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The Third-Noncommuting Graph of a Group

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ABSTRACT: Let G be a group and let $T^3(G)$ be the proper subgroup $\{h \in G|(gh)^3 = (hg)^3,\ for\ all\ g \in G\}$ of G. *The third-noncommuting graph* of G is the graph with vertex set $G \setminus T^3(G)$, where two vertices x and y are adjacent if $(xy)^3 \neq (yx)^3$. In this paper, at first we obtain some results for this graph for any group G. Then, we investigate the structure of this graph for some groups.

Key Words: nth-noncommuting graph, finite group, graph.

Contents

1. Introduction

The non-commuting graph of a group was introduced by Paul Erdös in 1975 [8] as follows:
Let G be a group and consider a graph Γ whose vertex set is G and join two distinct elements if they do not commute. In [1] this graph is called the non-commuting graph and to avoid isolated vertices, the vertex set of this graph is taken as the elements of the group outside its center. Neumann in [8] solved the problem that posed by Paul Erdös about this type of graph associated to groups: "The class of groups whose center has finite index coincides with the class of groups whose non-commuting graph contains no infinite complete subgraph". After that, some of the researchers have studied this context and similar problems up to now (see [2,5,6,7]). In [7] Mashkouri and Taeri have extended the concept of non-commuting graph of a group as follows:
Consider the word $w(x,y) := (xy)^n(yx)^{-n}$ for the positive integer n and G is a group which is not defined by the law $w(x,y) = 1$. The n^{th}*-noncommuting graph* of G which is denoted by $\Gamma^n(G)$ is the graph with the vertex and edge sets $V(\Gamma^n(G)) := \{x \in G|w(x,g) \neq 1,\ for\ some\ g \in G\}$ and $E(\Gamma^n(G)) := \{xy|x,y \in V(\Gamma^n(G))\ and\ w(x,y) \neq 1\}$, respectively. If we denote the subgroup $\{h \in G|w(h,g) = w(g,h) = 1,\ for\ all\ g \in G\}$ of G by $T^n(G)$, then $G \setminus T^n(G)$ is the vertex set of the $\Gamma^n(G)$. It is obvious that if $n = 1$, then $\Gamma^n(G)$ coincides with the non-commuting graph of G. Also if $n = 2$, then $\Gamma^n(G)$ is the second-noncommuting graph of G, that is studied in [7]. The equality $T^n(G) = C_G(G^n)$ has been represented in [7], where $G^n = \{x^n|x \in G\}$ as well.
The main goal of this paper is study the structure of $\Gamma^3(G)$. Here, if there is

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no ambiguous, we write Γ^3, $V(\Gamma^3)$ and $E(\Gamma^3)$ instead of $\Gamma^3(G)$, $V(\Gamma^3(G))$ and $E(\Gamma^3(G))$, respectively. We use the concepts of graph theory according to [3,4].

2. Main results

If N is a normal subgroup with index t in G, we know that G is the union of the left (or right) cosets of N in G, $G = N \cup x_1N \cup x_2N \cup ... \cup x_{t-1}N$, where the cosets x_iN are mutually disjoint, $x_i \in G \setminus N$, $(1 \leqslant i \leqslant t-1)$.
Throughout this paper, G is a group and $C_G(G^3)$ is its subgroup with index t.

Lemma 2.1. *Let G be a group and $t \geqslant 1$, then the elements of any coset of $C_G(G^3)$ in G, as a part of vertices of Γ^3 , are not adjacent together.*

Proof: Let $x \in G \setminus C_G(G^3)$ and $xg \in xC_G(G^3)$ where $g \in C_G(G^3)$. Since every two elements of $xC_G(G^3)$ can be written such as a and ag where $g \in C_G(G^3)$ and $a \in G \setminus C_G(G^3)$, it's sufficient to prove the equality $(x^2g)^3 = (xgx)^3$.
Now note that x and xg are not adjacent if and only if:
$(x^2g)^3 = (xgx)^3 \iff (x^2g)^3x^2 = (xgx)^3x^2 \iff x^2gx^2gx^2gx^2 = x^4gx^2gx^2g \iff (gx^2)^3 = (x^2g)^3$.
Since $g \in C_G(G^3)(= T^3(G))$, the last equality is true. Thus any two vertices of Γ^3 that include in a coset of $C_G(G^3)$ are not adjacent. □

Suppose that G is a group and $t > 2$. If $x, y \in G \setminus C_G(G^3)$ are adjacent, then according to Lemma 2.1, $xC_G(G^3)$ and $yC_G(G^3)$ are two distinct cosets of $C_G(G^3)$ in G. If xh is any element of $xC_G(G^3)$, then $(xhh^{-1}y)^3 = (h^{-1}yxh)^3 = (hh^{-1}yx)^3$ if and only if $(xy)^3 = (yx)^3$. Since x, y are adjacent, xh and $h^{-1}y$ are adjacent. Thus, every element of $xC_G(G^3)$ is adjacent to at least one element of $yC_G(G^3)[= C_G(G^3)y]$.

Lemma 2.2. *Let G be a group and $t \geqslant 2$. Suppose that $xC_G(G^3)$ and $yC_G(G^3)$ are two distinct cosets in G. If $C_G(G^3) \leq C_G(x) \cap C_G(y)$, then any vertex in $xC_G(G^3)$ is adjacent to all vertices in $yC_G(G^3)$ if and only if x and y are adjacent.*

Proof: Let xh and ky be in $xC_G(G^3)$ and $yC_G(G^3)$, respectively. Define $g = hk$, we have $(xhky)^3 = (kyxh)^3$ if and only if $(xgy)^3 = (yxg)^3$. Since $g \in C_G(x) \cap C_G(y)$, the last equivalent is true if and only if $g^3(xy)^3 = g^3(yx)^3$ if and only if $(xy)^3 = (yx)^3$. So x and y are adjacent if and only if ky and xh are adjacent. □

For an integer $k \geqslant 2$ and positive integers $n_1, n_2, \ldots, n_k$, a complete $k-$partite graph $K_{n_1,n_2,\ldots,n_k}$ is that graph G whose vertex set can be partitioned into k subsets $V_1, V_2, \ldots, V_k$ with $|Vi| = n_i$ for $1 \leqslant i \leqslant k$ such that $uv \in E(G)$ if $u \in V_i$ and $v \in V_j$, where $1 \leqslant i, j \leqslant k$ and $i \neq j$.

Remark 2.3. *According to Lemma 2.2, we can conclude that if $C_G(G^3) = Z(G)$, then two arbitrary elements of two distinct cosets of $C_G(G^3)$ in G are adjacent if and only if two elements of these cosets are adjacent.*

Theorem 2.4. *Let G be a group, $t \geqslant 1$ and $C_G(G^3) = Z(G)$, then the third-noncommuting graph of G is a complete $s-$partite graph, where $s \leq t-1$. In particular if $s = t-1$, then Γ^3 is $T_{s,|G \setminus C_G(G^3)|} = K_{|C_G(G^3)|,\ldots,|C_G(G^3)|}$.*

Proof: Since $C_G(G^3) = Z(G)$, the first condition of Remark 2.3 is held and according to adjacency or non adjacency of elements x and y, we have the following two cases, respectively:
a. Any element of a coset $xC_G(G^3)$ is adjacent to all elements of another coset, as vertices of Γ^3.
b. None of the elements of a coset is adjacent to an element of another coset, as vertices of Γ^3.
Therefore, since vertices of Γ^3 are union of cosets of $C_G(G^3)$ in G, the third-noncommuting graph of G is a complete multipartite graph.
Now, if the elements of distinct cosets of $C_G(G^3)$ lie in distinct parts of complete multipartite graph Γ^3, then Γ^3 is complete $(t-1)-$partite graph, where every part of this graph have $\mid C_G(G^3) \mid$ vertices, because the cardinality of $V(\Gamma^3)$ is equal to $(t-1) \mid C_G(G^3) \mid$. □

Corollary 2.5. *Let G be a group. If Γ^3 is a complete $(t-1)-$partite graph, then $x^2 \in C_G(G^3)$, for all $x \in G \setminus C_G(G^3)$, and also $\mid C_G(G^3) \mid \neq 1$.*

Proof: According to the hypothesis, all elements of a coset of $C_G(G^3)$ in G are adjacent to all elements of other cosets, as vertices of Γ^3. Since x and x^{-1} are not adjacent, $x^{-1} \in xC_G(G^3)$, for all $x \in G \setminus C_G(G^3)$. Therefore, there exists an element of $C_G(G^3)$, such as h, that $x^{-1} = xh$, thus $x^2 \in C_G(G^3)$.
Now, we show that $\mid C_G(G^3) \mid \neq 1$. Suppose that $\mid C_G(G^3) \mid = 1$, by the first part, any non-trivial element of G has order 2. Hence G is an abelian group, contradicting the hypothesis $\mid C_G(G^3) \mid = 1$. □

Remark 2.6. *If the conditions of the above theorem is satisfied, then the complement of the third-noncommuting graph of G is a disconnected graph and any of its connected component is complete. It's shown simply that $\mid C_G(G^3) \mid$ divides the order of these components.*
If $[G : C_G(G^3)] = 2$, then there is only one the non-trivial coset of $C_G(G^3)$ in G. By Lemma 2.1, since any vertex of Γ^3 is an element of the coset of $C_G(G^3)$ in G, the third-noncommuting graph of G is empty, in contradiction the hypothesis $C_G(G^3) \neq G$. Therefore $C_G(G^3)$ cannot has index 2 in G.
If $C_G(G^3)$ has index 3, then the group $G/C_G(G^3)$ has two non-trivial elements. It's clear that the union of these elements(sets) is the vertex set of Γ^3.

Theorem 2.7. *Let G be a group and $t = 3$, then there exist some $h \in C_G(G^3)$ such that $h^3 \in C_G(x)$, where $x \in G \setminus C_G(G^3)$.*

Proof: Let $xC_G(G^3)$ and $yC_G(G^3)$ be two non-trivial cosets of $C_G(G^3)$ in G. It's clear that $yx \in C_G(G^3)$. Suppose that for any $h \in C_G(G^3)$ we have $h^3 \in C_G(x)$, then Γ^3 has no edges, contradicting the hypothesis $C_G(G^3) \neq G$. □

In the following we prove some results:

Corollary 2.8. *Let G be a group and $t > 2$. If $Z(G)$ is maximal in G, then $\Gamma^3(G)$ is a complete $s-$partite graph, where $s \leq t-1$.*

Proof: Since $C_G(G^3)$ contains $Z(G)$, $Z(G)$ is a maximal subgroup of G, and $C_G(G^3) \neq G$, we conclude that $C_G(G^3) = Z(G)$. Thus, by Theorem 2.7 the proof is complete. □

Corollary 2.9. *Let G be a group that $[G : C_G(G^3)] = 6$. If there exist some $x \in G \setminus C_G(G^3)$ such that $x^2 \in G \setminus C_G(G^3)$, then Γ^3 is an $s-$partite graph where $s \leq 4$, and $h^3 \in C_G(x)$, for all $x \in C_G(G^3)$.*

Proof: Since $[G : C_G(G^3)] = 6$, the group $G/C_G(G^3)$ has a normal subgroup with index 2. Suppose that non-trivial elements of this subgroup are $xC_G(G^3)$ and $yC_G(G^3)$, where $x, y \in G \setminus C_G(G^3)$. Since the product of these two elements of $G/C_G(G^3)$ is the identity of $G/C_G(G^3)$, $ab \in C_G(G^3)$, where $a \in xC_G(G^3)$ and $b \in yC_G(G^3)$.
Since $| C_G(G^3) |=| xC_G(G^3) |=| yC_G(G^3) |$, for any element z of $C_G(G^3)$, there exists an element xh of $xC_G(G^3)$ and an element yk of $yC_G(G^3)$ corresponding to z. Therefore, for $1 \in C_G(G^3)$, there exists z and its inverse z^{-1} in $G \setminus C_G(G^3)$ such that $zC_G(G^3) = xC_G(G^3)$ and $z^{-1}C_G(G^3) = yC_G(G^3)$. Now let $z^{-1}h_1$ and h_2z be elements of $z^{-1}C_G(G^3)$ and $zC_G(G^3)$, respectively. Set $h = h_1h_2$, so $((h_2z)(z^{-1}h_1))^3 = ((z^{-1}h_1)(h_2z))^3$ if and only if $h^3 = z^{-1}h^3z$ if and only if $h^3 \in C_G(z)$. Thus none of the elements of $zC_G(G^3)$ is adjacent to an element of $z^{-1}C_G(G^3)$. Hence $\Gamma^3(G)$ is at most a $4-$partite graph. □

Now, we classify some graphs $\Gamma^3(G)$ which are planar:

Theorem 2.10. *Let G be a group such that $C_G(G^3) = Z(G)$ is non-trivial, then $\Gamma^3(G)$ is planar if and only if $G \cong S_6$, $G \cong D_8$, or $G \cong Q_8$.*

Proof: By Kuratowski's Theorem it's sufficient to obtain some subgraphs of subdivisions of K_5 and $K_{3,3}$ that are also the third-noncommuting graph of a group. By Theorem 2.4, since $C_G(G^3) = Z(G)$, Γ^3 is a complete $s-$partite graph, where $s \leq 4$. Maximum size of the number of vertices in any part of complete bipartite, $3-$partite and $4-$partite graphs are $4, 3$ and 1, respectively.
By [3, Corollary 3.2.8], we have the inequality $| E(\Gamma) | \leq | V(\Gamma) | -6$, for any planar graph Γ. So we have to consider three cases:
Case 1. The bipartite graph $K_{a,b}$: By the above observations, whole possible choices are $K_{1,b}$, $K_{2,2}$ and $K_{2,3}$. According to [Remark 2.6], $| C_G(G^3) |$ divides both a and b, thus we have $| C_G(G^3) |= 1$ and $| C_G(G^3) |= 2$.
Since $C_G(G^3)$ is non-trivial, $K_{a,b} = K_{2,2}$. So $| G |= 6$. Since G is not an abelian group, $G \cong S_6$.
Case 2. The $3-$partite graph $K_{a,b,c}$: By observations in the first case, the only possible choice is $K_{2,2,2}$. So $| G |= 8$. Since G is not an abelian group, $G \cong D_8$ or

$G \cong Q_8$.
Case 3. The 4−partite graph $K_{a,b,c,d}$: Since $K_{2,2,2,2}$ is not planar and $\mid C_G(G^3) \mid$ divides a, b, c, and d, there is no graph in this case. □

Now, we study the third-noncommuting graph of $G = U_{6n}$, where:

$$U_{6n} = \langle a, b | a^{2n} = b^3 = 1, ba = ab^{-1} \rangle = \{1, a, ..., a^{2n-1}, b, b^2, ab, ..., a^{2n-1}b^2\}$$

So $C_G(G^3) = \{1, a^2, a^4, ..., a^{2n-2}\}$. By Theorem 2.4, Γ^3 is a complete $s-$partite graph, where $s \leq 5$. For any two integers $1 \leq i, j \leq n$ and two distinct numbers $k, l \in \{0, 1, 2\}$ we have the following inequality:

$$((a^{2i-2}b^k)(a^{2j-1}b^l))^3 \neq ((a^{2j-1}b^l)(a^{2i-2}b^k))^3$$

Since the union of cosets of $\{1, a^2, a^4, ..., a^{2n-2}\}$ in G is the vertex set of Γ^3, the vertex set of Γ^3 is union of two following parts:
a. $\{a, a^2, ..., a^{2n-1}, ab, a^3b, ..., a^{2n-1}b^2\}$
b. $\{b, a^2b, ..., a^{2n-2}b, ..., a^{2n-2}b^2\}$
Therefore, the third-noncommuting graph of U_{6n} is isomorph to the complete bipartite graph $K_{2n,3n}$.

Theorem 2.11. *Suppose that G is a group and it's center is non-trivial. If p is a prime number and $\Gamma^3(G) \cong \Gamma^3(U_{6p})$, then $\mid G \mid= 6p$.*

Proof: Since $n(\Gamma^3(U_{6p})) =\mid U_{6p} \setminus C_{U_{6p}}(U_{6p}^3) \mid$, we obtain

$$n(\Gamma^3(U_{6p})) = 5 \mid C_{U_{6p}}(U_{6p}^3) \mid .$$

On the other hand, p is a prime number and $\mid C_{U_{6p}}(U_{6p}^3) \mid$ divides $2p$ and $3p$, because $\Gamma^3(G) \cong \Gamma^3(U_{6p}) = K_{2p,3p}$. Thus $\mid C_G(G^3) \mid= 1$ or $\mid C_G(G^3) \mid= p$. But the center of G is non-trivial, so $\mid C_G(G^3) \mid= p$. It follows that $\mid G \mid= 6p$. □

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