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Fixed Point Theorem in Fuzzy Metric Space

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ABSTRACT: In this paper we prove a fixed point theorem on a fuzzy set defining a new class of fuzzy metric space as *structure fuzzy metric space*

Key Words: Fixed point, fuzzy metric, continuous t-norm.

Contents

1. Introduction

After Zadeh [5] introduced fuzzy sets in 1965, many researchers from various areas have developed the theory of fuzzy sets and its applications. Deng [8],Erceg [9], George and Veeramani [7] etc gave initial foundations of different forms of fuzzy metric spaces. Grebiec [3] extended Banach's [11] and Edelstein's [13] fixed point theorem in fuzzy metric space. Kramosil and Michalek [12] investigated common fixed point theorems for compatible maps. The investigation of fixed point theorems are going on fuzzy metric spaces.

In this paper we will study a fixed point theorem from view point of a new class of fuzzy metric defined on a fuzzy set. This concept came to exist when the author was investigating properties in a generalized closed set of bitopological space using topological ideal. Often topological ideal is simply stated as ideal.

A non-empty collection I of subsets of a set X is said to be an ideal if it follows following two conditions

(1) If $A \in I$ and $B \subset A$ then $B \in I$.

(2) If $A \in I, B \in I$, then $A \cup B \in I$.

Fixed point theorems in any areas are most useful. Mathematical economists, physicists,computer scientists etc are using fixed point theorems in their respective research areas. Now a days fuzzy fixed point theorems are also playing crucial role in mathematical economics, social choices,auction theory. One remarkable

2000 *Mathematics Subject Classification:* 54H25, 47H10 application of convex topological fixed point theory can be found in the 1994's Nobel laureate John Fr. Nash's classic seminal paper of equilibrium point of "Non-cooperative games" [14]. His proof is based on Kakutani's fixed point theorem [15], which is the generalization of Brouwer's fixed point theorem .

2. Preliminary definitions

In this section we discuss some existing definitions.

Definition 2.1. ([6]) *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a continuous* t*-norm if * satisfies the following conditions*

*(a) * is commutative and associative;*

*(b) * is continuous;*

(c) $a * 1 = a \forall a \in [0,1]$*;*

(d) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *and* $a, b, c, d \in [0,1]$.

Definition 2.2. ([12]) *Let* X *be a non-empty set, * be a continuous* t*-norm and* $M : X^2 \times [0,\infty) \to [0,1]$ *be a fuzzy set. Consider the following conditions for all* $x, y, z \in X$ *and* $t, s \in [0,\infty)$*;*

(M1) $M(x,y,0) = 0$

(M2) $M(x,x,t) = 1$

(M3) $M(x,y,t) = 1 \Rightarrow x = y$

(M4) $M(x,y,t) = M(y,x,t)$

(M5) $M(x,y,t+s) \geq M(x,z,t) * M(z,y,s)$

(M6) $M(x,y,.) : [0,\infty) \to [0,1]$ *is left continuous*

Then $(X, M, *)$ *is said to be a fuzzy metric space.*

3. Main result

This section contains some new definitions, terminologies and they are used to prove one theorem in fuzzy fixed point theory.

Definition 3.1. *$(X, M, *)$ is said to be a structure fuzzy metric space (SFMS) if it satisfies conditions $(M1), (M3), (M4), (M5)$ and $(M6)$ of Definition 2.2.*

Example 3.2. *If $X = R$, define $a * b = ab$ and $M(x, y, t) = \frac{1}{2^{\frac{|x-y|+|x|+|y|}{t}}}$ then $(X, M, *)$ is a SFMS.*

Definition 3.3. *A sequence $< x_n >$ in a SFMS is said to be structure convergent if there exists $x \in X$ such that $\lim_{n\to\infty} M(x_n, x, t) = 1\ \forall t > 0$. Then x is said to be structure limit of $< x_n >$ and denoted by $\lim_{n\to\infty} x_n = x$.*

Definition 3.4. *A sequence $< x_n >$ in a SFMS $(X, M, *)$ is said to be structure Cauchy sequence if for each $t > 0$ and $r \in N$ such that $\lim_{n\to\infty} M(x_{n+r}, x_n, t) = 1$.*

$(X, M, *)$ is said to be structure complete if every structure Cauchy sequence in it is structure convergent.

Definition 3.5. *Let $(X, M, *)$ be a SFMS, f and h are self maps on X. Then f and h are said to be normalized at x if and only if $M(fhx, hfx, t) = 1 \forall t \in [0, \infty)$.*

The functions f and h are said to be normalized on X if f and h are normalized at all points x of X.

Definition 3.6. *The functions f and h are said to be common domain normalized (CDN) if they are normalized at the coincidence point of f and h.*

Remark 3.7. *A SFMS has a unique limit point.*

Proof: Proof is easy, so omitted. □

Now we discuss the main theorem of this section.

Theorem 3.8. *Let $(X, M, *)$ be a SFMS and let $f, h : X \to X$ be two mappings with the following conditions,*

(a) $f(X) \subset h(X)$

(b) Either $f(X)$ or $h(X)$ is structure complete

(c) $M(fx, fy, kt) \geq M(hx, hy, t)$ for all $x, y \in X$ and $0 < k < 1$,$t \in [0, \infty)$

(d) $\lim_{t\to\infty} M(x, y, t) = 1$

Then f and h have a coincidence point; moreover if f and h are CDN then f and h have a unique fixed point.

Proof:

By condition (a), for some $x_o \in X$; we have $x_1 \in X$ such that $fx_0 = hx_1 = y_1$(say). Thus by using mathematical induction, we have $fx_n = hx_{n+1} = y_{n+1}$ where $n \in N$ and $y_0 = hx_0$.

For $0 < k < 1$ and $t \in [0, \infty)$ we have $M(y_1, y_2, kt) = M(fx_o, fx_1, kt) \geq M(hx_o, hx_1, t) = M(y_0, y_1, t)$

$M(y_2, y_3, kt) = M(fx_1, fx_2, kt) \geq M(hx_1, hx_2, t) = M(y_1, y_2, t) \geq M(y_o, y_1, \frac{t}{k})$. Thus $M(y_2, y_3, t) \geq M(y_o, y_1, \frac{t}{k^2})$

Proceeding by mathematical induction we have $M(y_n, y_{n+1}, t) \geq M(y_o, y_1, \frac{t}{K^n})$

Thus for $r \in N, t \in [0, \infty)$ we have $M(y_n, y_{n+r}, t) \geq M(y_n, y_{n+1}, \frac{t}{2}) * M(y_{n+1}, y_{n+r}, \frac{t}{2}) \geq M(y_n, y_{n+1}, \frac{t}{2}) * M(y_{n+1}, y_{n+2}, \frac{t}{4}) * M(y_{n+2}, y_{n+r}, \frac{t}{4}) \geq \ldots \geq M(y_o, y_1, \frac{t}{2k^n}) * M(y_o, y_1, \frac{t}{4k^{n+1}}) * \ldots * M(y_o, y_1, \frac{t}{2^r k^{n+r-1}})$
If $n \to \infty$ then $\lim_{n\to\infty} M(y_n, y_{n+r}, t) = 1$

Thus $< y_n >$ is a *structure Cauchy sequence*. Let $h(X)$ is *structure complete*; then there exists $u \in h(X)$ such that $\lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} hx_{n+1} = u = \lim_{n\to\infty} fx_n$. Let $hp = u$ for some $p \in X$

Thus $M(fp, hp, kt) = \lim_{n\to\infty} M(fp, fx_n, kt) \geq \lim_{n\to\infty} M(hp, hx_n, t) = \lim_{n\to\infty} M(u, hx_n, t) = 1$. So, $fp = hp$ and it proves that f and h have a coincidence point.

Now let f and h are normalized at some coincidence point θ. Thus from definition, we have $M(fh\theta, hf\theta, t) = 1 \forall t \geq 0$. This condition $(M3)$ implies $fh\theta = hf\theta$.

Let $f\theta = h\theta = v$ then $M(fv, v, kt) = M(fv, f\theta, kt) \geq M(hv, h\theta, t) = M(hf\theta, f\theta, t) \geq M(hv, h\theta, \frac{t}{k}) \geq \ldots \geq M(hv, h\theta, \frac{t}{k^n})$. If $n \to \infty$ then we must have $fv = v$. In similar manner we can show that $hv = v$. Thus v is common fixed point of f and h. Proceeding in the same way we may work for $h(X)$.

Uniqueness of fixed point: Let λ be another common fixed point of f and h; then

$M(v, \lambda, kt) = M(fv, f\lambda, kt) \geq M(hv, h\lambda, t) = M(v, \lambda, t) = M(fv, f\lambda, t) \geq M(hv, h\lambda, \frac{t}{k}) = M(v, \lambda, \frac{t}{k}) \geq \ldots \geq M(v, \lambda, \frac{t}{k^n})$.Thus if $n \to \infty$ then $v = \lambda$. Hence the proof. □

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