{X}$ to measure the distance between elements $f(v) = (v,\, \partial_1 v,\, \partial_2 v)$ of $\mathbf{Y}$ and $f_h(v_h) = (v_h,\, \partial_{h1} v_h,\, \partial_{h2} v_h)$ of $\mathbf{Y}_h$.
Figure 5 (top–left) provides an example of a rectangular element mesh with a refined zone in the right upper corner. On the same figure we also represent the meshes H_{h0} and H_{hi} $(i = 1, 2)$ used to define the piecewise constant distributions v_h and $\partial_{hi} v_h$, $i = 1, 2$. Each of these meshes defines cells which are useful for distinct purposes. The elements are denoted by Ω_ℓ, $\ell \in J$ with boundaries $\partial\Omega_\ell$ and closures $V_\ell = \overline{\Omega}_\ell$. We similarly denote the cells of the other meshes by $\Omega_{\ell i}$, $\ell \in J_i$, $i = 0, 1, 2$ respectively, with boundaries $\partial\Omega_{\ell i}$ and closures $V_{\ell i} = \overline{\Omega}_{\ell i}$. The measures of these cells will be denoted by $|\,\Omega_\ell\,|$ and $|\,\Omega_{\ell i}\,|$, $i = 0, 1, 2$. It is of interest to note that each node of the mesh may be uniquely associated with a cell of H_{h0}; we shall therefore denote them by N_ℓ, $\ell \in J_0$.
Also we note that $\partial_{hi} v_h$ has the following property:

$$\int_P^Q \partial_{hi} v_h \, dx_i \;=\; v_h(Q) \,-\, v_h(P) \;\; (i = 1, 2) \tag{2.2}$$

for any pair of nodes $\{P,\; Q\}$ of the mesh.

At this stage, we do not introduce restrictions on the choice of the discrete mesh. It will be seen however in the convergence analysis that the grid lines of the element mesh should include all *material* discontinuities, i.e all lines of discontinuity of $p(x)$.

Before closing this section we note that triangular elements may also be introduced. In this way, the PCD method can accommodate any shape of the domain through the combined use of local mesh refinement and triangular elements, see Tahiri [15]. We note that the use of rectangular and triangular elements is not a restriction of the PCD discretization. Other elements and other forms of submeshes on such elements can be used, see Beauwens [6].



Element mesh H_{h0} mesh

H_{h1} mesh H_{h2} mesh

Figure 5: Discrete meshes with local mesh refinement

2.2. Discrete equations :

We now define the discrete problem to be solved in H_h by:

$$\text{find} \quad u_h \in H_h \quad \text{such that} \quad \forall\, v_h \in H_h \quad a_h(\,u_h,\, v_h\,) = (s,\, v_h)_\Omega \tag{2.3}$$

where

$$a_h(\,u_h\,,\, v_h\,) = (\,p(x)\,\partial_{h1}\,u_h\,,\ \partial_{h1}\,v_h\,)_\Omega \ +\ (\,p(x)\,\partial_{h2}\,u_h\,,\ \partial_{h2}\,v_h\,)_\Omega$$
$$+\ \ (\,q(x)\,u_h\,,\ v_h\,)_\Omega\ +\ (\,p(x)\,\omega(x)\,u_h\,,\ v_h\,)_{\Gamma_1}\,. \tag{2.4}$$

The discrete matrix is obtained as usual by introducing a basis $(\phi_i)_{\,i\in J_0}$ of the space H_h. Expanding the unknown u_h in this basis

$$u_h \ = \sum_{j\in J_0} \xi_j\ \phi_j$$

and expressing the variational condition (2.3) by:

$$a_h(\,u_h\,,\,\phi_i\,) \ = \sum_{j\in J_0} a_h(\,\phi_j\,,\,\phi_i\,)\ \xi_j \ = \ (\,s\,,\,\phi_i\,)_\Omega \quad \text{for all } i\in J_0$$

whence the linear system $\mathcal{A}\,\xi \ = \ b$ with stiffness matrix:
$\mathcal{A} = (\,a_{i\,j}\,) = (\,a_h(\,\phi_j\,,\,\phi_i\,)\,)$, right-hand side b with components
$b_i = (\,s\,,\,\phi_i\,)_\Omega\,,\ \ i\in J_0\,,$ and unknown vector ξ with components $\xi_j\ ,\ \ j\in J_0\,.$
The basis $(\phi_i)_{\,i\in J_0}$ of H_h is defined as usual through the conditions:

$$\phi_i \in H_h \quad \text{and} \quad \phi_i(N_j) = \delta_{i\,j}\,.$$

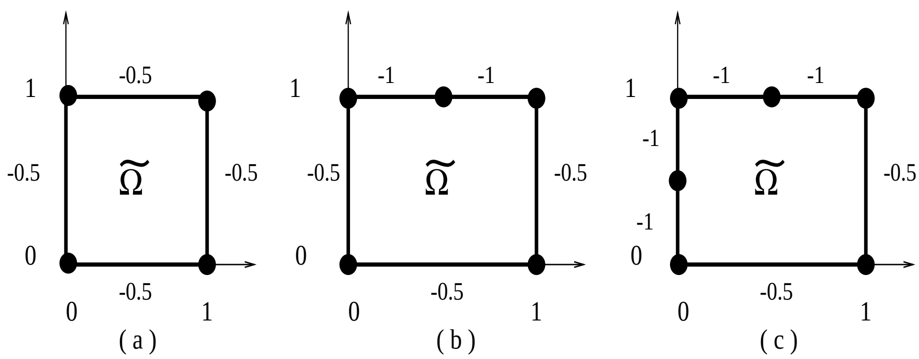


Figure 6: Element matrix graphs

The stiffness matrix is also built as usual by assembling element stiffness matrices. To this end a few reference elements $\widetilde{\Omega}$ may be used.

On Figure 6 we represent a reference elements $\widetilde{\Omega}$ for which it is readily seen that the element matrix graph is the element grid itself. On the same figure, we have indicated the offdiagonal element matrix entries $a_{ij}^{(\ell)}$ along the edges in the case where $p(x) = 1$. The diagonal entries follow from the formula:

$$a_{ii}^{(\ell)} = -\sum_{j \neq i} a_{ij}^{(\ell)} \; + \; \int_{\Omega} q(\tilde{x})\, \phi_i(\tilde{x})\, d\tilde{x} \ .$$

The values represented on Figure 6 are obtained by using equation (2.3) and by introducing the local basis of each reference element. We recall that this local basis is reduced to the characteristic function of each volume defined on the Figure 1 (a) , Figure 2 (a) and Figure 3 (a) respectively (in the case where $p(x) = 1$ and $q(x) = 0$).

3. Properties of the discrete space

3.1. Notation :

We investigate in this section general properties of the discrete space H_h and of the possible discrete representations of $v \in H$ in H_h. We sometimes need to assume that the discretization is *regular*. We hereby mean that there exist positive constants C_1 and C_2 independent of h such that:

$$C_1\, h \;\leq\; h_{\ell 1}\ ,\ h_{\ell 2} \;\leq\; C_2\, h \qquad \forall\, \ell \in J \tag{3.1}$$

$$C_1\, h^2 \;\leq\; |\,\Omega_{\ell\, 0}\,|,\ |\,\Omega_{\ell 1}\,|,\ |\,\Omega_{\ell 2}\,| \;\leq\; C_2\, h^2 \qquad \ell \in J_i\ ,\ i = 0,\, 1,\, 2 \tag{3.2}$$

By $x_\ell = (x_{\ell 1},\, x_{\ell 2}), \ell \in J_0$, we denote the grid points of the element mesh, by x_{ℓ_E} we denote the right neighbor of x_ℓ if it exists. By $x_{\ell +}$ (respectively $x_{\ell -}$) we denote the nearest neighbor of x_{ℓ_E} and both are located on the same vertical grid line ($x_{\ell +}$ the top neighbor of x_{ℓ_E} and $x_{\ell -}$ the bottom neighbor of x_{ℓ_E}).

We denote by J_R the subset of J containing the fine element subscripts (elements located in the refined zone). $J_{Ir}^i\,,\ (i = 1,\, 2)$ (Ir for irregular) denotes the subset of J containing the subscripts of the irregular element in x_i-direction ($i = 1,\, 2$). We denote by $(r_h v)_\ell$ the value of $r_h v$ on $\Omega_{\ell\, 0}$.
We split the domain Ω into two subdomains Ω_C (the coarse zone) and Ω_R (the refined zone) with $\Omega = \Omega_C \cup \Omega_R$.
Finally we denote by Ω_{Ir} the union of all irregular elements, $\Omega_{Ir} = \cup_\ell\, \Omega_\ell$ such that $\Omega_\ell \cap \Omega_R = \emptyset$ and $\overline{\Omega}_\ell \cap \partial\Omega_R \neq \emptyset\,,\ \ell \in (J \backslash J_R)$; Ω_{Ir} is a strip in Ω with a width $O(h)$ and has the interface boundary as part of its boundary.
The notation C is used throughout the paper to denote a generic positive constant independent of the mesh size.

3.2. Discrete Friedrichs inequalities :

The PCD discretization has the following properties which represent a discrete version of the first and the second Friedrichs inequalities and the trace inequality, for the proof we refer to Tahiri [15].

Lemma 3.1. *Let Ω be a bounded polygonal domain. We assume that $\Gamma_0 = \partial\Omega$. Then, there exists a constant $C > 0$, independent of h such that:*

$$\| v_h \|^2 \ \le \ C \ \left(\| \partial_{h1} v_h \|^2 \ + \ \| \partial_{h2} v_h \|^2 \right) \qquad \forall \ v_h \in H_h \ . \tag{3.3}$$

When Γ_0 has a positive measure and $\Gamma_0 \neq \partial\Omega$, one may prove the following lemmas.

Lemma 3.2. *Let Ω be a bounded polygonal domain. Then, there exists a constant $C > 0$, independent of h such that:*

$$\| v_h \|_h \ \le \ C \ \left(\| \partial_{h1} v_h \|^2 \ + \ \| \partial_{h2} v_h \|^2 \ + \ \| v_h \|_\Gamma^2 \right)^{1/2} \ \forall \ v_h \in H_h \ . \tag{3.4}$$

Lemma 3.3. *Let Ω be a bounded polygonal domain. Then, there exists a constant $C > 0$, independent of h such that:*

$$\int_\Gamma v_h(x)^2 \, ds \ = \ \| v_h \|_\Gamma^2 \ \le \ C \ \| v_h \|_h^2 \qquad \forall \ v_h \in H_h \ . \tag{3.5}$$

The results given in the previous lemmas are independent of the presence or not of the local mesh refinement. We note that, with the PCD discretization, for any pair of nodes of the mesh we can find a path connecting these nodes (succession of mesh grid segments). The proofs of the previous lemmas are based on this property and the property (2.2).

3.3. Discrete representation of $v \in H^1(\Omega)$:

Considering now discrete representation of elements of H, we first investigate L^2-orthogonal projections $r_h v \in H_h$ of $v \in H$.

Lemma 3.4. *Let Ω be a rectangular bounded open set in $\mathcal{R}^2$; $\forall v \in H^1(\Omega)$,*

$$\| v \ - \ r_{h0} v \| \ \le \ C \, h \, | \, v \, |_1 \quad \text{and} \ \lim_{h \to 0} \ \| \ f(v) \ - \ f_h(r_h v) \ \|_{\boldsymbol{X}} = 0 \tag{3.6}$$

Proof: For all $v \in H^1(\Omega)$, it known that $\| v \ - \ r_{h0} \, v \| \ \le \ C \, h \, | \, v \, |_1$ and $\| \partial_i v \ - \ r_{hi}(\partial_i v) \|$, $(i = 1, 2)$ converges to zero, see for example Brezzi and Fortin [7] and Douglas et al. [11].
We shall prove that $\| \partial_{h1}(r_h v) \ - \ r_{h1}(\partial_1 v) \|$ converges to zero. Since the space $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$ it is sufficient to prove this in $C^1(\overline{\Omega})$. Then we can write for all $v \in C^1(\overline{\Omega})$:
for all $\varepsilon \ > \ 0$ there exists a h_c such that for all $\ h \ \le \ h_c$:
$v(x) = v_\ell \ + \ C \, h \, \varepsilon$, $\forall x \in \overline{\Omega}_{\ell 0}$, $\forall \ell \in J_0$, where $v_\ell = v(x_\ell)$. Then,

$$(\, r_h v \,)_\ell = (\, r_h v \,)(x_\ell) = \frac{1}{|\, \Omega_{\ell 0} \,|} \int_{\Omega_{\ell 0}} v(x) \, dx \ = \ v_\ell \ + \ C \, h \, \varepsilon \tag{3.7}$$

Then,

$$\int_{\Omega_{\ell 1}} \partial_1 \, v(x) \, dx \ = \int_{\partial\Omega_{\ell 1}} v(x) \, n \ e_1 \, ds \ = \frac{|\, \Omega_{\ell 1} \,|}{h_{\ell 1}} \, (\, v_{\ell_E} \ - \ v_\ell \,) \ + \ C \, h^2 \, \varepsilon \tag{3.8}$$

where n is the unit outward normal vector on $\partial\Omega_{\ell 1}$ and e_1 is the unit vector in x_1 direction. Then, using (3.1), (3.2), (3.7) and (3.8) we get:

$$\int_{\Omega_{\ell 1}} \partial_1 v(x)\, dx \;=\; |\,\Omega_{\ell 1}\,|\; \frac{(r_h v)_{\ell_E} \;-\; (r_h v)_\ell}{h_{\ell 1}} \;+\; |\,\Omega_{\ell 1}\,|\; C\,\varepsilon$$

Then,

$$(\,r_{h1}(\partial_1 v)\,)_{\ell 1} = \frac{1}{|\,\Omega_{\ell 1}\,|} \int_{\Omega_{\ell 1}} \partial_1 v(x)\, dx \;=\; (\,\partial_{h1}(r_h v)\,)_{\ell 1} \;+\; C\,\varepsilon$$

Hence,

$$\|\,\partial_{h1}(r_h v) \;-\; r_{h1}(\partial_1 v)\,\|^2 = \sum_{J_1} |\Omega_{\ell 1}|\; \Big(\,(r_{h1}(\partial_1 v)\,)_{\ell 1} \;-\; (\partial_{h1}(r_h v)\,)_{\ell 1}\,\Big)^2$$

$$= \sum_{J_1} C\,\varepsilon\,|\Omega_{\ell 1}| \;=\; |\,\Omega\,|\; C\,\varepsilon$$

Thus, $\|\,\partial_{h1}(r_h v) \;-\; r_{h1}(\partial_1 v)\,\|$ converges to zero for $h \to 0$.
By the same argument $\|\,\partial_{h2}(r_h v) \;-\; r_{h2}(\partial_2 v)\,\|$ converges to zero too. Triangle inequality completes the proof. □

3.4. Discrete representation of $v \in H^1(\Omega) \cap C^0(\overline{\Omega})$:

We next consider the interpolant $v_I \in H_{h0}$ of $v \in H^1(\Omega) \cap C^0(\overline{\Omega})$, defined by:

$$v_I(\,x_\ell\,) = v(x_\ell) \qquad \text{for all nodes} \qquad x_\ell\,,\;\; \ell \in J_0 \tag{3.9}$$

We try to obtain similar results for v_I as those obtained for $r_{h0}v$ with $v \in H^1(\Omega)$.

Lemma 3.5. *Let Ω be a rectangular bounded open set in $\Re^2$;*
$\forall v \in H^1(\Omega) \cap C^0(\overline{\Omega})$,

$$\|\,v \;-\; v_I\,\| \;\leq\; C\,h\,|\,v\,|_1 \tag{3.10}$$

$$\lim_{h\to 0}\; \|\; f(v) \;-\; f_h(v_I)\;\|_{\boldsymbol{X}} = 0 \tag{3.11}$$

Proof: By a density argument it is sufficient to prove this in $C^1(\overline{\Omega})$.
For all $\ell \in J_0$ and for all $x \in \overline{\Omega}_{\ell 0}$, we have:

$$v(x) \;-\; v_I = v(x) \;-\; v(x_\ell) = \int_{S_1} \partial_1\, v\, dx_1 \;+\; \int_{S_2} \partial_2\, v\, dx_1$$

where S_1 is an horizontal segment, S_2 is a vertical segment and $S_1 \cup S_2$ is a path connecting x and x_ℓ. Then,

$$|v(x) - v_I| \leq \left(\int_{S_1} 1 dx_1\right)^{\frac{1}{2}} \left(\int_{S_1} |\partial_1 v|^2 dx_1\right)^{\frac{1}{2}} + \left(\int_{S_2} 1 dx_2\right)^{\frac{1}{2}} \left(\int_{S_2} |\partial_2 v|^2 dx_2\right)^{\frac{1}{2}}$$

$$\leq (h_{\ell 1})^{1/2} \left(\int_{S_1} |\partial_1 v|^2 dx_1 \right)^{1/2} + (h_{\ell 2})^{1/2} \left(\int_{S_2} |\partial_2 v|^2 dx_2 \right)^{1/2}$$

Then,

$$|v(x) - v_I|^2 \leq C h \left(\int_{S_1} |\partial_1 v|^2 dx_1 \right) + C h \left(\int_{S_2} |\partial_2 v|^2 dx_2 \right)$$

Integrating this inequality on the cell $\Omega_{\ell 0}$

$$\| v - v_I \|^2_{\Omega_{\ell 0}} \leq C h^2 \left(\| \partial_1 v \|^2_{\Omega_{\ell 0}} + \| \partial_2 v \|^2_{\Omega_{\ell 0}} \right) = C h^2 \, | v |^2_{1,\Omega_{\ell 0}}$$

Then, (3.10) follows immediately.

For the partial derivatives, we note that the closure $\overline{\Omega}_{\ell i} = V_{\ell i}$ $(i = 1, 2)$ are compact sets and that if $v \in C^1(V_{\ell i})$, then $(\partial_{hi} v_I)_{\ell i}$ converges uniformly to $\partial_i v$ in $V_{\ell i}$. Since uniform convergence implies L^2-convergence, this immediately shows that:

$$\lim_{h \to 0} \| \partial_i v - \partial_{hi} v_I \| = 0 \quad (i = 1, 2)$$

Hence, (3.11) follows for all $v \in H^1(\Omega) \cap C^0(\overline{\Omega})$. □

3.5. Case of higher regularity :

We now assume that:
$v \in H^2_L(\Omega) = H^1(\Omega) \cap C^0(\overline{\Omega}) \cap (\cup_j H^2(\Omega_j))$, where Ω_j is a subdomain of Ω (for example subdomains where $p(x)$ is continuous).
For all $v \in H^2_L(\Omega)$, the interpolant $v_I \in H_{h0}$ is defined by (3.9). For all $v \in H^2_L(\Omega)$, we have (see Brezzi and Fortin [7] and Douglas et al. [11].)

$$\| v - r_{h_0} v \| \leq C h \, | v |_1 \leq C h \left(\sum_j \| v \|^2_{2,\Omega_j} \right)^{1/2} \tag{3.12}$$

$$\| \partial_i v - r_{hi}(\partial_i v) \|^2 \leq Ch \sum_j | \partial_i v |^2_{1,\Omega_j} \leq Ch \sum_j \| v \|^2_{2,\Omega_j} \; (i = 1, 2) \tag{3.13}$$

$r_h v$ and $r_{hi}(\partial_i v)$ are defined by:

$$(r_h v)_\ell = \frac{1}{|\Omega_{\ell 0}|} \int_{\Omega_{\ell 0}} v(x) \, dx \quad , \quad \forall \ell \in J_0 \tag{3.14}$$

$$(r_{hi}(\partial_i v))_{\ell i} = \frac{1}{|\Omega_{\ell i}|} \int_{\Omega_{\ell i}} \partial_i v(x) \, dx \quad , \quad \forall \ell \in J_i \; (i = 1, 2) \tag{3.15}$$

Lemma 3.6. *Let Ω be a rectangular bounded open set of $\Re^2$, there exists a constant $C > 0$ (independent of h) such that: for all v in $H^2_L(\Omega)$*

$$\| f(v) - f_h(v_I) \|_{\boldsymbol{X}} \leq C h \left(\sum_j \| v \|^2_{2,\Omega_j} \right)^{\frac{1}{2}} \tag{3.16}$$

Proof: Using (3.10) we have:

$$\| v - v_I \| \leq C\, h \,|\, v\,|_1 \;\leq C\, h \left(\sum_j \| v \|^2_{2,\Omega_j} \right)^{\frac{1}{2}}$$

For the partial derivatives, by a density argument, it is sufficient to prove these in $C^2(\overline{\Omega}_{\ell i}) = C^2(V_{\ell i})\,,\ i = 1,\, 2.$
For all $\ell \in J_1$ and for all $v \in C^2(V_{\ell 1})\,,$ there exists $\theta \in]\,0,\,1\,[$ such that:

$$(\partial_{h1}\, v_I)_{\ell 1} = \frac{v_{\ell_E} - v_\ell}{h_{\ell 1}} = \partial_1\, v(x_\ell + \theta h_{\ell 1})$$

Furthermore, we have for all $x \in V_{\ell 1}$:

$$\partial_1 v(x) \,-\, \partial_1\, v(x_\ell + \theta h_{\ell 1}) = \int_{S_1} \partial_1\partial_1 v\, dx_1 \,+\, \int_{S_2} \partial_2\partial_1 v\, dx_2$$

where S_1 is an horizontal segment, S_2 is a vertical segment and $S_1 \cup S_2$ is a path connecting x and $(x_\ell + \theta h_{\ell 1})$. Then,

$$|\,\partial_1 v(x) \,-\, \partial_{h1}\, v_I\,|^2 \,\leq\, \left(\int_{S_1} |\,\partial_1\partial_1 v\,|\, dx_1 \,+\, \int_{S_2} |\,\partial_2\partial_1 v\,|\, dx_2 \right)^2$$

$$\leq \left(\int_{S_1} 1\, dx_1 \right) \left(\int_{S_1} |\,\partial_1\partial_1\, v\,|^2\, dx_1 \right) \,+\, \left(\int_{S_2} 1\, dx_2 \right) \left(\int_{S_2} |\,\partial_2\partial_1\, v\,|^2\, dx_2 \right)$$

Integrating this inequality on the cell $\Omega_{\ell 1}$

$$\|\,\partial_1 v(x) \,-\, \partial_{h1}\, v_I\,\|^2_{\Omega_{\ell 1}} \,\leq\, C\, h^2\,|\,v\,|^2_{2,\Omega_{\ell 1}} \,\leq\, C\, h^2\,|\,v\,|^2_{2,\Omega_j}$$

where Ω_j is the subdomain of Ω containing the cell $\Omega_{\ell 1}$. In the case where $V_{\ell 1}$ is a subset of $\Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are two subdomains of Ω, we consider the proof in $V_{\ell 1} \cap \overline{\Omega}_1$ and in $V_{\ell 1} \cap \overline{\Omega}_2$. We get:

$$\begin{aligned} \|\partial_1 v(x) - \partial_{h1} v_I\|^2_{\Omega_{\ell 1}} &= \|\partial_1 v(x) - \partial_{h1} v_I\|^2_{\Omega_{\ell 1}\cap\Omega_1} \,+\, \|\partial_1 v(x) - \partial_{h1} v_I\|^2_{\Omega_{\ell 1}\cap\Omega_2} \\ &\leq C\, h^2\,|\,v\,|^2_{2,\Omega_{\ell 1}\cap\Omega_1} \,+ C\, h^2\,|\,v\,|^2_{2,\Omega_{\ell 1}\cap\Omega_2} \end{aligned}$$

Then,

$$\|\,\partial_1 v(x) - \partial_{h1} v_I\,\|^2 = \sum_{J_1} \|\partial_1 v(x) - \partial_{h1} v_I\|^2_{\Omega_{\ell 1}} \,\leq\, C\, h^2 \left(\sum_j |\,v\,|^2_{2,\Omega_j} \right)$$

The same argument can be used for $\|\,\partial_2 v(x) - \partial_{h2} v_I\,\|$. Then (3.16) follows easily. For other proofs see Tahiri [15]. □

We note that the results presented in this section are independent of the presence or not of the local mesh refinement.

4. Convergence analysis

4.1. Convergence :

In this section we analyze the convergence of the solution u_h of the approximate problem (2.3) to the solution u of the continuous problem (1.1). The approximate bilinear form (2.4) is symmetric over $H_h \times H_h$. By using the Lemma 3.1 or the Lemmas 3.2 and 3.3, we prove the uniform coercivity of the approximate bilinear form. The coercivity of (2.4) implies that the stiffness matrix is positive definite. Moreover, it is also symmetric since (2.4) is a symmetric form.
The convergence is analyzed in two steps. First we give the error bounds induced by some specific approximation $u_d \in H_h$ of $u \in H$ ($u_d = r_h u$ or $u_d = u_I$). Such bounds have already been obtained for $r_h u$ (Lemma 3.4) and for u_I (Lemma 3.6). It remains to give an error bound for $\| u_d - u_h \|_h$ between u_d and the approximate solution u_h. Since H_h norm and a_h-norm are equivalent, we may equivalently try to bound:

$$\| u_d - u_h \|_{a_h} := \sup_{\substack{v_h \in H_h \\ v_h \neq 0}} \frac{| a_h(u_d, v_h) - a_h(u_h, v_h) |}{\|v_h\|_h} \tag{4.1}$$

$$= \sup_{\substack{v_h \in H_h \\ v_h \neq 0}} \frac{| a_h(u_d, v_h) - a(u, v_h) |}{\|v_h\|_h}$$

because, $\forall\, v_h \in H_h \quad a_h(u_h, v_h) = (s, v_h)_\Omega = a(u, v_h)$.
Since $s(x)$ is still defined on H_h and $s(x)$ is replaced by its value in (1.1). By introducing the following restrictions the expression $a(u, v_h)$ is well defined. It should be noticed that $\partial_1 v_h$ is reduced to Dirac distributions taken along the edges of the regions where v_h is constant (the vertical median line E_ℓ of the cell $\Omega_{\ell 1}$ in this case), weighted by the corresponding discontinuity of v_h. We required, in the definition of our BVP, that $p(x)$ be a bounded distribution, i.e. $p \in L^\infty(\Omega)$. The reason was that with $\partial_i u \in L^2(\Omega)$, we then also have $p(x)\, \partial_i u \in L^2(\Omega)$ and all operators are clearly defined. However, in the expression of the error, we now see that $\partial_1 v_h$ appear, with Dirac behaviors across specific lines and this is clearly incompatible with coefficients $p(x)$ that would be discontinuous across the same lines. To avoid such situations, we must introduce restrictions on the choice of the mesh, namely that material discontinuities (i.e. discontinuities of $p(x)$) should never match grid lines of the H_{h0} mesh. The best practical way to ensure this restriction is to require that material discontinuities be always grid lines of the element mesh.

Theorem 4.1. *Let Ω be a rectangular bounded open set of $\mathcal{R}^2$. Assume that the unique variational solution u of (1.1) belongs to $H^1(\Omega) \cap H^2(\Omega_{Ir})$.*
Then we have:

$$\lim_{h \to 0} \| u_h - r_h u \|_h = 0 \qquad \textit{and} \qquad \lim_{h \to 0} \| f(u) - f_h(u_h) \|_{\boldsymbol{X}} = 0$$

where u_h is the solution of the problem (2.3) with local mesh refinement and Ω_{Ir} is a strip in Ω.

Proof: We assume that $u \in H^1(\Omega)$, by density argument, the proof is only considered for $u \in C^1(\overline{\Omega})$. We have for all $v_h \in H_h$ and for $r_h u$:

$$a_h(r_h u, v_h) - a(u, v_h) = A_0 + A_1 + A_2$$

where

$$A_0 = (q(x) r_h u, v_h)_\Omega - (q(x) u, v_h)_\Omega$$
$$+ (p(x)\omega(x) r_h u, v_h)_{\Gamma_1} - (p(x)\omega(x) u, v_h)_{\Gamma_1} \quad (4.2)$$
$$A_1 = (p(x)\partial_{h1} r_h u, \partial_{h1} v_h)_\Omega - (p(x)\partial_1 u, \partial_1 v_h)_\Omega \quad (4.3)$$
$$A_2 = (p(x)\partial_{h2} r_h u, \partial_{h2} v_h)_\Omega - (p(x)\partial_2 u, \partial_2 v_h)_\Omega \quad (4.4)$$

$$|A_0| \leq \|q\|_\infty \|v_h\|_h \|u - r_h u\| + \|p\|_\infty \|\omega\|_\infty \|v_h\|_h \|u - r_h u\|_{\Gamma_1}$$

Using lemma 3.4, one may write:

$$\forall \varepsilon > 0, \exists h_c \quad \text{such that} \quad \forall h \leq h_c \quad \text{we have} : |A_0| \leq C\varepsilon \|v_h\|_h$$

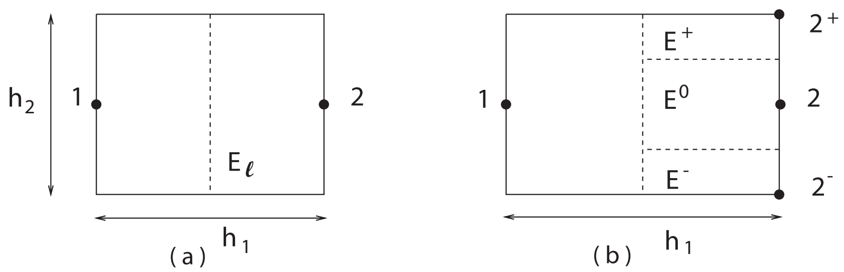


Figure 7: (a) Regular cell $\Omega_{\ell 1}$ (b) Irregular cell $\Omega_{\ell 1}$

The other terms A_i $(i = 1, 2)$ are most easily analyzed on the cells $\Omega_{\ell i}$ $(i = 1, 2)$ of the H_{hi} meshes $(i = 1, 2)$. Being similar in both cases, we consider $i = 1$. Suppose first that we have a regular case (Fig. 7 (a)). Then, the contribution of an arbitrary cell $\Omega_{\ell 1}$ is :

$$A_1^\ell = (p(x)\partial_{h1} r_h u, \partial_{h1} v_h)_{\Omega_{\ell 1}} - (p(x)\partial_1 u, \partial_1 v_h)_{\Omega_{\ell 1}}$$

Since,

$$(p(x)\partial_1 u, \partial_1 v_h)_{\Omega_{\ell 1}} = \int_{E_\ell} p(x)\partial_1 u (v_2 - v_1) ds$$

$$= \frac{1}{h_2}\int_{E_\ell} p(x)\partial_1 u\, ds\left(\frac{v_{h2} - v_{h1}}{h_1}\right) h_1 h_2$$

Then,

$$A_1^\ell = \left(\langle p(x) \rangle_{\Omega_{\ell 1}} \, \partial_{h1} \, r_h u \, \partial_{h1} \, v_h \; - \; \langle p(x) \, \partial_1 \, u \rangle_{E_\ell} \, \partial_{h1} \, v_h \right) \, h_1 \, h_2$$

$$= \left(\langle p(x) \rangle_{\Omega_{\ell 1}} \, \partial_{h1} \, r_h u \; - \; \langle p(x) \, \partial_1 \, u \rangle_{E_\ell} \right) \, \frac{v_{h2} \; - \; v_{h1}}{h_1} \; h_1 \, h_2$$

where $\langle \, \rangle_Q$ denotes average on Q, here the H_{h1} mesh cell $\Omega_{\ell 1}$ or its vertical median line E_ℓ, defined by:

$$\langle \, f(x) \, \rangle_Q \; = \; \frac{1}{|\, Q \,|} \int_Q f(x) \, dx$$

Note that,

$$\left| \, \langle p(x) \rangle_{\Omega_{\ell 1}} \, \partial_{h1} \, r_h u \; - \; \langle p(x) \, \partial_1 \, u \rangle_{E_\ell} \, \right| \; \leq \; C \, \left| \, \partial_{h1} \, r_h u \; - \; \langle \partial_1 \, u \rangle_{E_\ell} \, \right|$$

Taylor expansion and (3.7) give that: $\forall \, \varepsilon \, > \, 0 \, , \, \exists \, h_c$ such that $\forall \, h \; \leq \; h_c$

$$\int_{E_\ell} \partial_1 \, u \, ds \; = |\, E_\ell \,| \left(\frac{(r_h u)_2 \; - \; (r_h u)_1}{h_1} \right) + \; C \, h \, \varepsilon$$

Therefore,

$$\langle \partial_1 \, u \rangle_{E_\ell} \; = \; \frac{(r_h u)_2 \; - \; (r_h u)_1}{h_1} \; + \; C \, \varepsilon \; = (\, \partial_{h1} \, r_h u \,)_{\ell 1} \; + \; C \, \varepsilon$$

and then:

$$\left| \, \partial_{h1} \, r_h u \; - \; \langle \partial_1 \, u \rangle_{E_\ell} \, \right| \; \leq \; C \, \varepsilon$$

It follows that:

$$|\, A_1^\ell \,| \; \leq \; C \, \varepsilon \, \left| \, \frac{v_{h2} \; - \; v_{h1}}{h_1} \, \right| \, h_1 \, h_2 \; = \; C \, \varepsilon \, \left| \, (\, \partial_{h1} \, v_h \,)_{\ell 1} \, \right| \; |\, \Omega_{\ell 1} \,|$$

Now we consider an irregular case (Fig. 7 (b)), the contribution of this irregular cell $\Omega_{\ell 1}$ is:

$$A_1^\ell = (\, p(x) \, \partial_{h1} \, r_h u \, , \; \partial_{h1} \, v_h \,)_{\Omega_{\ell 1}} \; - \; (\, p(x) \, \partial_1 \, u \, , \; \partial_1 \, v_h \,)_{\Omega_{\ell 1}}$$

$$= \Bigg(\, \langle p(x) \rangle_{\Omega_{\ell 1}} \, \partial_{h1} \, r_h u \, \partial_{h1} \, v_h \; - \; \frac{1}{4} \, \langle p(x) \, \partial_1 \, u \rangle_{E^-} \left(\frac{v_{h2^-} \; - \; v_{h1}}{h_1} \right)$$

$$- \; \frac{1}{2} \, \langle p(x) \, \partial_1 \, u \rangle_{E^0} \left(\frac{v_{h2} \; - \; v_{h1}}{h_1} \right) - \; \frac{1}{4} \, \langle p(x) \, \partial_1 \, u \rangle_{E^+} \left(\frac{v_{h2^+} \; - \; v_{h1}}{h_1} \right) \Bigg) \, h_1 \, h_2$$

$$= \; \left(\, \langle p(x) \rangle_{\Omega_{\ell 1}} \, \partial_{h1} \, r_h u \; - \; \langle p(x) \, \partial_1 \, u \rangle_{E_\ell} \, \right) \; \left(\frac{v_{h2} \; - \; v_{h1}}{h_1} \right) \; h_1 \, h_2$$

$$- \left(\, \langle p(x) \, \partial_1 \, u \rangle_{E^+} \left(\frac{v_{h2^+} \; - \; v_{h2}}{h_1} \right) + \langle p(x) \, \partial_1 \, u \rangle_{E^-} \left(\frac{v_{h2^-} \; - \; v_{h2}}{h_1} \right) \, \right) \frac{h_1 \, h_2}{4}$$

Then,

$$| A_1^\ell | \leq C \left| \langle p(x) \rangle_{\Omega_{\ell 1}} \partial_{h1} r_h u - \langle p(x) \partial_1 u \rangle_{E_\ell} \right| \left| \frac{v_{h2} - v_{h1}}{h_1} \right| h_1 h_2$$

$$+ C \left| \langle \partial_1 u \rangle_{E^+} \right| \left| \frac{v_{h2^+} - v_{h2}}{h_1} \right| \frac{h_1 h_2}{4} + C \left| \langle \partial_1 u \rangle_{E^-} \right| \left| \frac{v_{h2^-} - v_{h2}}{h_1} \right| \frac{h_1 h_2}{4}$$

$$\leq C \varepsilon \left| (\partial_{h1} v_h)_{\ell 1} \right| | \Omega_{\ell 1} | + C \left| \int_{E^+} \partial_1 u \, ds \right| | v_{h2^+} - v_{h2} |$$

$$+ C \left| \int_{E^-} \partial_1 u \, ds \right| \left| v_{h2} - v_{h2^-} \right|$$

By $v_{\ell+}$ (respectively $v_{\ell-}$) we denote the value of v_h at the grid point $x_{\ell+}$ (respectively $x_{\ell-}$) and by $E_\ell = E^- \cup E^0 \cup E^+$ we denote the vertical median of the cell $\Omega_{\ell 1}$. We add together all terms A_1^ℓ , regular and irregular, and using (3.1) and (3.2) we get:

$$| A_1 | \leq C \varepsilon | \Omega |^{1/2} \| v_h \|_h + \sum_{\ell \in J^1_{Ir}} \left| \int_{E_\ell} \partial_1 u \, ds \right| | v_{\ell_E} - v_{\ell+} |$$

$$\leq C\varepsilon \|v_h\|_h + C \left(\sum_{\ell \in J^1_{Ir}} \left| \frac{v_{\ell_E} - v_{\ell+}}{h_{\ell 2}} \right|^2 |\Omega_{\ell 2+}| \right)^{1/2} \times \left(\sum_{\ell \in J^1_{Ir}} \left(\int_{E_\ell} \partial_1 u \, ds \right)^2 \right)^{1/2}$$

$$\leq C \varepsilon \| v_h \|_h + C \| v_h \|_h \left(\sum_{\ell \in J^1_{Ir}} \left(\int_{E_\ell} \partial_1 u \, ds \right)^2 \right)^{1/2}$$

Since we assume that u belongs to $H^2(\Omega_{Ir})$, we use the *Cauchy* $-$ *Schwarz* inequality and the trace inequality on the strip Ω_{Ir}. Then,

$$| A_1 | \leq C \varepsilon \| v_h \|_h + C \| v_h \|_h \left(\sum_{\ell \in J^1_{Ir}} |E_\ell| \, \| u \|^2_{1, E_\ell} \right)^{1/2}$$

$$\leq C \varepsilon \| v_h \|_h + C \| v_h \|_h \left(h \| u \|^2_{2, \Omega_{Ir}} \right)^{1/2}$$

Similarly, the same results are derived for A_2.
Finally we have $\forall \varepsilon > 0$, $\exists h_c$ such that $\forall h \leq h_c$

$$\| r_h u - u_h \|_h \leq C \| r_h u - u_h \|_{a_h} \leq \sup_{\substack{v_h \in H_h \\ v_h \neq 0}} \frac{| A_0 | + | A_1 | + | A_2 |}{\|v_h\|_h}$$

Hence,

$$\| r_h u - u_h \|_h \leq C \| r_h u - u_h \|_{a_h} \leq C \varepsilon$$

Triangle inequality completes the proof. □

It is of interest to consider here the case where $u_d = u_h$, the exact solution of the approximate problem, in the particular case where $q(x) = 0$. In this case indeed the error must be zero showing that u_h is such that:

$$\partial_{h1}\, u_h \;=\; \frac{1}{\langle\, p(x)\,\rangle_{\Omega_{\ell 1}}}\,\langle\, p(x)\, \partial_1\, u\,\rangle_{E_\ell}$$

Beyond giving us a relation between the exact and approximate solution, it shows that in this case, the method reduces to the control volume method under its corner mesh version. In more general cases, we may still consider our approach as an interpretation of this box method.
The presented method is not a control volume method and cannot in general be correctly framed in that way. However, in the case of a rectangular mesh (whether uniform or not) it can be recovered via the corner mesh version of the control volume method and it does conserve the mass in that case. This does not seem feasible in more general cases as for example along a refined zone.

Remark 4.1. *For an element with two refined sides, as on Fig. 3, the presented analysis is still valid. Indeed, the Dirac distribution $\partial_1 v_h$ is also weighted by the inner product of the unit outward normal vector on the boundary cell with the unit vector in x_1-direction.*

4.2. Convergence rate :

Under additional regularity for u the exact solution of (1.1), we derive a bound on the convergence rate. Now, we assume that u belongs to $H^2_L(\Omega)$. The nodal values of u are well defined for each point of Ω. We assign the unknowns of the approximate solution u_h to the nodes x_ℓ. We denote by u_ℓ the nodal value of u at the grid point x_ℓ.
Also here, the error analysis is done in two steps. The first one is $\|\, f(u) - f_h(u_I)\,\|_{\mathbf{X}}$ that is given in Lemma 3.6 and the second one is the error $\|\, u_I \,-\, u_h\,\|_h$. As previously we try to bound $\|\, u_I \,-\, u_h\,\|_{a_h}$ defined in (4.1) with $u_d = u_I$.

Theorem 4.2. *Let Ω be a rectangular bounded open set of $\mathcal{R}^2$. Assume that the unique variational solution u of (1.1) belongs to $H^2_L(\Omega)$, there exists a constant $C > 0$ (independent of h), such that:*

$$\|\, f(u) \,-\, f_h(u_h)\,\|_{\mathbf{X}} \,\leq\, C\, h \left(\sum_j \|\, u\,\|^2_{2,\,\Omega_j} \right)^{\frac{1}{2}}$$

where u_h is the solution of the problem (2.3) without local refinement.

Theorem 4.3. *Let Ω be a rectangular bounded open set of $\mathcal{R}^2$. Assume that the unique variational solution u of (1.1) belongs to $H^2_L(\Omega)$, there exists a constant*

$C > 0$ (independent of h), such that:

$$\|f(u) - f_h(u_h)\|_{\boldsymbol{X}} \leq Ch \left(\sum_j \|u\|^2_{2,\Omega_j} \right)^{\frac{1}{2}} + Ch^{1/2} \left(\sum_j \|u\|^2_{2,\Omega_j \cap \Omega_{Ir}} \right)^{\frac{1}{2}} \tag{4.5}$$

where u_h is the solution of the problem (2.3) with local mesh refinement and Ω_{Ir} is a strip in Ω.

Proof: The proofs of Theorems 4.2 and 4.3 are similar, we give only the proof of Theorem 4.3. We have for all $v_h \in H_h$ and for $u_d = u_I$:

$$a_h\,(\,u_I\,,\,v_h\,) \;-\; a\,(\,u\,,\,v_h\,) \;=\; A_0 \;+\; A_1 \;+\; A_2$$

where A_i $(i = 0,\,1,\,2)$ are defined in (4.2), (4.3) and (4.4) with u_I instead of $r_h u$. By Lemma 3.6 one may write:

$$|\,A_0\,| \;\leq\; C\;\|\,v_h\,\|_h\;\|\,u\;-\;u_I\,\| \;\leq\; C\,h\;\|\,v_h\,\|_h \left(\sum_j \|\,u\,\|^2_{2,\Omega_j} \right)^{1/2} \tag{4.6}$$

As previously, we consider the contribution of an arbitrary regular cell $\Omega_{\ell 1}$:

$$A_1^{\ell} \;=\; \left(\,\langle\, p(x)\,\rangle_{\Omega_{\ell 1}}\;\partial_{h1}\,u_I \;-\; \langle\, p(x)\,\partial_1\,u\,\rangle_{E_\ell}\,\right)\;\frac{v_{h2}\;-\;v_{h1}}{h_1}\;h_1\,h_2$$

Then,

$$|\,A_1^{\ell}\,| \;\leq\; C\;\left|\;\partial_{h1}\,u_I \;-\; \langle\,\partial_1\,u\,\rangle_{E_\ell}\,\right|\;\left|\;\frac{v_{h2}\;-\;v_{h1}}{h_1}\;\right|\;h_1\,h_2$$

Note that x_ℓ and x_{ℓ_E} are in $V_{\ell 1}$, and E_ℓ is a subset of $V_{\ell 1}$.
Taylor expansion gives on a closed domain K , for all $x\,,x_0 \in K$:

$$\begin{aligned} v(x) \;&=\; v(x_0) + \nabla v(x_0)\cdot(x - x_0) \\ &+\; \int_0^1 H(v)(tx + (1-t)x_0)(x - x_0)\cdot(x - x_0)\,(\,1\;-\;t\,)\,dt \end{aligned} \tag{4.7}$$

where $H(v)(x)$ denotes the $Hessian$ matrix of v at point x.
Using (4.7) for $(x = x_\ell\,,x_0 = x)$ and for $(x = x_{\ell_E}\,,x_0 = x)$, $\forall\,x \in\; E_\ell$, subtracting one from the other:

$$(u(x_{\ell_E})\;-\;u(x_\ell)) = \nabla u(x)\cdot(x_{\ell_E} - x_\ell)\;+\;\phi_{\ell_E}\;-\;\phi_\ell$$

where:

$$\phi_\ell = \int_0^1 H(v)(tx_\ell + (1-t)x)(x_\ell - x)\cdot(x_\ell - x)\,(\,1\;-\;t\,)\,dt\;;$$

$$\phi_{\ell_E} = \int_0^1 H(v)(tx_{\ell_E} + (1-t)x)(x_{\ell_E} - x) \cdot (x_{\ell_E} - x)\,(\,1\,-\,t\,)\,dt$$

Integrating over E_ℓ and since $\nabla u(x) \cdot (x_{\ell_E} - x_\ell) = (h_{\ell 1}\,\partial_1\,u(x)\,,\,0)$, we get:

$$h_1 \mid E_\ell \mid \left(\frac{(u_I)_2 \,-\, (u_I)_1}{h_1} \right) = \; h_1 \int_{E_\ell} \partial_1 u\, ds \;+\; \int_{E_\ell} \phi_{\ell_E}\, ds - \int_{E_\ell} \phi_\ell\, ds$$

Then,

$$\partial_{h1}\, u_I = \frac{(u_I)_2 \,-\, (u_I)_1}{h_1} = \langle\, \partial_1\, u\,\rangle_{E_\ell} + \frac{1}{h_1\, h_2} \int_{E_\ell} \phi_{\ell_E}\; ds \;\;-\;\; \frac{1}{h_1\, h_2} \int_{E_\ell} \phi_\ell\; ds$$

Hence,

$$\mid A_1^\ell \mid \;\leq\; C \; \left(\int_{E_\ell} \mid \phi_{\ell_E} \mid ds \;+\; \int_{E_\ell} \mid \phi_\ell \mid ds \right) \left| \; \frac{v_{h2} \,-\, v_{h1}}{h_1} \; \right|$$

Now, we give a bound for $\int_{E_\ell} |\phi_\ell|\; ds$ and $\int_{E_\ell} |\phi_{\ell_E}|\; ds$.

$$\int_{E_\ell} |\phi_\ell|\; ds \leq \int_{E_\ell} \int_0^1 |H(u)(tx_\ell + (1-t)x)(x_\ell - x) \cdot (x_\ell - x)\,(1-t)\,|\, ds\, dt$$

$$\leq\; C\; h^2 \int_{E_\ell} \int_0^1 |H(u)(tx_\ell + (1-t)x)\,(1-t)\,|\, ds\, dt$$

Using a change of variable $z = (z_1, z_2) = tx_\ell + (1-t)x$ for all $x \in E_\ell$, whence $dz_2 = (1-t)\, ds$.
For all $x \in E_\ell$, $z_1 = x_{\ell 1} + (1-t)\, h_1/2$, using an other change of variable $(1-t) = 2(z_1 - x_{\ell 1})/h_1$, therefore $dz_1 = -h_1\, dt/2$. Then,

$$\int_{E_\ell} |\phi_\ell|\; ds \leq\; C\; h^2 \int_{E_\ell^t} \int_0^1 |H(u)(z)|\, dz_2\, dt$$

where E_ℓ^t is the transformation of E_ℓ by the first change of variable. Since $E_\ell^t \subset E_\ell$ and using the second change of variable, we obtain:

$$\int_{E_\ell} |\phi_\ell|\; ds \leq\; C\; h^2 \int_{E_\ell} \int_{x_{\ell 1}}^{x_{\ell 1} + \frac{h_1}{2}} |H(u)(z)|\; \frac{2}{h_1}\; dz_1\, dz_2$$

We denote by $\Omega_{\ell 1}^1$ the left half part of $\Omega_{\ell 1}$, and using *Cauchy–Schwarz* inequality:

$$\int_{E_\ell} \mid \phi_\ell \mid\; ds \;\leq\; C\; h\; \|\, u\,\|_{2,\,\Omega_{\ell 1}^1} \left(\int_{E_\ell} \int_{x_{\ell 1}}^{x_{\ell 1} + \frac{h_1}{2}} \; dz \right)^{1/2}$$

$$\leq\; C\; h\; \|\, u\,\|_{2,\,\Omega_{\ell 1}^1}\; |\,\Omega_{\ell 1}^1\,|^{\,1/2} \;\leq\; C\; h\; \|\, u\,\|_{2,\,\Omega_{\ell 1}}\; |\Omega_{\ell 1}|^{\,1/2}$$

In similar way, we can get the same error bound for $\int_{E_\ell} |\phi_{\ell_E}|\, ds$.
Therefore,

$$| A_1^\ell | \leq C\, h \,\| u \|_{2,\Omega_{\ell 1}} \,|\Omega_{\ell 1}|^{1/2} \left| \frac{v_{h2} - v_{h1}}{h_1} \right| = C\, h \,\| u \|_{2,\Omega_{\ell 1}} \,\| \partial_{h1} v_h \|_{\Omega_{\ell 1}}$$

Now we consider an irregular case (Fig. 7 (b)), the contribution of this irregular cell $\Omega_{\ell 1}$ is:

$$A_1^\ell = (\,p(x)\,\partial_{h1}\,u_I\,,\ \partial_{h1}\,v_h\,)_{\Omega_{\ell 1}} \;-\; (\,p(x)\,\partial_1\,u\,,\ \partial_1\,v_h\,)_{\Omega_{\ell 1}}$$

$$= \Bigg(\langle\, p(x)\,\rangle_{\Omega_{\ell 1}}\ \partial_{h1}\,u_I\,\partial_{h1}\,v_h \;-\; \frac{1}{4}\ \langle\, p(x)\,\partial_1\,u\,\rangle_{E^-} \left(\frac{v_{h2^-} - v_{h1}}{h_1} \right)$$

$$- \frac{1}{2} \langle\, p(x)\,\partial_1\,u\,\rangle_{E^0} \left(\frac{v_{h2} - v_{h1}}{h_1} \right) - \frac{1}{4} \langle\, p(x)\,\partial_1\,u\,\rangle_{E^+} \left(\frac{v_{h2^+} - v_{h1}}{h_1} \right) \Bigg)\ h_1\,h_2$$

$$= \left(\langle\, p(x)\,\rangle_{\Omega_{\ell 1}}\ \partial_{h1}\,u_I \;-\; \langle\, p(x)\,\partial_1\,u\,\rangle_{E_\ell} \right) \left(\frac{v_{h2} - v_{h1}}{h_1} \right)\ h_1\,h_2$$

$$- \left(\langle\, p(x)\,\partial_1\,u\,\rangle_{E^+} \left(\frac{v_{h2^+} - v_{h2}}{h_1} \right) + \langle\, p(x)\,\partial_1\,u\,\rangle_{E^-} \left(\frac{v_{h2^-} - v_{h2}}{h_1} \right) \right) \frac{h_1\,h_2}{4}$$

Then,

$$| A_1^\ell | \;\leq\; C\ \left|\, \partial_{h1}\,u_I \;-\; \langle\, \partial_1\,u\,\rangle_{E_\ell} \right| \left| \frac{v_{h2} - v_{h1}}{h_1} \right|\ h_1\,h_2$$

$$+\,C\ \left| \langle\, \partial_1\,u\,\rangle_{E^+} \right| \left| \frac{v_{h2^+} - v_{h2}}{h_1} \right| \frac{h_1\,h_2}{4} \;+\; C\ \left| \langle\, \partial_1\,u\,\rangle_{E^-} \right| \left| \frac{v_{h2^-} - v_{h2}}{h_1} \right| \frac{h_1\,h_2}{4}$$

$$\leq\; C\ h\ \| u \|_{2,\,\Omega_{\ell 1}}\ \| \partial_{h1} v_h \|_{0,\,\Omega_{\ell 1}} \;+\; C\ \left| \int_{E^+} \partial_1\,u\,ds \right|\ \left| v_{h2^+} - v_{h2} \right|$$

$$+\,C\ \left| \int_{E^-} \partial_1\,u\,ds \right|\ \left| v_{h2} - v_{h2^-} \right|$$

In the case where E_ℓ is a subset of $\Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are two subdomains of Ω, we consider the proof in $E_\ell \cap \overline{\Omega}_1$ and in $E_\ell \cap \overline{\Omega}_2$.
We add together all terms A_1^ℓ, regular and irregular, and using (3.1) and (3.2) we get:

$$|A_1| \leq Ch \left(\sum_j \sum_{J_1} \|u\|_{2,\Omega_{\ell 1} \cap \Omega_j} \|\partial_{h1} v_h\|_{\Omega_{\ell 1}} \right) + \sum_{\ell \in J^1_{Ir}} \left| \int_{E_\ell} \partial_1\,u\,ds \right| |v_{\ell_E} - v_{\ell^+}|$$

$$\leq\; C\,h\ \| v_h \|_h\ \left(\sum_j \| u \|^2_{2,\,\Omega_j} \right)^{1/2}$$

$$+ C \left(\sum_{\ell \in J^1_{Ir}} \left| \frac{v_{\ell_E} - v_{\ell+}}{h_{\ell 2}} \right|^2 |\Omega_{\ell 2+}| \right)^{1/2} \left(\sum_{\ell \in J^1_{Ir}} \left(\int_{E_\ell} \partial_1 u \, ds \right)^2 \right)^{1/2}$$

$$\leq Ch \| v_h \|_h \left(\sum_j \| u \|^2_{2,\Omega_j} \right)^{1/2} + C \| v_h \|_h \left(\sum_{\ell \in J^1_{Ir}} \left(\int_{E_\ell} \partial_1 u \, ds \right)^2 \right)^{1/2}$$

Using the $Cauchy - Schwarz$ inequality and the trace inequality, we get:

$$| A_1 | \leq C h \| v_h \|_h \left(\sum_j \| u \|^2_{2,\Omega_j} \right)^{1/2} + C \| v_h \|_h \left(\sum_{\ell \in J^1_{Ir}} |E_\ell| \| u \|^2_{1,E_\ell} \right)^{1/2}$$

$$\leq C h \| v_h \|_h \left(\sum_j \| u \|^2_{2,\Omega_j} \right)^{1/2} + C \| v_h \|_h \left(h \sum_j \| u \|^2_{2,\Omega_j \cap \Omega_{Ir}} \right)^{1/2}$$

$$\leq C h \| v_h \|_h \left(\sum_j \| u \|^2_{2,\Omega_j} \right)^{1/2} + C h^{1/2} \| v_h \|_h \left(\sum_j \| u \|^2_{2,\Omega_j \cap \Omega_{Ir}} \right)^{1/2}$$

Similarly, one may write for A_2:

$$| A_2 | \leq Ch \| v_h \|_h \left(\sum_j \| u \|^2_{2,\Omega_j} \right)^{1/2} + C h^{1/2} \| v_h \|_h \left(\sum_j \| u \|^2_{2,\Omega_j \cap \Omega_{Ir}} \right)^{1/2}$$

Therefore,

$$\| u_I - u_h \|_h \leq C \| u_I - u_h \|_{a_h} \leq \sup_{\substack{v_h \in H_h \\ v_h \neq 0}} \frac{| A_0 | + | A_1 | + | A_2 |}{\|v_h\|_h}$$

Finally,

$$\| u_I - u_h \|_h \leq C \| u_I - u_h \|_{a_h}$$

$$\leq C h \left(\sum_j \| u \|^2_{2,\Omega_j} \right)^{1/2} + C h^{1/2} \left(\sum_j \| u \|^2_{2,\Omega_j \cap \Omega_{Ir}} \right)^{1/2}$$

Triangle inequality and Lemma 3.6 complete the proof. □

Remark 4.2. *The presented method is well adapted for multilevel local refinement. Successive local mesh refinement can be handled in similar way. The results established here are still valid.*

5. Concluding remarks

The first and main issue of the present work is the introduction of the PCD discretization method that relies on the systematic use of piecewise constant distributions to represent the unknown distribution as well as its derivatives, each one on a specific mesh. This method was only formulated for rectangular grids because the choice of the appropriate meshes is rather simple in that case and this choice led us to a new Ritz-Galerkin formulation of the corner mesh box scheme. Its extension to more general (irregular) meshes clearly raises geometrical difficulties but a priori no basic principle objection.

Staying still with rectangular meshes we have investigated the question of feasibility of introducing local mesh refinement without slave nodes, with the issue of getting the most compact possible discrete stencil. The main question to solve here was to prove the convergence of the resulting scheme.

Our conclusion is that, provided that the solution belongs to $H^1(\Omega) \cap H^2(\Omega_{Ir})$, there is always convergence in energy norm. Further, if the solution is locally H^2-regular, we then showed an $O(h^{1/2})$ convergence rate.

One can also use slave node techniques to improve the convergence at the expense of increased complexity, see Tahiri [15].
In our study of this method, we have used both theoretical and numerical investigations. The present work summarizes our theoretical findings. For the numerical results, we refer to Tahiri [15,17]. Our numerical results are in agreement with the theoretical results presented here. Furthermore, the numerical results show that the L^2-error $\| u - u_h \|$ has an $O(h)$-convergence rate independently of the presence or not of the local mesh refinement.

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References

1. R. A. Adams, Sobolev Spaces, Academic Press, New York, (1975).
2. J. P. Aubin, Approximation des espaces de distributions et ses opérateurs différentiels, Bull. Soc. Math. France, 12 (1967), 1-139.
3. J. P. Aubin, Behavior of the error of the approximate solution of boundary value problems for linear elliptic operators by Galerkin's and finite difference methods, Ann. Scuola N. Sup. Pisa, 21 (1967), 599-637.
4. J. P. Aubin, Approximation of Elliptic Boundary-Value Problems, Wiley-Interscience, New York, (1972).
5. R. E. Bank and D. J. Rose, Some error estimates for the box method, SIAM J. Num. Anal. 24 (1987), 777-787.

6. R. Beauwens, Forgivale variational crimes, Lecture Notes in Computer Science, 2542 (2003), 3-11.

7. F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, New York, (1991).

8. Z. Cai, J. Mandel and S. McCormick, The finite volume element method for diffusion equations on general triangulations, SIAM J. Numer. Anal. 28 (1991), 392-402.

9. Z. Cai and S. McCormick, On the accuracy of the finite volume element method for diffusion equations on composite grids, SIAM J. Numer. Anal. 27 (1990), 636-655.

10. J. Cea, Approximation variationnelle des problèmes aux limites, Ann. Inst. Fourier, 14 (1964), 345-444.

11. J. Douglas and J. E. Roberts, Global estimates for mixed methods for second order elliptic equations, Math. Comp. 44 (1985), 39-52.

12. R. E. Ewing, R. D. Lazarov and P. S. Vassilevski, Local refinement techniques for elliptic problems on cell-centred grid, I: Error analysis, Math. Comp. 56 (1991), 437-461.

13. T. Gallouët, R. Herbin and M. H. Vignal, Error estimates on the approximate finite volume solution of convection diffusion equations with general boundary conditions, SIAM J. Num. Anal. 37 (2000), 1935-1972.

14. V. Girault, Theory of a finite difference method on irregular networks, SIAM J. Num. Anal. 11 (1974), 260-282.

15. A. Tahiri, A compact discretization method for diffusion problems with local mesh refinement, PhD thesis, Service de Métrologie Nucléaire, ULB, Brussels, Belgium, September (2002).

16. A. Tahiri, The PCD method, Lecture Notes in Computer Science, 2542 (2003), 563-571.

17. A. Tahiri, Local mesh refinement with the PCD method, Adv. Dyn. Syst. Appl. 8(1) (2013), 124-136.

18. R. Temam, Analyse Numérique, Presses Univ. de France, Paris, (1970).

19. P. S. Vassilevski, S. I. Petrova and R. D. Lazarov, Finite difference schemes on triangular cell-centred grids with local refinement, SIAM J. Sci. Stat. Comput. 13 (1992), 1287-1313.

20. M. H. Vignal, Schémas volumes finis pour des équations elliptiques ou hyperboliques avec conditions aux limites, convergence et estimations d'erreur, Thèse de Doctorat, ENS de Lyon, France, (1997).

21. A. Weiser and M. F. Wheeler, On convergence of block-centered finite differences for elliptic problems, SIAM J. Num. Anal. 25 (1988), 351-375.

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