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Continuous Wavelet Transform on Local Fields *

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abstract: The main objective of this paper is to define the mother wavelet on local fields and study the continuous wavelet transform (CWT) and some of its basic properties. Its inversion formula, the Parseval relation and associated convolution are also studied.

Key Words: Continuous Wavelet Transform , Local field, Fourier transform.

Contents

1. Introduction

A local field means an algebraic field and a topological space with the topological properties of locally compact, non-discrete, complete and totally disconnected, denoted by $\mathbb{K}$ [8]. The additive and multiplicative groups of $\mathbb{K}$ are denoted by $\mathbb{K}^+$ and $\mathbb{K}^*$, respectively. We may choose a Haar measure dx for $\mathbb{K}^+$. If $\alpha \neq 0$ $(\alpha \in \mathbb{K})$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha|dx$ and call $|\alpha|$ the absolute value or valuation of α. Let $|0| = 0$. The absolute value has the following properties:
(i) $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$;
(ii) $|xy| = |x||y|$;
(iii) $|x + y| \leq max(|x|, |y|)$.
The last one of these properties is called the ultrametric inequality. The set $\mathfrak{D} = \{x \in \mathbb{K} : |x| \leq 1\}$ is called the ring of integers in $\mathbb{K}$. It is the unique maximal compact subring of $\mathbb{K}$. Define $\mathfrak{P} = \{x \in \mathbb{K} : |x| < 1\}$. The set $\mathfrak{P}$ is called the prime ideal in $\mathbb{K}$. The prime ideal in $\mathbb{K}$ is the unique maximal ideal in $\mathfrak{D}$. It is principal and prime.
If $\mathbb{K}$ is a local field, then there is a nontrivial, unitary, continuous character χ on $\mathbb{K}^+$ and $\mathbb{K}^*$ is self-dual.
χ is fixed character on $\mathbb{K}^+$ that is trivial on $\mathfrak{D}$ but is nontrivial on $\mathfrak{P}^{-1}$. It follows

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that χ is constant on cosets of $\mathfrak{D}$ and that if $y \in \mathfrak{P}^k$, then $\chi_y(\chi_y(x)) = \chi(xy)$ is constant on cosets of $\mathfrak{P}^{-k}$

Definition 1.1. *The Fourier transform of $f \in L^1(\mathbb{K})$ is denoted by $\hat{f}(\xi)$ and defined by [9]*

$$\hat{f}(\xi) = \int_{\mathbb{K}} f(x)\overline{\chi_\xi(x)}dx = \int_{\mathbb{K}} f(x)\chi(-\xi x)dx, \ \xi \in \mathbb{K}, \tag{1.1}$$

and the inverse Fourier transform by

$$f(x) = \int_{\mathbb{K}} \hat{f}(\xi)\chi_x(\xi)dx, \ x \in \mathbb{K}. \tag{1.2}$$

Some important properties of the Fourier transform can be proved easily :
(i) $||\hat{f}||_{L^\infty(\mathbb{K})} \leq ||f||_{L^1(\mathbb{K})}$.
(ii) If $f \in L^1(\mathbb{K})$, then $\hat{f}$ is uniformly continuous.
(iii) **Parseval formula**:If $f \in L^1(\mathbb{K}) \cap L^2(\mathbb{K})$, then $||\hat{f}||_{L^2(\mathbb{K})} = ||f||_{L^2(\mathbb{K})}$
(iv) If the convolution of f and g is defined as

$$(f * g)(t) = \int_{\mathbb{K}} f(x)g(t-x)dx, \tag{1.3}$$

then

$$F((f * g)) = F(f).F(g). \tag{1.4}$$

The article is divided in four sections. In section 2, we propose the definition of mother wavelet and define the continuous wavelet transform (CWT). In section 3, we discuss some basic properties of CWT. In section 4, we prove the Plancherel formula , inversion formula and also define the convolution associated with CWT.

2. Continuous Wavelet transform on local fields

Similar to $L^2(\mathbb{R})$ [1,3,5], we define the wavelet on local fields and define the continuous wavelet transform.

Definition 2.1. ***Admissible wavelet on local fields***
The function $\psi(x) \in L^2(\mathbb{K})$ is said to be an admissible wavelet on local fields if $\psi(x)$ satisfies the following admissibility condition:

$$c_\psi = \int_{\mathbb{K}} \frac{|\hat{\psi}(\xi)|^2}{|\xi|}d\xi < \infty, \tag{2.1}$$

where $\hat{\psi}$ is the Fourier transform of ψ.

Remark 2.2. *If $|\hat{\psi}(\xi)|$ is continuous near $\xi = 0$, then the existence of integral (2.1) guarantees that , $\hat{\psi}(0) = 0$. Since the Fourier transform of mother wavelet $\psi \in L^1(\mathbb{K}) \cap L^2(\mathbb{K})$ is bounded and uniformly continuous, we have*

$$\begin{aligned} 0 = \hat{\psi}(0) &= \int_{\mathbb{K}} \psi(x)\chi_0(x)dx \\ &= \int_{\mathbb{K}} \psi(x)dx. \end{aligned}$$

This means that the integral of mother wavelet is zero:

Theorem 2.3. *If ψ is a mother wavelet and $\phi \in L^1(\mathbb{K})$, then the convolution function $\psi * \phi$ is a mother wavelet.*

Proof: Since

$$\begin{aligned} \int_{\mathbb{K}} |(\psi * \phi)(x)|^2 dx &= \int_{\mathbb{K}} \Big| \int_{\mathbb{K}} \psi(x-y)\phi(y)dy \Big|^2 dx \\ &\leq \int_{\mathbb{K}} \left(\int_{\mathbb{K}} |\psi(x-y)||\phi(y)|^{\frac{1}{2}} ||\phi(y)|^{\frac{1}{2}} dy| \right)^2 dx \\ &\leq \int_{\mathbb{K}} \left(\int_{\mathbb{K}} |\psi(x-y)||\phi(y)|dy \int_{\mathbb{K}} |\phi(y)|dy \right) dx \\ &= \int_{\mathbb{K}} |\phi(y)|dy \int_{\mathbb{K}} \int_{\mathbb{K}} |\psi(x-y)|^2 |\phi(y)|dydx \\ &= \left(\int_{\mathbb{K}} |\phi(y)|dy \right)^2 \int_{\mathbb{K}} |\psi(x)|^2 dx \\ &= ||\phi||^2_{L^1(\mathbb{K})} ||\psi||^2_{L^2(\mathbb{K})}. \end{aligned}$$

Therefore $(\psi * \phi)(x) \in L^2(\mathbb{K})$. Moreover

$$\begin{aligned} c_{\psi * \phi} &= \int_{\mathbb{K}} \frac{|\psi \hat{*} \phi(\xi)|^2}{|\xi|} d\xi \\ &= \int_{\mathbb{K}} \frac{|\hat{\psi}(\xi)|^2 |\hat{\phi}(\xi)|^2}{|\xi|} d\xi \\ &\leq ||\hat{\phi}||^2_{L^\infty(\mathbb{K})} \int_{\mathbb{K}} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi. \end{aligned}$$

This completes the proof of the theorem □

Definition 2.4. ***Continuous wavelet transform (CWT) on local fields*** *For $\psi(x) \in L^2(\mathbb{K})$ and $a, b \in \mathbb{K}, a \neq 0$, we define the unitary linear operator:*

$$U_a^b : L^2(\mathbb{K}) \to L^2(\mathbb{K}),$$

by

$$U_a^b(\psi(x)) = \psi_{a,b}(x) = \frac{1}{|a|^{\frac{1}{2}}}\ \psi\left(\frac{x-b}{a}\right), \tag{2.2}$$

ψ is called mother wavelet and $\psi_{a,b}(x)$ are called daughter wavelets, where a is a dilation parameter, b is a translation parameter.
The Fourier transform of $\psi_{a,b}(x)$ is given by

$$\hat{\psi_{a,b}}(\xi) = |a|^{\frac{1}{2}}\ \hat{\psi}(a\xi)\chi_b(\xi), \tag{2.3}$$

where $\hat{\psi}$ is the Fourier transform of ψ.
The CWT on local fields

$$K_\psi : L^2(\mathbb{K}) \to L^2(\mathbb{K}\times\mathbb{K}),$$

of a function $f \in L^2(\mathbb{K})$ with respect to a mother wavelet ψ is defined by

$$\begin{aligned} f \mapsto K_\psi f(a,b) &= (f, \psi_{a,b})_{L^2(\mathbb{K})} \\ &= \int_{\mathbb{K}} f(x)\overline{\psi_{a,b}(x)}dx \\ &= \int_{\mathbb{K}} f(x)\frac{1}{|a|^{\frac{1}{2}}}\ \overline{\psi\left(\frac{x-b}{a}\right)}dx. \end{aligned} \tag{2.4}$$

3. Basic Properties of CWT on local fields

Before giving the fundamental properties of CWT, we list their basic properties.

Theorem 3.1. *Let ψ and φ be two wavelets and f, g be two functions belonging to $L^2(\mathbb{K})$, then*
(1) ***Linearity***

$$K_\psi(\eta f + \vartheta g)(a,b) = \eta K_\psi(f)(a,b) + \vartheta K_\psi(g)(a,b), \tag{3.1}$$

where η and ϑ are any two scalers.
*(ii)****Shift property***

$$(K_\psi f(x-\varsigma))(a,b) = (K_\psi f)(a, b-\varsigma), \tag{3.2}$$

where ς is any scaler.
(iii) ***Scaling property*** *If $\sigma \neq 0$ is any scaler, then the CWT of the scaled function $f_\sigma(x) = \frac{1}{\sigma}f(\frac{1}{\sigma})$ is*

$$(K_\psi f_\sigma)(a,b) = K_\psi(f)(\frac{a}{\sigma}, \frac{b}{\sigma}). \tag{3.3}$$

(iv) ***Symmetry***

$$(K_\psi f)(a,b) = \overline{(K_f(\psi))(\frac{1}{a}, -\frac{1}{b})}. \tag{3.4}$$

(iv) ***Parity***

$$(K_{P\psi}Pf)(a,b) = (K_\psi f)(a,-b), \tag{3.5}$$

where P is a parity operator define by $Pf(x) = f(-x)$.

Proof: The proof is the straight forward application of CWT. □

Theorem 3.2. *Show that the continuous wavelet transform can also expressed as*

$$(K_\psi f)(a,b) = \left(f * \frac{1}{\sqrt{a}}\overline{\psi}\left(\frac{x}{a}\right)\right)(b), \tag{3.6}$$

where the $*$ is defined as

$$(f * g)(t) = \int_{\mathbb{K}} f(x)g(t-x)dx. \tag{3.7}$$

Proof: From definition of CWT we have

$$\begin{aligned}(K_\psi f)(a,b) &= \int_{\mathbb{K}} f(x)\frac{1}{a^{\frac{1}{2}}}\ \overline{\psi\left(\frac{x-b}{a}\right)}dx \\ &= \left(f * \frac{1}{\sqrt{a}}\overline{\psi}\left(\frac{x}{a}\right)\right)(b). \end{aligned} \tag{3.8}$$

□

Theorem 3.3. *If f is homogeneous function of degree n, then*

$$(K_\psi f)(\lambda a, \lambda b) = \lambda^n |\lambda|^{\frac{1}{2}} (K_\psi f)(a,b), \tag{3.9}$$

where the λ is scaler .

Proof: From definition of CWT we have

$$\begin{aligned}(K_\psi f)(\lambda a, \lambda b) &= \int_{\mathbb{K}} f(x)\frac{1}{|\lambda a|^{\frac{1}{2}}}\ \overline{\psi\left(\frac{x-\lambda b}{\lambda a}\right)}dx \\ &= \int_{\mathbb{K}} f(\lambda x)\frac{1}{|a|^{\frac{1}{2}}}\ \overline{\psi\left(\frac{x-b}{a}\right)}|\lambda|dx \\ &= \lambda^n |\lambda|^{\frac{1}{2}} (K_\psi f)(a,b). \end{aligned} \tag{3.10}$$

□

4. Main Properties of the CWT

This section describes important properties of the CWT, such as the Plancherel , inversion formula and associated convolution fist, we establish the Plancherel theorem.

Theorem 4.1. *(**CWT Plancherel**) Let $f, g \in L^2(\mathbb{K})$. Then we have*

$$(\ (K_\psi f)(a,b), (K_\psi g)(a,b) \ )_{L^2(\mathbb{K}\times\mathbb{K})} = c_\psi (f,g)_{L^2(\mathbb{K})}, \tag{4.1}$$

where c_ψ is given in (2.1).

Proof: By using parseval formula for Fourier we can write the wavelet transform as

$$\begin{aligned} (K_\psi f)(a,b) &= \int_{\mathbb{K}} f(x) \frac{1}{|a|^{\frac{1}{2}}} \overline{\psi\left(\frac{x-b}{a}\right)} dx \\ &= (f, \psi_{a,b}) \\ &= (\hat{f}, \hat{\psi}_{a,b}) \\ &= \int_{\mathbb{K}} \hat{f}(\xi)|a|^{\frac{1}{2}} \ \overline{\hat{\psi}(a\xi)}\overline{\chi_b(\xi)} d\xi. \end{aligned} \tag{4.2}$$

Similarly

$$\overline{K_\psi(g)(a,b)} = \int_{\mathbb{K}} \overline{\hat{f}(\xi)}|a|^{\frac{1}{2}} \ \hat{\psi}(a\xi)\chi_b(\xi) d\xi. \tag{4.3}$$

Now, by using above (4.2) and (4.3) we get

$$\begin{aligned} \int_{\mathbb{K}}\int_{\mathbb{K}} K_\psi(f)(a,b)\overline{K_\psi(g)(a,b)} \frac{dadb}{|a|^2} &= \int_{\mathbb{K}}\int_{\mathbb{K}} |a| \frac{dadb}{|a|^2} \int_{\mathbb{K}} \hat{f}(\xi)\overline{\hat{\psi}(a\xi)\chi_b(\xi)} d\xi \\ &\quad \times \int_{\mathbb{K}} \overline{\hat{g}(\upsilon)}\hat{\psi}(a\upsilon)\chi_b(\upsilon) d\upsilon \\ &= \int_{\mathbb{K}}\int_{\mathbb{K}} \frac{dadb}{|a|} \overline{\int_{\mathbb{K}} \overline{\hat{f}(\xi)}\hat{\psi}(a\xi)\chi_b(\xi) d\xi} \\ &\quad \times \int_{\mathbb{K}} \overline{\hat{g}(\upsilon)}\hat{\psi}(a\upsilon)\chi_b(\upsilon) d\upsilon \\ &= \int_{\mathbb{K}}\int_{\mathbb{K}} \overline{F(\overline{\hat{f}(\xi)}\hat{\psi}(a\xi))(b)} F(\overline{\hat{g}(\upsilon)}\hat{\psi}(a\upsilon))(b) \frac{dadb}{|a|} \\ &= \int_{\mathbb{K}}\int_{\mathbb{K}} \hat{f}(\xi)\overline{\hat{\psi}(a\xi)}\overline{\hat{g}(\xi)}\hat{\psi}(a\xi) \frac{d\xi da}{|a|} \\ &= \int_{\mathbb{K}} \hat{f}(\xi)\overline{\hat{g}(\xi)} \left(\int_{\mathbb{K}} \overline{\hat{\psi}(a\xi)}\hat{\psi}(a\xi) \frac{da}{|a|} \right) d\xi \\ &= \int_{\mathbb{K}} \hat{f}(\xi)\overline{\hat{g}(\xi)} \left(\int_{\mathbb{K}} \frac{|\hat{\psi}(a\xi)|^2}{|a|} da \right) d\xi \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{K}} \hat{f}(\xi)\overline{\hat{g}(\xi)} \left(\int_{\mathbb{K}} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega \right) d\xi \\
&= c_\psi (\hat{f}, \hat{g})_{L^2(\mathbb{K})} \\
&= c_\psi (f, g)_{L^2(\mathbb{K})}. \qquad (4.4)
\end{aligned}$$

□

Theorem 4.2. ***(Inversion Formula)*** *Let $f \in L^2(\mathbb{K})$. Then we have*

$$f(x) = \frac{1}{c_\psi} \int_{\mathbb{K}} \int_{\mathbb{K}} K_\psi(f)(a,b)\psi_{a,b}(x) \frac{dadb}{|a|^2}, \qquad (4.5)$$

where c_ψ is given in (2.1).

Proof: Let $h(x) \in L^2(\mathbb{K})$ be any function, then by using above theorem, we have

$$\begin{aligned}
c_\psi (f, g)_{L^2(\mathbb{K})} &= \int_{\mathbb{K}} \int_{\mathbb{K}} (K_\psi f)(a,b)\overline{K_\psi(h)(a,b)} \frac{dadb}{|a|^2} \\
&= \int_{\mathbb{K}} \int_{\mathbb{K}} (K_\psi f)(a,b)\overline{\int_{\mathbb{K}} h(x)\overline{\psi_{a,b}(x)} dx} \frac{dadb}{|a|^2} \\
&= \int_{\mathbb{K}} \int_{\mathbb{K}} \int_{\mathbb{K}} (K_\psi f)(a,b)\psi_{a,b}(x)\overline{h(x)} \frac{dadbdx}{|a|^2} \\
&= \left(\int_{\mathbb{K}} \int_{\mathbb{K}} (K_\psi f)(a,b)\psi_{a,b}(x) \frac{dadb}{|a|^2}, h(x) \right).
\end{aligned}$$

Hence the result follows. □

If $f = h$,

$$||f||^2_{L^2(\mathbb{K})} = \int_{\mathbb{K}} \int_{\mathbb{K}} |(K_\psi f)(a,b)|^2 \frac{dadb}{|a|^2}. \qquad (4.6)$$

Moreover the wavelet transform is isometry from $L^2(\mathbb{K})$ to $L^2(\mathbb{K} \times \mathbb{K})$.

4.1. Associated convolution for CWT on local fields

Using Pathak and Pathak techniques [5], we define the basic function $D(x, y, z)$, translation τ_x and associated convolution # operator for CWT.
The basic function $D(x, y, z)$ for (2.4) is defined as

$$\begin{aligned}
K_\phi[D(x, y, z)](a, b) &= \int_{\mathbb{K}} D(x, y, z)\overline{\phi_{a,b}(t)} dt \\
&= \overline{\psi_{a,b}(z)}\ \overline{\chi_{a,b}(y)}, \qquad (4.7)
\end{aligned}$$

where ψ, ϕ and χ are three wavelets satisfying certain conditions (2.1).
Now, by using(4.5) we get,

$$D(x,y,z) = C_\phi^{-1} \int_{\mathbb{K}} \int_{\mathbb{K}} \overline{\psi_{a,b}(z)}\ \overline{\chi_{a,b}(y)}\ \phi_{a,b}(x)\, |a|^{-2} dadb. \tag{4.8}$$

The translation τ_x is defined as [5]

$$\begin{aligned} (\tau_x h)(y) &= h^*(x,y) = \int_{\mathbb{K}} D(x,y,z) h(z) dz \\ &= C_\phi^{-1} \int_{\mathbb{K}} \int_{\mathbb{K}} \int_{\mathbb{K}} \overline{\psi_{a,b}(z)}\ \overline{\chi_{a,b}(y)}\ \phi_{a,b}(x)\, h(z) |a|^{-2} dadbdz. \end{aligned}$$

The associated convolution is defined as

$$\begin{aligned} (h\#g)(x) &= \int_{\mathbb{K}} h^*(x,y) g(y) dy \\ &= \int_{\mathbb{K}} \int_{\mathbb{K}} D(x,y,z)\, h(z)\, g(y) dydz \\ &= C_\phi^{-1} \int_{\mathbb{K}} \int_{\mathbb{K}} \int_{\mathbb{K}} \int_{\mathbb{K}} \overline{\psi_{a,b}(z)}\ \overline{\chi_{a,b}(y)}\ \phi_{a,b}(x) h(z) g(y)\, |a|^{-2}\, dadbdzdy, \end{aligned} \tag{4.9}$$

by using the inversion formula we can write the above equation as

$$\begin{aligned} (h\#g)(x) &= C_\phi^{-1} \int_{\mathbb{K}} \int_{\mathbb{K}} (K_\psi h)(a,b)(K_\chi g)(a,b) \phi_{a,b}(x) |a|^{-2} dadb \\ &= K_\phi^{-1} \left[(K_\psi h)(a,b)(K_\chi g)(a,b) \right](x); \end{aligned}$$

so that

$$K_\phi[h\#g](a,b) = (K_\psi h)(a,b)(K_\chi g)(a,b)(x). \tag{4.10}$$

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References

1. Daubechies,I, *Ten Lectures on Wavelets* ,CBMS/NSF Ser. Appl. Math., vol. 61, SIAM, (1992).
2. L. Debnath, *Wavelet Transforms and Their Applications*, Birkhauser, Boston, (2002).
3. C.K. Chui, *An Introduction to Wavelets*, Academic Press, 1992.
4. M. Holschneider, *Wavelet analysis over Abelian groups*, Appl. Comput. Harmon. Anal. 2, 52-60, (1995).
5. R. S. Pathak and Ashish Pathak . *On convolution for wavelets transform* , International journal of wavelets, multiresolution and information processing , 6(5): 739-747, (2008).

6. Ashish Pathak, P Yadav, M M Dixit, *On convolution for general novel fractional wavelet transform* , Journal of Advanced Research in Scientific Computing, 7(1), 30-37, (2015).

7. D. Ramakrishnan and R. J. Valenza, *Fourier Analysis on Number Fields, Graduate Texts in Mathematics* , Springer-Verlag, New York, 1999.

8. H. Jiang, D. Li and N. Jin, *Multiresolution analysis on local fields*, J. Math. Anal. Appl., 294 ,523- 532 , (2004.)

9. M.H. Taibleson, *Fourier Analysis on Local Fields*, Princeton Univ. Press, 1975.

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