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Some Results and Characterizations for Mannheim Offsets of Ruled Surfaces

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abstract: In this study, we give dual characterizations for Mannheim offsets of ruled surfaces in terms of their integral invariants and obtain a new characterization for the Mannheim offsets of a developable surface, i.e., we show that the striction lines of developable Mannheim offset surfaces are Mannheim partner curves. Furthermore, we obtain the relationships between the area of projections of spherical images for Mannheim offsets of ruled surfaces and their integral invariants.

Key Words: Ruled surface; Mannheim offset; E. Study Mapping; integral invariants.

Contents

1. Introduction

Ruled surfaces are the surfaces which are generated by moving a straight line continuously in the space and these surfaces are one of the most important topics of differential geometry. A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line and is used in many areas of sciences such as Computer Aided Geometric Design (CAGD), mathematical physics, moving geometry, kinematics for modeling the problems and model-based manufacturing of mechanical products. Especially, the offsets of ruled surfaces have an important role in (CAGD). Some studies dealing with offsets of the surfaces have been given in ref. [1,7,9,10,11,12,13]. The well-known offsets of ruled surfaces are Bertrand offsets which were defined by Ravani and Ku by giving a generalization of the theory of Bertrand curves for trajectory ruled surfaces on line geometry [13]. Küçük and Gürsoy have studied integral invariants of Bertrand trajectory ruled surfaces in dual space and given relations between the invariants of offset surfaces [7].

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Furthermore, similar to the Bertrand curves, in [8] a new definition of special curve pairs has been given by Liu and Wang: Let C and C^* be two space curves. C is said to be a Mannheim partner curve of C^* if there exists a one to one correspondence between their points such that the binormal vector of C is the principal normal vector of C^*. Orbay, Kasap and Aydemir have given a generalization of the theory of Mannheim curves for ruled surfaces and called Mannheim offsets [9]. They have obtained some conditions characterizing developable Mannheim offset surfaces.

In this paper, we examine the Mannheim offsets of trajectory ruled surfaces in view of their integral invariants. Using dual representations of ruled surfaces, we give a result obtained in [9] in a short form and also we obtain some new results. Moreover, we show that if the Mannheim offsets of trajectory ruled surfaces are developable, then their striction lines are Mannheim partner curves. Furthermore, we give some characterizations for Mannheim offsets of trajectory ruled surfaces in terms of integral invariants of closed trajectory ruled surfaces. Finally, we obtain relationships between the area of projections of spherical images of Mannheim offsets of trajectory ruled surfaces and their integral invariants.

2. Differential Geometry of Ruled Surfaces in E^3

Let I be an open interval in the real line R, $k = k(s)$ be a regular curve in E^3 defined on I and $\vec{q} = \vec{q}(s)$ be a unit direction vector of an oriented line in E^3. Then we have the following parametrization for a ruled surface

$$\varphi_q(s, v) = \vec{k}(s) + v\,\vec{q}(s). \tag{2.1}$$

The parametric s-curve of this surface is a straight line of surface which is called ruling. For $v = 0$, the parametric v-curve of this surface is $\vec{k} = \vec{k}(s)$ which is called base curve or generating curve of the surface. In particular, if the direction of ruling $\vec{q}$ is constant, the ruled surface is said to be cylindrical and non-cylindrical otherwise [6].

The striction point on a ruled surface is the foot of common normal between two consecutive rulings. The set of the striction points constitute a curve $\vec{c} = \vec{c}(s)$ lying on the ruled surface and is called striction curve. The parametrization of the striction curve $\vec{c} = \vec{c}(s)$ on a ruled surface is given by

$$\vec{c}(s) = \vec{k}(s) - \frac{\left\langle d\vec{q},\ d\vec{k}\right\rangle}{\left\langle d\vec{q},\ d\vec{q}\right\rangle}\vec{q}(s). \tag{2.2}$$

So that, the base curve of ruled surface is its striction curve if and only if $\left\langle d\vec{q},\ d\vec{k}\right\rangle = 0$ [6].

The distribution parameter (or drall) of the ruled surface in (2.1) is given by

$$\delta_q = \frac{\left\langle d\vec{k},\ \vec{q} \times d\vec{q}\right\rangle}{\left\langle d\vec{q}, d\vec{q}\right\rangle}. \tag{2.3}$$

If $\delta_q = 0$, then the normal vectors of ruled surface are collinear at all points of same ruling and at the nonsingular points belonging to ruling, the tangent planes are identical. We then say that the tangent plane contacts the surface along a ruling. Such a ruling is called a torsal ruling. If $\delta_q \neq 0$, then the tangent planes are distinct at all points of the same ruling which is called nontorsal.

A ruled surface whose all rulings are torsal is called a developable ruled surface. The remaining ruled surfaces are called skew ruled surfaces. Thus, from (2.3) a ruled surface is developable if and only if at all its points the distribution parameter is zero, i.e., $\delta_q = 0$ [6,13].

Let $\left\{\vec{q}, \vec{h} = \frac{d\vec{q}/ds}{\|d\vec{q}/ds\|}, \vec{a} = \vec{q} \times \vec{h}\right\}$ be a moving othonormal trihedron making a spatial motion along a closed space curve $\vec{k}(s)$, $s \in \mathrm{R}$, in E^3. In this motion, an oriented line fixed in the moving system generates a closed ruled surface if the whole moving frame comes to its initial position and this surface is called closed trajectory ruled surface (CTRS) in E^3 [7]. A parametric equation of a closed trajectory ruled surface generated by $\vec{q}$-axis is

$$\varphi_q(s,v) = \vec{k}(s) + v\,\vec{q}(s), \quad \varphi(s+2\pi,\, v) = \varphi(s,v), \;\; s,v \in \mathrm{R}. \tag{2.4}$$

Consider the moving orthonormal system $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$. Then, the axes of the trihedron intersect at the striction point of $\vec{q}$-generator of φ_q-CTRS. The structural equations of this motion are

$$\begin{cases} d\vec{q} = k_1\vec{h} \\ d\vec{h} = -k_1\vec{q} + k_2\vec{a} \\ d\vec{a} = -k_2\vec{h} \end{cases} \tag{2.5}$$

and

$$\frac{d\vec{c}}{ds} = \cos\sigma\vec{q} + \sin\sigma\vec{a}, \tag{2.6}$$

where $\vec{c} = \vec{c}(s)$ is the striction line of φ_q-CTRS and the differential forms k_1, k_2 and σ are the natural curvature, the natural torsion and the striction of φ_q-CTRS, respectively [6,7]. Here, the striction is restricted as $-\pi/2 < \sigma < \pi/2$ for the orientation on φ_q-CTRS and s is arc-length of striction line.

The pole vector and the Steiner vector of the motion are given by

$$\vec{p} = \frac{\vec{\psi}}{\left\|\vec{\psi}\right\|}, \quad \vec{d} = \oint \vec{\psi}, \tag{2.7}$$

respectively, where $\vec{\psi} = k_2\vec{q} + k_1\vec{a}$ is the instantaneous Pfaffian vector of the motion.

The pitch of φ_q-CTRS is defined by

$$\ell_q = \oint dv = -\oint \left\langle d\vec{k}, \vec{q}\right\rangle, \tag{2.8}$$

and the angle of pitch of φ_q-CTRS is given as follows

$$\lambda_q = -\oint \left\langle d\vec{h}, \vec{a} \right\rangle = -\left\langle \vec{q}, \vec{d} \right\rangle = 2\pi - a_q = \oint g_q, \tag{2.9}$$

where a_q is the measure of spherical surface area bounded by spherical image of φ_q-CTRS and g_q is the geodesic curvature of this image. The pitch and the angle of pitch are well-known real integral invariants of closed trajectory ruled surface [2,3,4,5].

The area vector of a closed space curve x in E^3 is given by

$$\vec{v}_x = \oint \vec{x} \times d\vec{x} \tag{2.10}$$

and the area of projection of a closed space curve x in direction of the generator of a CTRS -$y(s,v)$ is

$$2f_{x,y} = \langle \vec{v}_x, \vec{y} \rangle\,. \tag{2.11}$$

(See [4]).

3. Dual Numbers and Dual Vectors

In this section, we give a brief summary of theory of dual numbers and dual vectors. For more details, one can see references [2,5,14].

Dual numbers had been introduced by W.K. Clifford (1845-1879). A dual number has the form $\bar{a} = (a, a^*) = a + \varepsilon a^*$ where a and a^* are real numbers and $\varepsilon = (0,1)$ is dual unit. The product of dual numbers $\bar{a} = (a, a^*) = a + \varepsilon a^*$ and $\bar{b} = (b, b^*) = b + \varepsilon b^*$ is defined by

$$\bar{a}\bar{b} = (a, a^*)(b, b^*) = (ab, ab^* + a^*b) = ab + \varepsilon(ab^* + a^*b). \tag{3.1}$$

Then it is seen that $\varepsilon^2 = 0$ while $\varepsilon \neq 0$. We denote the set of dual numbers by D and write

$$D = \left\{ \bar{a} = a + \varepsilon a^* : \quad a, a^* \in \mathrm{R},\ \varepsilon^2 = 0 \right\}. \tag{3.2}$$

Clifford showed that dual numbers form a ring, but not a field. The pure dual numbers εa^* are zero divisors, i.e., $(\varepsilon a^*)(\varepsilon b^*) = 0$ while $(\varepsilon a^*) \neq 0$, $(\varepsilon b^*) \neq 0$. However, the other laws of algebra of dual numbers are same as the laws of algebra of complex numbers.

Now let f be a differentiable function with dual variable $\bar{x} = x + \varepsilon x^*$. Then the Maclaurine series generated by f is given by

$$f(\bar{x}) = f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x), \tag{3.3}$$

where $f'(x)$ is derivative of $f(x)$.

Let D^3 be the set of all triples of dual numbers, i.e. ,

$$D^3 = \left\{ \tilde{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) : \quad \bar{a}_i \in D, \quad i = 1, 2, 3 \right\}. \tag{3.4}$$

Then the set D^3 is called dual space. The elements of D^3 are called dual vectors. A dual vector $\tilde{a}$ may be expressed in the form $\tilde{a} = \vec{a} + \varepsilon\vec{a}^* = (\vec{a}, \vec{a}^*)$, where $\vec{a}$ and $\vec{a}^*$ are vectors of R^3. Then for any vectors $\tilde{a} = \vec{a} + \varepsilon\vec{a}^*$ and $\tilde{b} = \vec{b} + \varepsilon\vec{b}^*$ in D^3, the scalar product and the vector product are defined by

$$\left\langle \tilde{a}, \tilde{b} \right\rangle = \left\langle \vec{a}, \vec{b} \right\rangle + \varepsilon \left(\left\langle \vec{a}, \vec{b}^* \right\rangle + \left\langle \vec{a}^*, \vec{b} \right\rangle \right), \tag{3.5}$$

and

$$\tilde{a} \times \tilde{b} = \vec{a} \times \vec{b} + \varepsilon \left(\vec{a} \times \vec{b}^* + \vec{a}^* \times \vec{b} \right), \tag{3.6}$$

respectively, where $\left\langle \vec{a}, \vec{b} \right\rangle$ and $\vec{a} \times \vec{b}$ are the inner product and the vector product in R^3.

The norm of a dual vector $\tilde{a}$ is defined by

$$\|\tilde{a}\| = \sqrt{\langle \tilde{a}, \tilde{a} \rangle} = \|\vec{a}\| + \varepsilon \frac{\langle \vec{a}, \vec{a}^* \rangle}{\|\vec{a}\|}. \tag{3.7}$$

A dual vector $\tilde{a}$ with norm 1 is called dual unit vector. The set of dual unit vectors is

$$\tilde{S}^2 = \left\{ \tilde{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) \in D^3 : \ \langle \tilde{a}, \ \tilde{a} \rangle = 1 + \varepsilon 0 \right\}, \tag{3.8}$$

which is called dual unit sphere.

E. Study used dual numbers and dual vectors in his research on the geometry of lines and kinematics. He devoted a special attention to the representation of directed lines by dual unit vectors and defined the mapping that is known by his name: There exists a one-to-one correspondence between vectors (points) of dual unit sphere $\tilde{S}^2$ and directed lines of space R^3. By the aid of this correspondence, the properties of the spatial motion of a line can be derived. Hence, the geometry of ruled surfaces is represented by the geometry of dual curves on the dual unit sphere in D^3. If the ruled surface is closed then the corresponding dual curve can be closed.

The angle $\bar{\theta} = \theta + \varepsilon\theta^*$ between two dual unit vectors $\tilde{a}, \ \tilde{b}$ is called dual angle and defined by

$$\left\langle \tilde{a}, \tilde{b} \right\rangle = \cos\bar{\theta} = \cos\theta - \varepsilon\theta^* \sin\theta. \tag{3.9}$$

By considering The E. Study Mapping, the geometric interpretation of dual angle is that θ is real angle between lines $L_1, \ L_2$ corresponding to dual unit vectors $\tilde{a}, \ \tilde{b}$, respectively, and θ^* is the shortest distance between those lines.

Let now K be a moving dual unit sphere generated by a dual orthonormal system

$$\left\{ \tilde{q}, \ \tilde{h} = \frac{d\tilde{q}}{\|d\tilde{q}\|}, \ \tilde{a} = \tilde{q} \times \tilde{h} \right\}, \ \tilde{q} = \vec{q} + \varepsilon\vec{q}^*, \ \tilde{h} = \vec{h} + \varepsilon\vec{h}^*, \ \tilde{a} = \vec{a} + \varepsilon\vec{a}^*, \tag{3.10}$$

and K' be a fixed dual unit sphere with the same center. Then, the derivative equations of dual spherical closed motion of K with respect to K' are

$$\begin{cases} d\tilde{q} = \bar{k}_1\tilde{h} \\ d\tilde{h} = -\bar{k}_1\tilde{q} + \bar{k}_2\tilde{a} \\ d\tilde{a} = -\bar{k}_2\tilde{h} \end{cases} \tag{3.11}$$

where $\bar{k}_1(s) = k_1(s) + \varepsilon k_1^*(s)$, $\bar{k}_2(s) = k_2(s) + \varepsilon k_2^*(s)$, $(s \in \mathrm{R})$ are dual curvature and dual torsion, respectively. From the E. Study mapping, during the spherical motion of K with respect to K', the dual unit vector $\tilde{q}$ draws a dual curve on dual unit sphere K' and this curve represents a ruled surface with ruling $\vec{q}$ in line space R^3.

Dual vector $\tilde{\psi} = \vec{\psi} + \varepsilon\vec{\psi}^* = \bar{k}_2\tilde{q} + \bar{k}_1\tilde{a}$ is called the instantaneous Pfaffian vector of motion and the vector $\tilde{P}$ given by $\tilde{\psi} = \left\|\tilde{\psi}\right\| \tilde{P}$ is called dual pole vector of motion. Then the vector

$$\tilde{d} = \oint \tilde{\psi} \tag{3.12}$$

is called dual Steiner vector of closed motion [5].

By considering the E. Study mapping, the dual equations (3.11) correspond to real equations (2.5) and (2.6) of a closed spatial motion in R^3. So, the differentiable dual closed curve $\tilde{q} = \tilde{q}(s)$ corresponds to a closed trajectory ruled surface in line space and denoted by φ_q-CTRS.

A dual integral invariant of a φ_q-CTRS can be given in terms of real integral invariants as follows and is called dual angle of pitch of a φ_q-CTRS

$$\bar{\wedge}_q = -\oint \left\langle d\tilde{h}, \tilde{a} \right\rangle = -\left\langle \tilde{q}, \tilde{d} \right\rangle = 2\pi - \bar{a}_q = \oint \bar{g}_q = \lambda_q - \varepsilon\ell_q \tag{3.13}$$

where $\tilde{d} = \vec{d} + \varepsilon\vec{d}^*$, $\bar{a}_q = a_q + \varepsilon a_q^*$ and $\bar{g}_q = g_q + \varepsilon g_q^*$ are the dual Steiner vector of motion, the measure of dual spherical surface area and the dual geodesic curvature of spherical image of φ_q-CTRS, respectively.

4. Mannheim Offsets of Trajectory Ruled Surfaces

In this section, by considering dual representations of ruled surfaces we give definition and characterizations of Mannheim offsets of trajectory ruled surfaces. First, we give the following definition.

Definition 4.1. *Let φ_q and φ_{q_1} be two trajectory ruled surfaces generated by dual vectors $\tilde{q}$ and $\tilde{q}_1$ of the dual orthonormal frames $\left\{\tilde{q}(s),\ \tilde{h}(s),\ \tilde{a}(s)\right\}$ and $\left\{\tilde{q}_1(s_1),\ \tilde{h}_1(s_1),\ \tilde{a}_1(s_1)\right\}$, respectively. Then φ_q and φ_{q_1} are called Mannheim offsets of trajectory ruled surfaces, if*

$$\tilde{a} = \tilde{h}_1 \tag{4.1}$$

holds at the corresponding points of the striction curves of surfaces, where s and s_1 are arc-lengths of the striction lines of φ_q and φ_{q_1}, respectively.

By this definition, the relation between the trihedrons

$$\left\{ \tilde{q},\ \tilde{h} = \frac{d\tilde{q}}{\|d\tilde{q}\|},\ \tilde{a} = \tilde{q} \times \tilde{h} \right\} \tag{4.2}$$

and

$$\left\{ \tilde{q}_1,\ \tilde{h}_1 = \frac{d\tilde{q}_1}{\|d\tilde{q}_1\|},\ \tilde{a}_1 = \tilde{q}_1 \times \tilde{h}_1 \right\} \tag{4.3}$$

of trajectory ruled surfaces φ_q and φ_{q_1} can be given as follows

$$\begin{pmatrix} \tilde{q}_1 \\ \tilde{h}_1 \\ \tilde{a}_1 \end{pmatrix} = \begin{pmatrix} \cos\bar{\theta} & \sin\bar{\theta} & 0 \\ 0 & 0 & 1 \\ \sin\bar{\theta} & -\cos\bar{\theta} & 0 \end{pmatrix} \begin{pmatrix} \tilde{q} \\ \tilde{h} \\ \tilde{a} \end{pmatrix} \tag{4.4}$$

where $\bar{\theta} = \theta + \varepsilon\theta^*$, $(0 \leq \theta \leq \pi,\ \ \theta^* \in \mathrm{R})$ is dual angle between dual generators $\tilde{q}$ and $\tilde{q}_1$ of Mannheim trajectory ruled surfaces φ_q and φ_{q_1}. The angle θ is called offset angle and the real number θ^* is called offset distance. Then, $\bar{\theta} = \theta + \varepsilon\theta^*$ is called dual offset angle of the Mannheim trajectory ruled surfaces φ_q and φ_{q_1}. If $\theta = 0$ and $\theta = \pi/2$, then the Mannheim offsets are said to be oriented offsets and right offsets, respectively. Thus, we can give the followings.

Theorem 4.2. *Let φ_q and φ_{q_1} be the Mannheim trajectory ruled surfaces. The offset angle and offset distance are given by*

$$\theta = -\int k_1 ds + c, \quad \theta^* = -\int k_1^* ds + c^*, \tag{4.5}$$

respectively where k_1 and k_1^* are real and dual parts of dual curvature $\bar{k}_1$ of φ_q and c, c^* are real constants.

Proof: From (4.4) we have

$$\tilde{q}_1 = \cos\bar{\theta}\tilde{q} + \sin\bar{\theta}\tilde{h}. \tag{4.6}$$

Differentiating (4.6) and by using (3.11) and (4.4) we may write

$$\frac{d\tilde{q}_1}{ds} = -\left(\frac{d\bar{\theta}}{ds} + \bar{k}_1\right)\tilde{a}_1 + \bar{k}_2 \sin\bar{\theta}\tilde{h}_1. \tag{4.7}$$

Since $\frac{d\tilde{q}_1}{ds}$ is linearly dependent with $\tilde{h}_1$, from (4.7) we get

$$\bar{\theta} = -\int \bar{k}_1 ds + \bar{c}, \tag{4.8}$$

where $\bar{c} = c + \varepsilon c^*$ is a dual constant. Separating (4.8) into real and dual parts we have equalities in (4.5). □

Theorem 4.3. *The closed trajectory ruled surfaces φ_q and φ_{q_1} form a Mannheim offset with constant dual offset angle $\bar{\theta}$ if and only if the following relationship holds*

$$\bar{\lambda}_{q_1} = \bar{\lambda}_q \cos\bar{\theta} + \bar{\lambda}_h \sin\bar{\theta}. \tag{4.9}$$

Proof: Let the closed trajectory ruled surfaces φ_q and φ_{q_1} form a Mannheim offset with constant dual offset angle $\bar{\theta}$. Then from (3.13) and (4.4), we have

$$\begin{aligned}\bar{\lambda}_{q_1} &= -\oint \left\langle d\tilde{h}_1, \tilde{a}_1 \right\rangle \\ &= -\oint \left\langle d\tilde{a}, \quad (\sin\bar{\theta})\tilde{q} - (\cos\bar{\theta})\tilde{h} \right\rangle \\ &= -\oint \langle d\tilde{a}, \tilde{q}\rangle \sin\bar{\theta} + \oint \left\langle d\tilde{a}, \tilde{h} \right\rangle \cos\bar{\theta}\end{aligned}$$

Since $\bar{\lambda}_q = \oint \left\langle d\tilde{a}, \tilde{h} \right\rangle$, $\bar{\lambda}_h = -\oint \langle d\tilde{a}, \tilde{q}\rangle$, from last equality the dual angle of pitch of φ_{q_1}-CTRS is obtained as follows

$$\bar{\lambda}_{q_1} = \bar{\lambda}_q \cos\bar{\theta} + \bar{\lambda}_h \sin\bar{\theta}.$$

Conversely, if (4.9) holds, it is easily seen that φ_q and φ_{q_1}-CTRS form a Mannheim offset with constant dual offset angle.

Equality (4.9) is a dual characterization for Mannheim offsets of CTRS with constant dual offset angle in terms of their dual integral invariants. Separating (4.9) into the real and dual parts, we obtain

$$\begin{cases} \lambda_{q_1} = \lambda_q \cos\theta + \lambda_h \sin\theta, \\ \ell_{q_1} = (\ell_q - \theta^* \lambda_h)\cos\theta + (\ell_h + \theta^* \lambda_q)\sin\theta, \end{cases} \tag{4.10}$$

respectively. Then, we may give the following corollaries.

□

Corollary 4.4. *If φ_q and φ_{q_1} are oriented closed Mannheim trajectory ruled surfaces, i.e., $\theta = 0$, then the relationships between real integral invariants of φ_q and φ_{q_1}-CTRS are given by,*

$$\lambda_{q_1} = \lambda_q, \quad \ell_{q_1} = \ell_q - \theta^* \lambda_h. \tag{4.11}$$

Furthermore, from (3.13) the measure of spherical surface areas bounded by spherical images of φ_q and φ_{q_1}-CTRS Mannheim offsets are the same, i.e.,

$$a_{q_1} = a_q \ \text{and} \ a^*_{q_1} = -a^*_q + \theta^*(2\pi - a_h). \tag{4.12}$$

Corollary 4.5. *If φ_q and φ_{q_1} are right closed Mannheim trajectory ruled surfaces, i.e., $\theta = \pi/2$, then the relationships between real integral invariants of φ_q and φ_{q_1}-CTRS are given as follows*

$$\lambda_{q_1} = \lambda_h, \quad \ell_{q_1} = \ell_h + \theta^* \lambda_q. \tag{4.13}$$

Then, the measure of spherical surface areas bounded by spherical images of φ_{q_1} and φ_h-CTRS are the same, i.e.,

$$a_{q_1} = a_h \text{ and } a^*_{q_1} = -(a^*_h + \theta^*(2\pi - a_q)). \tag{4.14}$$

Corollary 4.6. *If $\theta^* = 0$, i.e., the generators $\vec{q}$ and $\vec{q}_1$ of the Mannheim offset surfaces intersect, then we have*

$$\begin{cases} \lambda_{q_1} = \lambda_q \cos\theta + \lambda_h \sin\theta \\ \ell_{q_1} = \ell_q \cos\theta + \ell_h \sin\theta \end{cases} \tag{4.15}$$

In this case, φ_q and φ_{q_1}-CTRS intersect along their striction lines. It means that their striction lines are the same.

Let now consider that what the condition for developable Mannheim offset of a CTRS is. Let φ_q and φ_{q_1}-CTRS be the Mannheim offset surfaces and let $\vec{\alpha}(s)$ and $\vec{\beta}(s_1)$ be striction lines of φ_q and φ_{q_1}-CTRS, respectively. Then, we can write

$$\vec{\beta}(s) = \vec{\alpha}(s) + \theta^* \vec{a}(s), \tag{4.16}$$

where s is the arc-length of $\vec{\alpha}(s)$. Assume that φ_q-CTRS is developable. Then from (2.3) and (2.6) we have

$$\delta_q = \frac{\left\langle \cos\sigma\vec{q} + \sin\sigma\vec{a},\ \ \vec{q}\times k_1\vec{h} \right\rangle}{\left\langle k_1\vec{h}, k_1\vec{h} \right\rangle} = \frac{\sin\sigma}{k_1} = 0, \tag{4.17}$$

which gives that $\sigma = 0$. Thus, from (2.6) we have

$$\frac{d\vec{\alpha}}{ds} = \vec{q}. \tag{4.18}$$

Hence, along the striction line $\vec{\alpha}(s)$, the real orthogonal frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ coincides with the Frenet frame $\left\{\vec{T}, \vec{N}, \vec{B}\right\}$ of $\alpha(s)$ and the differential forms k_1 and k_2 turn into the curvature κ_α and torsion τ_α of the striction line $\alpha(s)$, respectively. Then from (2.5), (4.16) and (4.18) we have

$$\frac{d\vec{\beta}}{ds} = \vec{q} - \theta^* \tau_\alpha \vec{h}. \tag{4.19}$$

On the other hand from (2.5) and (4.4) we obtain

$$\frac{d\vec{q}_1}{ds} = -\left(\frac{d\theta}{ds} + \kappa_\alpha\right)\sin\theta\vec{q} + \left(\frac{d\theta}{ds} + \kappa_\alpha\right)\cos\theta\vec{h} + \tau_\alpha \sin\theta\vec{a}. \tag{4.20}$$

By using (4.5) and the fact that $k_1 = \kappa_\alpha$, from (4.20) it follows

$$\frac{d\vec{q}_1}{ds} = \tau_\alpha \sin\theta\vec{a}. \tag{4.21}$$

Then from (4.19) and (4.21) we have

$$\begin{aligned}
\delta_{q_1} &= \frac{\left\langle d\vec{\beta},\ \vec{q}_1 \times d\vec{q}_1 \right\rangle}{\langle d\vec{q}_1, d\vec{q}_1 \rangle} \\
&= \frac{\left\langle \vec{q} - (\theta^* \tau_\alpha)\vec{h},\quad (\tau_\alpha \sin\theta)\vec{q}_1 \times \vec{a} \right\rangle}{\langle d\vec{q}_1, d\vec{q}_1 \rangle} \\
&= \frac{\tau_\alpha \sin\theta \left\langle \vec{q} - (\theta^* \tau_\alpha)\vec{h},\quad \vec{a}_1 \right\rangle}{(\tau_\alpha \sin\theta)^2} \\
&= \frac{\left\langle \vec{q} - (\theta^* \tau_\alpha)\vec{h},\quad (\sin\theta)\vec{q} - (\cos\theta)\vec{h} \right\rangle}{\tau_\alpha \sin\theta}
\end{aligned}$$

$$\delta_{q_1} = \frac{\sin\theta + \theta^* \tau_\alpha \cos\theta}{\tau_\alpha \sin\theta}. \tag{4.22}$$

Thus, from (4.18) and (4.22) it can be stated that if the Mannheim offsets of φ_q and φ_{q_1} ruled surfaces are developable then the following relationship holds

$$\sin\theta + \theta^* \tau_\alpha \cos\theta = 0. \tag{4.23}$$

Equality (4.23) has been also found in [9] with a different way.

If (4.23) holds, along the striction line $\beta(s_1)$, the real orthogonal frame $\left\{\vec{q}_1, \vec{h}_1, \vec{a}_1\right\}$ coincides with the Frenet frame $\left\{\vec{T}_1, \vec{N}_1, \vec{B}_1\right\}$. Thus, the following theorem may be given.

Theorem 4.7. *If φ_q and φ_{q_1} are developable Mannheim offset surfaces then their striction lines are Mannheim partner curves.*

From (4.23) we can also give the following special cases by assuming $\tau_\alpha \neq 0$:

Corollary 4.8. $\theta = 0$, *i.e., the Mannheim offsets of developable trajectory ruled surfaces φ_q and φ_{q_1} are oriented.*

$\Leftrightarrow$ *Their generators are coincident, i.e., $\theta^* = 0$.*

$\Leftrightarrow$ *The Mannheim offsets of developable trajectory ruled surfaces φ_q and φ_{q_1} are coincident.*

Corollary 4.9. $\theta = \pi/4 \Leftrightarrow$ *there is a relationship between the torsion of $\alpha(s)$ and offset distance as follows*

$$\tau_\alpha \theta^* = -1 \tag{4.24}$$

If φ_q-CTRS is developable then from the equations (2.8), (4.4) and (4.19) the pitch ℓ_{q_1} of φ_{q_1}-CTRS is

$$\begin{aligned}
\ell_{q_1} &= -\oint \left\langle d\vec{\beta},\ \vec{q}_1 \right\rangle \\
&= -\oint \left\langle \vec{q} - (\theta^* \tau_\alpha)\vec{h},\quad (\cos\theta)\vec{q} + (\sin\theta)\vec{h} \right\rangle ds \\
&= -\oint (\cos\theta - \theta^* \tau_\alpha \sin\theta)\, ds
\end{aligned}$$

Then we can give the following corollary:

Corollary 4.10. *If φ_q-CTRS is developable then the relation between the pitch ℓ_{q_1} of φ_{q_1}-CTRS and the torsion of striction line $\alpha(s)$ of φ_q-CTRS is given by*

$$\ell_{q_1} = -\oint (\cos\theta - \theta^* \tau_\alpha \cos\theta) ds. \tag{4.25}$$

Let now consider the area of projections of Mannheim offsets. The dual area vectors of the spherical images of φ_q and φ_{q_1} surfaces are

$$\begin{cases} \tilde{w}_q = \tilde{d} + \bar{\Lambda}_q \tilde{q} \\ \tilde{w}_{q_1} = \tilde{d} + \bar{\Lambda}_{q_1} \tilde{q}_1 \end{cases} \tag{4.26}$$

respectively [4]. Then, the dual area of projection of spherical image of φ_{q_1}-CTRS in the direction $\tilde{q}$, generators of φ_q-offsets, is

$$\begin{aligned} 2\bar{f}_{\tilde{q}_1,\tilde{q}} &= \langle \tilde{w}_{q_1},\ \tilde{q} \rangle = \left\langle \tilde{d} + \bar{\Lambda}_{q_1} \tilde{q}_1,\ \tilde{q} \right\rangle \\ &= \left\langle \tilde{d},\ \tilde{q} \right\rangle + \bar{\Lambda}_{q_1} \cos\bar{\theta} \end{aligned}$$

$$2\bar{f}_{\tilde{q}_1,\tilde{q}} = -\bar{\Lambda}_q + \bar{\Lambda}_{q_1} \cos\bar{\theta}. \tag{4.27}$$

Separating (4.27) into real and dual parts we have the following theorem

Theorem 4.11. *The relationships between the area of projections of spherical images of the Mannheim offsets and their integral invariants are given as follows*

$$\begin{cases} 2f_{q_1,q} = -\lambda_q + \lambda_{q_1} \cos\theta, \\ 2f^*_{q_1,q} = \ell_q - \ell_{q_1} \cos\theta - \lambda_{q_1} \theta^* \sin\theta. \end{cases} \tag{4.28}$$

Corollary 4.12. *If φ_q and φ_{q_1}-CTRS are the oriented surfaces, i.e., $\theta = 0$, then from (4.28) we have*

$$2f_{q_1,q} = -\lambda_q + \lambda_{q_1}, \quad 2f^*_{q_1,q} = \ell_q - \ell_{q_1}. \tag{4.29}$$

Corollary 4.13. *If φ_q and φ_{q_1}-CTRS are the right Mannheim offsets, i.e., $\theta = \pi/2$, then from (4.28) we have*

$$2f_{q_1,q} = -\lambda_q, \quad 2f^*_{q_1,q} = \ell_q - \lambda_{q_1} \theta^*. \tag{4.30}$$

Similarly, from (4.26) the dual area of projection of spherical image of φ_{q_1}-CTRS in direction $\tilde{h}$ is

$$\begin{aligned} 2\bar{f}_{\tilde{q}_1,\tilde{h}} &= \left\langle \tilde{w}_{q_1}, \tilde{h} \right\rangle = \left\langle \tilde{d} + \bar{\Lambda}_{q_1} \tilde{q}_1, \tilde{h} \right\rangle \\ &= \left\langle \tilde{d}, \tilde{h} \right\rangle + \bar{\Lambda}_{q_1} \sin\bar{\theta} \end{aligned}$$

$$2\bar{f}_{q_1,h} = -\bar{\Lambda}_h + \bar{\Lambda}_{q_1} \sin\bar{\theta}. \tag{4.31}$$

Separating (4.31) into real and dual parts we have the followings:

Corollary 4.14. *If φ_q and φ_{q_1}-CTRS are Mannheim offsets then we have*

$$\begin{cases} 2f_{q_1,h} = -\lambda_h + \lambda_{q_1}\sin\theta, \\ 2f^*_{q_1,h} = \ell_h - \ell_{q_1}\sin\theta + \lambda_{q_1}\theta^*\cos\theta. \end{cases} \tag{4.32}$$

Corollary 4.15. *If φ_q and φ_{q_1}-CTRS are the oriented Mannheim surfaces, i.e., $\theta = 0$, then from (4.32) we have*

$$2f_{q_1,h} = -\lambda_h, \quad 2f^*_{q_1,h} = \ell_h + \lambda_{q_1}\theta^*. \tag{4.33}$$

Corollary 4.16. *If φ_q and φ_{q_1}-CTRS are right Mannheim offsets, i.e., $\theta = \pi/2$, then from (4.30) we have*

$$2f_{q_1,h} = -\lambda_h + \lambda_{q_1}, \quad 2f^*_{q_1,h} = \ell_h - \ell_{q_1}. \tag{4.34}$$

Similarly, the dual area of projection of spherical image of φ_{q_1}-CTRS in direction $\tilde{a}$ is

$$\begin{aligned} 2\bar{f}_{\tilde{q}_1,\tilde{a}} &= \langle \tilde{w}_{q_1}, \tilde{a}\rangle = \left\langle \tilde{d} + \bar{\lambda}_{q_1}\tilde{q}_1, \ \tilde{a}\right\rangle \\ &= \left\langle \tilde{d}, \tilde{h}_1\right\rangle \end{aligned}$$

$$2\bar{f}_{q_1,a} = -\bar{\lambda}_a = -\bar{\lambda}_{h_1}. \tag{4.35}$$

Separating (4.35) into real and dual parts we have the following corollary:

Corollary 4.17. *If φ_q and φ_{q_1}-CTRS are Mannheim offsets then we have*

$$f_{q_1,a} = -\lambda_{h_1} = -\lambda_a, \quad f^*_{q_1,a} = \ell_{h_1} = \ell_a. \tag{4.36}$$

Example 4.18. *Let consider the hyperboloid of one sheet plotted in Fig. 1, given by the parametrization*

$$\varphi(s,v) = (\cos s, \ \sin s, \ 0) + v\,(-\sin s, \ \cos s, \ 1). \tag{4.37}$$

From E. Study Mapping, this surface corresponds to following dual curve,

$$\tilde{q}(s) = \frac{1}{\sqrt{2}}(-\sin s, \ \cos s, \ 1) + \varepsilon\frac{1}{\sqrt{2}}(\sin s, \ -\cos s, \ 1). \tag{4.38}$$

From (4.38) dual curvature of the surface is $\bar{k}_1 = \frac{\sqrt{2}}{2}(1-\varepsilon)$. Then from Theorem 4.2, offset angle and offset distance are obtained as $\theta(s) = -\frac{\sqrt{2}}{2}s + c_1$, $\theta^*(s) = \frac{\sqrt{2}}{2}s + c_2$, respectively, where c_1, c_2 are real constants. Then by taking special values such as $c_1 = c_2 = 0$, a Mannheim offset of the surface is given by

$$\begin{aligned} \varphi_1(s,v) = {} & \left(\cos s + \tfrac{s}{2}\sin s, \sin s - \tfrac{s}{2}\cos s, \tfrac{s}{2}\right) \\ & + v\left(-\tfrac{\sqrt{2}}{2}\cos\left(-\tfrac{\sqrt{2}}{2}s\right)\sin s - \sin\left(-\tfrac{\sqrt{2}}{2}s\right)\cos s,\right. \\ & \left. \tfrac{\sqrt{2}}{2}\cos\left(-\tfrac{\sqrt{2}}{2}s\right)\cos s - \sin\left(-\tfrac{\sqrt{2}}{2}s\right)\sin s, \ \tfrac{\sqrt{2}}{2}\cos\left(-\tfrac{\sqrt{2}}{2}s\right)\right) \end{aligned} \tag{4.39}$$

which is plotted in Fig. 2.

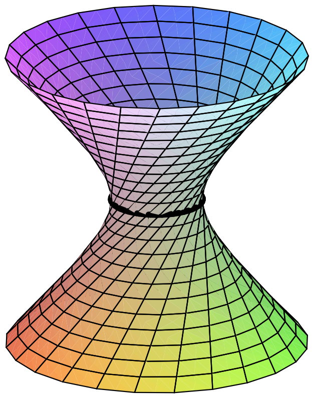

Figure 1: The surface $\varphi(s, v)$

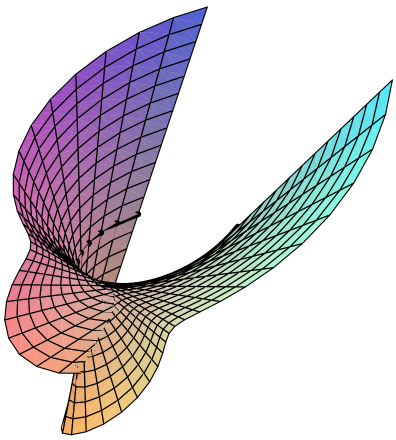

Figure 2: The Mannheim offset surface $\varphi_1(s, v)$

In Figures 1 and 2, the curves plotted in black are the striction lines of the surfaces.

5. Conclusion

Ruled surfaces have an important role in differential geometry and science since these surfaces are used in computer aided geometric design and kinematics. Considering this importance, in this paper, the characterizations for Mannheim offsets of ruled surfaces are given in dual space. New relations between the invariants of Mannheim offsets of ruled surfaces are obtained. Furthermore, it is shown that the striction lines of developable Mannheim offsets are Mannheim partner curves.

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