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Existence of Renormalized Solutions for $p(x)$-Parabolic Equations with three Unbounded Nonlinearities

Youssef Akdim, Nezha El gorch and Mounir Mekkour

abstract: In this article, we study the existence of a renormalized solution for the nonlinear $p(x)$-parabolic problem associated to the equation:

$$\frac{\partial b(x,u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f - \operatorname{div}F \text{ in } Q = \Omega \times (0,T)$$

with $f \in L^1(Q)$, $b(x,u_0) \in L^1(\Omega)$ and $F \in (L^{P'(\cdot)}(Q))^N$.
The main contribution of our work is to prove the existence of a renormalized solution in the Sobolev space with variable exponent. The critical growth condition on $H(x,t,u,\nabla u)$ is with respect to∇u, no growth with respect to u and no sign condition or the coercivity condition.

Key Words: Variable exponent Sobolev,Young's Inequality,Renomalized Solution, Parabolic problems,Tree unbounded nonlinearities.

Contents

1. Introduction

In the present paper we establish the existence of a renormalized solution for a class of nonlinear $p(x)$-parabolic equation of the type:

$$(\mathcal{P}) \begin{cases} \frac{\partial b(x,u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f - \operatorname{div}F & \text{in } Q = \Omega \times (0,T) \\ \text{b(x,u)}|_{t=0} = b(x,u_0) & \text{in } \Omega \\ \text{u} = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}$$

In the problem $(\mathcal{P})$, Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$, T is a positive real number, while $b(x,u_0) \in L^1(\Omega)$, $f \in L^1(Q)$ and $F \in (L^{P'(\cdot)}(Q))^N$.
The operator $-\operatorname{div}(a(x,t,u,\nabla u)$ is a Leray–Lions operator defined on

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$L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$ (see assumption (3.3)-(3.5) of section 3) which is coercive $b(x,u)$ is an unbounded function of u, H is a non linear lower order term.
The notion of renormalized solutions was introduced by R. J. Diperna and P. L. Lions [12] for the study of the Boltzmann equation, it was then used by L. Boccardo and al [11] when the right hand side is in $W^{-1,p'}(\Omega)$ and by J. M Rakoston [16] when the right hand side is in $L^1(\Omega)$.
For the degenerated parabolic equations the existence of weak solutions have been proved by L. Aharouch and al [2] in the case where $a(x,t,u,\nabla u)$is strictly monotone $H=0$, $F=0$ and $f\in L^{p'}(0,T,W^{-1,p'}(\Omega,W^*))$, see also the existence and uniqueness of a renormalized solution proved by Y. Akdim and al [5] in the case where $a(x,t,s,\xi)$ is independent of s, $H=0$ and $F=0$.
In the case $H(x,t,u,\nabla u)=\text{div}\phi(u)$ and $F=0$, the existence of renormalized solution has been established by H. Redwane in the classical Sobolev space and in Orlicz space [20,22] and by Y. Akdim and al [4] in the degenerate Sobolev space without the sign condition and the coercivity condition on the term $H(x,t,u,\nabla u)=\text{div}(\phi(x,t,u))$ and $F=0$, the existence of renormalized solutions has been established by A.Aberqi and al [1] in the classical Sobolev space.
Recently while $b(x,u)=u$, $a(x,t,u,\nabla u)=|\nabla u|^{p(x)-2}\nabla u$ and $F=0$, C. Zhang and S. Zhou [24] proved the existence of renormalized and entropy solutions with L^1-data and see also M. Bendahmane, P. Wittbold, A. Zimmermann [8] proved the existence of renormalized solutions for a nonlinear parabolic equation with L^1-data. The notion was then adapted to an elliptic version of problem $(\mathcal{P})$ by E. Azroul, M. B Benboubker and M. Rhoudaf [7] where the right hand side is in $L^1(\Omega)+W^{-1,p'(\cdot)}(\Omega)$ and $H(x,u,\nabla u)$ satisfying a sign condition on u.
It is our purpose to prove the existence of a renormalized solution of variable exponent Sobolev spaces for the problem $(\mathcal{P})$ setting without the sign condition and without the coercivity condition, the critical growth condition on H is only with respect to ∇u and not with respect to u (see assumption H2), where the right hand side is assumed to satisfy: f belongs to $L^1(Q)$ and $F\in(L^{P'(\cdot)}(Q))^N$.
This article is organized as follows: In Section 2 we collect some important propositions and results of variable exponent Lebesgue–Sobolev spaces that will be used throughout the paper. In Section 3 we make precise all the assumption on b,a,H,f and $b(x,u_0)$ and give the definition of a renormalized solution of the problem $(\mathcal{P})$ for which our problem has a solution. In Section 4 we establish the existence of such a solution (Theorem 4.1). In Section 5 we give the proof of theorem 4.2, lemma 4.6 and proposition 4.8 (see appendix). Section 6 is devoted to an example which illustrates our abstract result.

2. Mathematical preliminaries on variable exponent Sobolev spaces

2-1 Sobolev space with exponent variable

In this section we recall some definitions and basic properties of the generalised Lebegue–Sobolev spaces with variable exponent $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(.)}(\Omega)$, we refer to Fan and Zhao [13] for further properties of variable exponent Lebesgue - Sobolev spaces.

Let Ω be a bounded open subset of $\mathbb{R}^N(N \geq 2)$, we say that a real-valued continuous function $p(.)$ is log-Höder continuous in Ω if

$$|p(x) - p(y)| \leq \frac{C}{|\log|x-y||} \quad \forall x, y \in \bar{\Omega} \quad \textit{such that } |x-y| < \frac{1}{2},$$

with possible different constant C. We denote
$C_+(\bar{\Omega}) = \{\text{log-Höder continuous function } \ p : \bar{\Omega} \to \mathbb{R} \quad \textit{with } 1 < p^- \leq p^+ < N\}$,
where

$$p^- = \min\{p(x) : x \in \bar{\Omega}\} \quad \textit{and} \quad p^+ = \max\{p(x) : x \in \bar{\Omega}\}$$

we denote by $P(\Omega)$ the set of Lebesgue measurable function
$P(\Omega) = \{u : \Omega \to \mathbb{R} \quad \textit{measurable}\}$ and $P^+(\Omega) = \{u : \Omega \to [1, \infty) \quad \textit{measurable}\}$.
We define the variable exponent Lebesgue space for $p \in C_+(\bar{\Omega})$ by

$$L^{p(.)}(\Omega) = \{u \in P(\Omega) : \int_\Omega |u(x)|^{p(x)} < \infty\},$$

this space is endowed with the (Luxembourg) norm define by the formula

$$||u||_{L^{p(.)}(\Omega)} = ||u||_{p(.)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\}.$$

If $1 < p^- \leq p^+ < \infty$ then $L^{p(.)}(\Omega)$ is a uniformly convex Banach space and therefore reflexive and if $p \in P^+(\Omega) \cap L^\infty(\Omega)$, then $L^{p(.)}(\Omega)$ is separable space.
We denote by $L^{p'(.)}(\Omega)$ the conjugate space of $L^{p(.)}(\Omega)$ where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$, see [14,23].

Proposition 2.1. *(Young's Inequality) Let* $p,\ p' \in C_+(\bar{\Omega})$, *where* p' *the conjugate, i.e.,* $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$. *For all* $a, b > 0$, *we have*

$$ab \leq \frac{a^{p(x)}}{p(x)} + \frac{b^{p'(x)}}{p'(x)}.$$

Proposition 2.2. *(Generalised Hölder Inequality)see [13,18]*

i) *For any functions* $u \in L^{p(.)}(\Omega)$ *and* $v \in L^{p'(.)}(\Omega)$, *we have* $|\int_\Omega uv dx| \leq (\frac{1}{p^-} + \frac{1}{p'^-})||u||_{p(.)}||v||_{p'(.)} \leq 2||u||_{p(.)}||v||_{p'(.)}$.

ii) *For all* $p,\ q \in C_+(\bar{\Omega})$ *such that* $p(x) \leq q(x)$ *a.e. in* Ω, *we have* $L^{q(.)} \hookrightarrow L^{p(.)}$ *and the embedding is continuous.*

Lemma 2.3. *(See [13])If we denote* $\rho(u) = \int_\Omega |u(x)|^{p(x)} dx \ \ \forall u \in L^{p(.)}(\Omega)$ *then,*

$$\min\left\{||u||^{p^-}_{p(.)}, ||u||^{p^+}_{p(.)}\right\} \leq \rho(u) \leq \max\left\{||u||^{p^-}_{p(.)}, ||u||^{p^+}_{p(.)}\right\}.$$

Proposition 2.4. *See([14,23]) For $u \in L^{p(\cdot)}(\Omega)$ and $\{u_k\}_{k\in N} \subset L^{p(\cdot)}(\Omega)$ then, the following assertions hold*

$$u \neq 0 \Rightarrow [||u||_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1], \tag{2.1}$$

$$||u||_{p(.)} > 1 \Rightarrow ||u||_{p(.)}^{p^-} \leq \rho(u) \leq ||u||_{p(.)}^{p^+}, \tag{2.2}$$

$$||u||_{p(.)} < 1 \Rightarrow ||u||_{p(.)}^{p^+} \leq \rho(u) \leq ||u||_{p(.)}^{p^-}, \tag{2.3}$$

$$\lim_{k\to\infty} ||u_k||_{p(.)} = 0 \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = 0, \tag{2.4}$$

$$\lim_{k\to\infty} ||u_k||_{L^{p(\cdot)}(\Omega)} = \infty \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = \infty. \tag{2.5}$$

Lemma 2.5. *. Let $f_n \to f$ a.e and $f_n \rightharpoonup f$ in $L^{p(\cdot)}(\Omega)$. Then,*

$$\lim_{n\to\infty} \int_\Omega |f_n|^{p(x)} dx - \int_\Omega |f - f_n|^{p(x)} dx = \int_\Omega |f|^{p(x)} dx.$$

Theorem 2.6. *For any function $u \in L^{p(\cdot)}(\Omega)$ and $u_n \in L^{p(\cdot)}(\Omega)$, we have then, the following are equivalent assertions*

i) $\lim_{n\to\infty} ||u_n - u||_{p(.)} = 0$

ii) $\lim_{k\to\infty} \rho(u_n - u) = 0$

iii) *u_n converge to u in measure and $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.*

Which share the same type of properties as $L^{p(\cdot)}(\Omega)$, we define also the variable Sobolev space by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \ \text{ and } \quad |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

where the norm is defined by

$$||u||_{1,p(.)} = ||u||_{p(.)} + ||\nabla u||_{p(.)} \quad \forall u \in W^{1,p(\cdot)}(\Omega).$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$, i.e.,

$$W_0^{1,p(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}$$

and we define the Sobolev exponent by $p^*(x) = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$.

Proposition 2.7. *[14]*

i) *Assuming $1 < p^- \leq p^+ < \infty$ the spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.*

ii) *If $q \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \Omega$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.*

iii) *Poincaré inequality: there exists a constant $C > 0$, such that*

$$||u||_{p(.)} \leq C||\nabla u||_{p(.)} \quad \forall u \in W_0^{1,p(.)}(\Omega).$$

Remark 2.8. *By (iii) of Proposition 2.4, we deduce that $||\nabla u||_{p(.)}$ and $||u||_{1,p(.)}$ are equivalent norms in $W_0^{1,p(.)}(\Omega)$.*

We will also use the standard notation for Bochner spaces, i.e., if $q \geq 1$ and X is a Banach space then $L^q(0,T;X)$ denotes the space of strongly measurable function $u : (0,T) \to X$ for which $t \mapsto \|u(t)\|_X \in L^q((0,T)$. Morever, $C([0,T];X)$ denotes the space of continuous function $u : [0,T] \to X$ endowed with the norm $\|u\|_{C([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X$.

$$L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) = \Big\{u : (0,T) \to W_0^{1,p(x)}(\Omega)\ measurable; \\ \Big(\int_0^T \|u(t)\|^{p^-}_{W_0^{1,p(x)}(\Omega)} dt\Big)^{\frac{1}{p^-}} < \infty\Big\}$$

and we define the space

$$L^\infty(0,T;X) = \Big\{u : (0,T) \to X\ measurable,\ \exists\, C > 0\ /\|u(t)\|_X \leq\ C\ a.e.\Big\},$$

where the norm is defined by

$$\|u\|_{L^\infty(0,T;X)} = \inf\Big\{C > 0; \|u(t)\|_X \leq\ C\ a.e.\Big\}.$$

We introduce the functional space see [8]

$$V = \Big\{f \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)); |\nabla f| \in L^{p(.)}(Q)\Big\}, \tag{2.6}$$

which endowed with the norm

$$\|f\|_V = \|\nabla f\|_{L^{p(.)}(Q)}$$

or, the equivalent norm

$$\||f|\|_V = \|f\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} + \|\nabla f\|_{L^{p(.)}(Q)},$$

is a separable and reflexive Banach space. The equivalence of the two norms is an easy consequence of the continuous embedding $L^{p(.)}(Q) \hookrightarrow L^{p^-}(0,T;L^{p(.)}(\Omega))$ and the Poincaré inequality. We state some further properties of V in the following lemma.

Lemma 2.9. *Let V be defined as in (2.6) and its dual space be denote by V^*. Then,*

i) *we have the following continuous dense embeddings*

$$L^{p^+}(0,T;W_0^{1,p(.)}(\Omega)) \hookrightarrow V \hookrightarrow L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)).$$

In particular, since $D(Q)$ is dense in $L^{p^+}(0,T;W_0^{1,p(.)}(\Omega))$, it is dense in V and for the corresponding dual spaces, we have

$$L^{(p^-)'}(0,T;(W_0^{1,p(.)}(\Omega))^*) \hookrightarrow V^* \hookrightarrow L^{(p^+)'}(0,T;(W_0^{1,p(.)}(\Omega))^*).$$

Note that, we have the following continuous dense embeddings

$$L^{p^+}(0,T;L^{p(.)}(\Omega)) \hookrightarrow L^{p(.)}(Q) \hookrightarrow L^{p^-}(0,T;L^{p(.)}(\Omega)).$$

ii) *One can represent the elements of V^* as follows: if $T \in V^*$, then there exists $F = (f_1, ..., f_N) \in (L^{P'(.)}(Q))^N$ such that $T = divF$ and*

$$\langle T, \xi\rangle_{V^*,V} = \int_0^T \int_\Omega F.\nabla\xi dxdt,$$

for any $\xi \in V$. Moreover, we have

$$\|T\|_{V^*} = \max\Big\{\|f_i\|_{L^{p(.)}(Q)}, i = 1, ..., n\Big\}.$$

Remark 2.10. *The space $V \cap L^\infty(Q)$, is endowed with the norm defined by the formula*

$$\|v\|_{V\cap L^\infty(Q)} = \max\Big\{\|v\|_V, \|v\|_{L^\infty(Q)}\Big\},\ v \in V \cap L^\infty(Q),$$

is a Banach space. In fact, it is the dual space of the Banach space $V^* + L^1(Q)$ endowed with the norm

$$\|v\|_{V^*+L^1(Q)} = \inf\Big\{\|v_1\|_{V^*} + \|v_2\|_{L^1(Q)};\ v = v_1 + v_2, v_1 \in V^*, v_2 \in L^1(Q)\Big\}.$$

2-2 Some Technical Results.

Lemma 2.11. *([2]) Assume (3.3) -(3.5) and let $(u_n)_n$ be a sequence in $L^{p^-}(0,T;L^{p(.)}(\Omega))$ such that $u_n \rightharpoonup u$ weakly in $L^{p^-}(0,T;L^{p(.)}(\Omega))$ and*

$$\int_Q \Big(a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)\Big)\nabla(u_n - u)dx \to 0.$$

Then, $u_n \to u$ strongly in $L^{p^-}(0,T;L^{p(.)}(\Omega))$.

Lemma 2.12. *([8])Let $g \in L^{p(.)}(Q)$ and $g_n \in L^{p(.)}(Q)$ with $\|g_n\|_{p(x)} \le C$ for $1 < p(x) < \infty$, if $g_n(x) \to g(x)$ a.e. on Q. Then, $g_n \rightharpoonup g$ in $L^{p(.)}(Q)$.*

Lemma 2.13. *See [19]*
$W = \Big\{u \in V; u_t \in V^* + L^1(Q)\Big\} \hookrightarrow C([0,T];L^1(\Omega))$
and

$$W \cap L^\infty(Q) \hookrightarrow C([0,T];L^2(\Omega)).$$

Definition 2.14. *A monotone map $T : D(T) \to X^*$ is called maximal monotone if its graph*

$$G(T) = \Big\{(u, T(u)) \in X \times X^* \text{ for all } u \in D(T)\Big\},$$

is not a proper subset of any monotone set in $X \times X^*$.
Let us consider the operator $\frac{\partial}{\partial t}$ which induces a linear map L from the subset

$$D(L) = \Big\{v \in X : v' \in X^*, v(0) = 0\Big\} \text{ of } X \text{ in to } X^* \text{by}$$

$$\Big\langle Lu, v\Big\rangle_X = \int_0^T \langle u'(t), v(t)\rangle_V dt \quad u \in D(L),\ v \in X.$$

Definition 2.15. *See [5] A mapping S is called pseudo–monotone with $u_n \rightharpoonup u$ and $Lu_n \rightharpoonup Lu$ and $\lim_{n\to\infty} \sup\langle S(u_n), u_n - u\rangle \leq 0$, that we have* $\lim_{n\to\infty} \sup \Big\langle S(u_n), u_n - u\Big\rangle = 0$ *and $S(u_n) \rightharpoonup S(u)$ as $n \to \infty$.*

3. Essential Assumption

Throughout the paper, we assume that the following assumptions hold true.

ASSUMPTION (H1)

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 1)$, $p \in C_+(\bar{\Omega})$ and $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for every $x \in \Omega$, $b(x, .)$ is a strictly increasing C^1 function with

$$b(x, 0) = 0. \tag{3.1}$$

Next, for any $k > 0$, there exist $\lambda_k > 0$ and functions $A_k \in L^\infty(\Omega)$ and $B_k \in L^{p(.)}(\Omega)$ such that

$$\lambda_k \leq \frac{\partial b(x, s)}{\partial s} \leq A_k(x) \text{ and } \Big|D_x\Big(\frac{\partial b(x, s)}{\partial s}\Big)\Big| \leq B_k(x). \tag{3.2}$$

for almost every $x \in \Omega$ and every s such that $|s| \leq k$, we denote by $D_x(\partial b(x, s) \setminus \partial s)$ the gradient of $\partial b(x, s) \setminus \partial s$ defined in the sense of distributions.

ASSUMPTION (H2)

We consider a Leray–Lions operator defined by the formula

$$Au = -\text{div}(a(x, t, u, \nabla u)),$$

where $a : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, i.e., (measurable with respect to x in Ω for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous with respect to $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) which satisfies the following conditions there exist $k \in L^{p'(.)}(Q)$ *and* $\alpha > 0$, $\beta > 0$ such that, for almost every $(x, t) \in Q$ all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

$$|a(x, t, s, \xi)| \leq \beta\Big[k(x, t) + |s|^{p(x)-1} + |\xi|^{p(x)-1}\Big] \tag{3.3}$$

$$\Big[a(x, t, s, \xi) - a(x, t, s, \eta)\Big](\xi - \eta) > 0 \ \ \forall\, \xi \neq \eta \tag{3.4}$$

$$a(x, t, s, \xi) \cdot \xi \geq \alpha|\xi|^{p(x)}. \tag{3.5}$$

ASSUMPTION (H3)
Let $H : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e. $(x,t) \in Q$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$|H(x,t,s,\xi)| \leq \gamma(x,t) + g(s)|\xi|^{p(x)}. \tag{3.6}$$

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$ is a bounded continuous positive function that belongs to $L^1(\mathbb{R})$, while $\gamma \in L^1(Q)$.
We recall that, for $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(.)$ defined by

$$T_k(s) = \begin{cases} s & if \quad |s| \leq k \\ k\frac{s}{|s|} & if \quad |s| > k. \end{cases}$$

Definition 3.1. *Let* $f \in L^1(Q)$, $F \in (L^{P'(.)}(Q))^N$ *and* $b(.,u_0) \in L^1(\Omega)$
A real–valued function u *defined on* Q *is a renormalized solution of problem* $(\mathcal{P})$ *if*

$$T_k(u) \in L^{p^-}(0,T; W_0^{1,p(.)}(\Omega)) \text{ for all } k \geq 0,\ b(x,u) \in L^\infty(0,T; L^1(\Omega)), \tag{3.7}$$

$$\int_{\{m \leq |u| \leq m+1\}} a(x,t,u,\nabla u)\nabla u dx dt \to 0 \ as \ m \to \infty, \tag{3.8}$$

$$\begin{aligned} &\frac{\partial B_S(x,u)}{\partial t} - div\Big(S'(u)a(x,t,u,\nabla u)\Big) + S''(u)a(x,t,u,\nabla u)\nabla u \\ &+H(x,t,u,\nabla u)S'(u) = fS'(u) - div\Big(s'(u)F\Big) + s''(u)F\nabla u \ in \ D'(Q), \end{aligned} \tag{3.9}$$

for all $S \in W^{2,\infty}(\mathbb{R})$ *which are piecewise* C^1 *and such that* S' *has a compact support in* $\mathbb{R}$*, where* $B_S(x,z) = \int_0^z \frac{\partial b(x,r)}{\partial r} S'(r)dr$ *and*

$$B_S(x,u)\mid_{t=0} = B_S(x,u_0) \quad in \quad \Omega. \tag{3.10}$$

Remark 3.2. *Equation (3.9) is formally obtained through pointwise multiplication of problem* $(\mathcal{P})$ *by* $S'(u)$*. However, while* $a(x,t,u,\nabla u)$ *and* $H(x,t,u,\nabla u)$ *do not in general make sense in* $(\mathcal{P})$*, all the terms in (3.9) have a meaning in* $D'(Q)$*. Indeed, if* M *is such that supp* $S' \subset [-M,M]$*, the following identifications are made in (3.9)*

- $S(u)$ *belongs to* $V \cap L^\infty(Q)$*. Since* S *is a bounded function.*
- $S'(u)$ $a(x,t,u,\nabla u)$ *identifies with* $S'(u)$ $a(x,t,T_M(u),\nabla T_M(u))$ *a.e. in* Q*.*

for any $\varphi \in D(Q)$*, using Hölder inequality*

$$\begin{aligned} \int_Q S'(u)a(x,t,u,\nabla u)\nabla\varphi dx dt &= \int_Q S'(u)a(x,t,T_M(u),\nabla T_M(u))\nabla\varphi dx dt \\ &\leq C_M \|S'\|_{L^\infty(Q)} \max\Big\{\Big(\int_Q |\nabla T_M(u)|^{p(x)}\omega(x)\Big)^{\frac{1}{p'^-}}, \\ &\Big(\int_Q |\nabla T_M(u)|^{p(x)}\omega(x)\Big)^{\frac{1}{p'^+}}\Big\}\|\nabla\varphi\|_{L^{p(.)}(Q,\omega^*)}, \end{aligned}$$

where $M > 0$ is that $suppS' \subset [-M, M]$. As $D(Q)$ is dense in V, we deduce that

$$div(S'(u)a(x,t,u,\nabla u)) \in V^*.$$

- $S''(u)\ a(x,t,u,\nabla u)\nabla u$ *identifies with* $S''(u)\ a(x,u,T_M(u),\nabla T_M(u))\nabla T_M(u)$ *and*
$$S''(u)a(x,u,T_M(u),\nabla T_M(u))\nabla T_M(u) \in L^1(Q).$$
- $S'(u)\ H(x,t,u,\nabla u)$ *identifies with* $S'(u)H(x,t,T_M(u),\nabla T_M(u))$ *a.e. in* Q. *Since* $|T_M(u)| \le M$ *a.e. in* Q *and* $S'(u) \in L^\infty(Q)$, *we see from (3.6) and (3.7) that*
$$S'(u)H(x,t,T_M(u),\nabla T_M(u)) \in L^1(Q).$$
- $S'(u)\ f$ *belongs to* $L^1(Q)$ *while* $S'(u)F$ *belongs to* $(L^{p'(\cdot)}(Q))^N$.
- $S''(u)\ F\nabla u$ *identifies with* $S''(u)\ F\nabla T_M(u)$, *which belongs to* $L^1(Q)$.

The above considerations show that equation (3.9) hold in $D'(Q)$ and that

$$\frac{\partial B_S(x,u)}{\partial t} \in V^* + L^1(Q).$$

Due to the properties of S and (3.9) $\frac{\partial S(u)}{\partial t} \in V^* + L^1(Q)$ using lemma 2.13, which implies that $S(u) \in C^0([0,T]; L^1(\Omega))$, so that the initial condition (3.10) makes sense, since, due to the properties of S (increasing) and (3.2), we have

$$\Big|(B_S(x,r) - B_S(x,r')\Big| \le A_k(x)\Big|S(r) - S(r')\Big| \textit{ for all } r, r' \in \mathbb{R}. \tag{3.11}$$

4. Existence Results.

In this section, we establish the following existence theorem:

Theorem 4.1. *Let $f \in L^1(Q)$, $F \in (L^{p'(\cdot)}(Q))^N$, $p(.) \in C_+(\bar{\Omega})$ and assume that u_0 is a measurable function such that $b(., u_0) \in L^1(\Omega)$. Assume that (H1)–(H3) hold true. Then, there exists a renormalized solution u of problem $(\mathcal{P})$ in the sense of Definition (3.1).*

Proof. The proof is in five steps.

STEP 1: **Approximate problem** :

For $n > 0$, we define approximations of b, H, f, F and u_0. First, set

$$b_n(x,r) = b(x, T_n(r)) + \frac{1}{n}r. \tag{4.1}$$

b_n is a Carathéodory function and satisfies (3.2) : there exist $\lambda_n > 0$ and functions $A_n \in L^\infty(\Omega)$ and $B_n \in L^{p(\cdot)}(\Omega)$ such that

$$\lambda_n \le \frac{\partial b_n(x,s)}{\partial s} \le A_n(x) \text{ and } \Big|D_x\Big(\frac{\partial b_n(x,s)}{\partial s}\Big)\Big| \le B_n(x) \text{ a.e. in } \Omega,\ s \in \mathbb{R}.$$

Next, set

$$\begin{aligned} H_n(x,t,s,\xi) &= \frac{H(x,t,s,\xi)}{1+\frac{1}{n}|H(x,t,s,\xi)|}, \\ \textit{Note that}\quad |H_n(x,t,s,\xi)| &\leq |H(x,t,s,\xi)| \\ \textit{and}\quad |H_n(x,t,s,\xi)| &\leq n \text{ for all } (s,\xi)\in\mathbb{R}\times\mathbb{R}^N. \end{aligned}$$

and select f_n, u_{0n} and b_n so that

$$f_n \in L^{p'(.)}(Q) \text{ and } f_n \to f \text{ a.e. in } Q \text{ and strongly in } L^1(Q) \text{ as } n \to \infty, \tag{4.2}$$

$$u_{0n} \in D(\Omega), \quad \|b_n(x,u_{0n})\|_{L^1(\Omega)} \leq \| b(x,u_0)\|_{L^1(\Omega)}, \tag{4.3}$$

$$b_n(x,u_{0n}) \to b(x,u_0) \text{ a.e. in } \Omega \text{ and strongly in } L^1(\Omega). \tag{4.4}$$

Let us now consider the approximate problem

$$(\mathcal{P}_n)\begin{cases} \frac{\partial b_n(x,u_n)}{\partial t} - \operatorname{div}(a(x,t,u_n,\nabla u_n)) + H_n(x,t,u_n,\nabla u_n) = f_n - \operatorname{div}F \text{ in } D'(Q), \\ b_n(x,u_n)\mid_{t=0} = b_n(x,u_{0n}) \quad \text{in } \Omega \\ u_n = 0 \quad \text{on } \partial\Omega\times(0,T). \end{cases}$$

Theorem 4.2. *Let $f_n \in L^{p'^-}(0,T;W^{-1,p'(.)}(\Omega))$, $p(.)\in C_+(\overline{\Omega})$ for fixed n, the approximate problem $(\mathcal{P}_n)$ has at least one weak solution $u_n \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$.*

Proof. See Appendix.
In view of Theorem 4.2, there exists at least one weak solution $u_n \in L^{p^-}(0;T;W_0^{1,p(.)}(\Omega))$ of the problem (P_n).(see [15]).
STEP 2: **A Priori Estimates**:

Proposition 4.3. *Let u_n a solution of the approximate problem $(\mathcal{P}_n)$. Then, there exists a constant C(which does not depend on the n and k) such that*

$$\|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} \leq C\,k \quad \forall\, k>0.$$

Proof.
Let $\varphi \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \cap L^\infty(Q)$, with $\varphi > 0$, Choosing $v = \exp(G(u_n))\varphi$ as a test function in $(\mathcal{P}_n)$ where $G(s) = \int_0^s(\frac{g(r)}{\alpha})dr$. (the function g appears in (3.6)), we have

$$\begin{aligned} \int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,\nabla u_n)\nabla(\exp(G(u_n))\varphi)dxdt \\ + \int_Q H_n(x,t,u_n,\nabla u_n)\exp(G(u_n))\varphi dxdt \\ = \int_Q f_n\exp(G(u_n))\varphi dxdt + \int_Q F\nabla(\exp(G(u_n))\varphi)dxdt. \end{aligned}$$

In view of (3.6), we obtain

$$\begin{aligned}\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp(G(u_n))\varphi dxdt \\ + \int_Q a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla\varphi dxdt \\ \leq \int_Q \gamma(x,t)\exp(G(u_n))\varphi dxdt + \int_Q f_n \exp(G(u_n))\varphi dxdt \\ + \int_Q g(u_n)|\nabla u_n|^{p(x)}\exp(G(u_n))\varphi dxdt + \int_Q F\nabla(\exp(G(u_n))\varphi)dxdt\end{aligned}$$

By using (3.5), we obtain

$$\begin{aligned}\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla\varphi dxdt \\ \leq \int_Q \gamma(x,t)\exp(G(u_n))\varphi dxdt + \int_Q f_n \exp(G(u_n))\varphi dxdt \\ + \int_Q F\nabla(\exp(G(u_n)))\varphi dxdt + \int_Q F\exp(G(u_n))\nabla\varphi dxdt \qquad (4.5)\end{aligned}$$

for all $\varphi \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \cap L^\infty(Q)$, with $\varphi > 0$.
On the other hand, taking $v = \exp(-G(u_n))\varphi$ as a test function in $(\mathcal{P}_n)$, we deduce as in (4.5), that

$$\begin{aligned}\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(-G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,\nabla u_n)\exp(-G(u_n))\nabla\varphi dxdt \\ + \int_Q \gamma(x,t)\exp(-G(u_n))\varphi dxdt \geq \int_Q f_n \exp(-G(u_n))\varphi dxdt \\ + \int_Q F\nabla(\exp(-G(u_n)))\varphi dxdt + \int_Q F\exp(-G(u_n))\nabla\varphi dxdt \quad (4.6)\end{aligned}$$

for all $\varphi \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \cap L^\infty(Q)$, with $\varphi > 0$.
Letting $\varphi = T_k(u_n)^+\chi(0,\tau)$, for every $\tau \in [0,T]$ in (4.5), we have

$$\begin{aligned}\int_\Omega B^n_{k,G}(x,u_n(\tau))dx + \int_{Q^\tau} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+ dxdt \\ \leq \int_{Q^\tau} \gamma(x,t)\exp(G(u_n))T_k(u_n)^+ dxdt + \int_{Q^\tau} f_n \exp(G(u_n))T_k(u_n)^+ dxdt \\ + \int_{Q^\tau} F\nabla T_k(u_n)^+ \exp(G(u_n))dxdt \qquad (4.7) \\ + \int_{Q^\tau} FT_k(u_n)^+ \exp(G(u_n))\nabla u_n \frac{g(u_n)}{\alpha}dxdt + \int_\Omega B^n_{k,G}(x,u_{0n})dx,\end{aligned}$$

where
$$B^n_{k,G}(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s} T_k(s)^+ \exp(G(s))ds.$$

Due to the definition of $B^n_{k,G}$ and $|G(u_n)| \leq \exp(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha})$, we have

$$0 \leq \int_\Omega B^n_{k,G}(x,u_{0n})dx \leq k \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\|b(.,u_0\|_{L^1(\Omega)}. \tag{4.8}$$

Using (4.8), $B^n_{k,G}(x,u_n) \geq 0$ and Young's Inequality, we obtain

$$\begin{aligned}
&\int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+)\exp(G(u_n))\nabla T_k(u_n)^+ dxdt \\
&\leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big] \\
&+\frac{1}{\alpha}\int_{Q^\tau} FT_k(u_n)^+\exp(G(u_n))g(u_n)\nabla u_n dxdt \\
&+\int_{Q^\tau} F\Big[\exp(G(u_n))\Big]^{1-\frac{1}{p(x)}}\Big[\exp(G(u_n))\Big]^{\frac{1}{p(x)}}\nabla T_k(u_n)^+ dxdt
\end{aligned}$$

then

$$\begin{aligned}
&\int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+)\exp(G(u_n))\nabla T_k(u_n)^+ dxdt \\
&\qquad\leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big] \\
&+\frac{1}{\alpha}\int_{Q\tau} FT_k(u_n)^+\exp(G(u_n))\nabla u_n g(u_n)\nabla u_n dxdt \\
&+\int_{Q^\tau}\frac{F\Big[\exp(G(u_n))\Big]^{\frac{1}{p'(x)}}}{\Big[\frac{\alpha}{2}p(x)\Big]^{\frac{1}{p(x)}}}\Big[\frac{\alpha}{2}p(x)\Big]^{\frac{1}{p(x)}}\Big|\nabla T_k(u_n)^+\Big|\Big[\exp(G(u_n))\Big]^{\frac{1}{p(x)}}dxdt
\end{aligned}$$

using and Young's Inequality, we obtain

$$\begin{aligned}
&\int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+)\exp(G(u_n))\nabla T_k(u_n)^+ dxdt \\
&\qquad\leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big] \\
&+\frac{1}{\alpha}\int_{Q\tau} FT_k(u_n)^+\exp(G(u_n))\nabla u_n g(u_n)\nabla u_n dxdt \\
&+\int_{Q^\tau}\frac{|F|^{p'(x)}\exp(G(u_n))}{\Big[\frac{p'(x)\alpha}{2}p(x)\Big]^{\frac{p'(x)}{p(x)}}}dxdt + \frac{\alpha}{2}\int_Q\Big|\nabla T_k(u_n)^+\Big|^{p(x)}\exp(G(u_n))dxdt
\end{aligned}$$

then,

$$\int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+)\exp(G(u_n))\nabla T_k(u_n)^+ dxdt$$

$$\leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}+\|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big]$$

$$+\frac{1}{\alpha}\int_{Q^\tau} FT_k(u_n)^+\exp(G(u_n))\nabla u_n g(u_n)\nabla u_n dxdt$$

$$+C\int_Q |F|^{p'(x)}dxdt+\frac{\alpha}{2}\int_Q \Big|\nabla T_k(u_n)^+\Big|^{p(x)}\exp(G(u_n))dxdt$$

and since

$$\int_Q |F|^{p'(x)}dxdt \quad = \quad \rho(F)\leq \max\Big\{||F||^{p^-}_{(L^{P'(\cdot)}(Q))^N},||F||^{p^+}_{(L^{P'(\cdot)}(Q))^N}\Big\}=C'$$

$$\textit{then}\int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+)\exp(G(u_n))\nabla T_k(u_n)^+ dxdt$$

$$\leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}+\|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big]$$

$$+C_1+\frac{\alpha}{2}\int_Q |\nabla T_k(u_n)^+|^{p(x)}\exp(G(u_n))dxdt$$

$$+\frac{1}{\alpha}\int_{Q^\tau} Fg(u_n)\exp(G(u_n))\nabla u_n\chi_{\{u_n>0\}}dxdt$$

Thanks to (3.5), we have

$$\frac{\alpha}{2}\int_{Q^\tau} |\nabla T_k(u_n)^+|^{p(x)}\exp(G(u_n))dxdt$$

$$\leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}+\|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big]+C_1$$

$$+\frac{1}{\alpha}\int_{Q^\tau} Fg(u_n)\exp(G(u_n))\nabla u_n\chi_{\{u_n>0\}}dxdt. \tag{4.9}$$

Let us observe that if we take $\varphi=\rho(u_n)=\int_0^{u_n} g(s)\chi_{\{s>0\}}ds$ in (4.5) and use (3.5), we obtain

$$\Big[\int_\Omega B_g^n(x,u_n)dx\Big]_0^T+\alpha\int_Q |\nabla u_n|^{p(x)}g(u_n)\chi_{\{u_n>0\}}\exp(G(u_n))dxdt$$

$$\leq \Big(\int_0^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}\Big]$$

$$+\int_Q F\nabla u_n g(u_n)\chi_{\{u_n>0\}}\exp(G(u_n))dxdt$$

$$+\Big(\int_0^\infty g(s)ds\Big)\int_Q |F\nabla u_n|\frac{g(u_n)}{\alpha}\exp(G(u_n))\chi_{\{u_n>0\}}dxdt,$$

where

$$B_g^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho(s)\exp(G(s))ds,$$

which implies, using $B_g^n(x,r) \geq 0$ and Young's Inequality, we obtain

$$\begin{aligned}
\alpha \int_{\{u_n>0\}} & |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
& \leq \|g\|_\infty \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\Big[\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)} + \|b(x,u_0\|_{L^1(\Omega)}\Big] \\
& \quad + C_2 + \frac{\alpha}{2}\int_{\{u_n>0\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
& \quad + C_3\|g\|_\infty \\
& \quad + \frac{\alpha}{2}\|g\|_\infty \int_{\{u_n>0\}} |\nabla u_n|^{p(x)} \frac{g(u_n)}{\alpha}\exp(G(u_n))dxdt
\end{aligned}$$

$$\textit{then} \int_{\{u_n>0\}} g(u_n)|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt \leq C_4.$$

Similarly, taking $\varphi = \int_{u_n}^0 g(s)\chi_{\{s<0\}}ds$ as a test function in (4.6), we conclude that

$$\int_{\{u_n<0\}} g(u_n)|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt \leq C_5.$$

Consequently,

$$\int_Q g(u_n)|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt \leq C_6. \tag{4.10}$$

Above, $C_1, ..., C_6$ are constants independent of n, we deduce that

$$\int_Q |\nabla T_k(u_n)^+|^{p(x)} dxdt \leq k\ C_7. \tag{4.11}$$

Similarly to (4.11), we take $\varphi = T_k(u_n)^-\chi(0,\tau)$ in (4.6) to deduce that

$$\int_Q |\nabla T_k(u_n)^-|^{p(x)} dxdt \leq k\ C_8. \tag{4.12}$$

Combining (4.11), (4.12) and lemma 2.3, we conclude that

$$\int_0^T \min\Big\{\|\nabla T_k(u_n)\|_{p(.)}^{p^+}, \|\nabla T_k(u_n)\|_{p(.)}|^{p^-}\Big\}dt \leq \rho(\nabla T_k(u_n)) \leq kC_9.$$

$$\|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} \leq\ k\ C_{10}. \tag{4.13}$$

Where C_8, C_9, C_{10} are constants independent of n. Thus, $T_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$ independently of n for any $k>0$. Then, we deduce from (4.7), (4.8) and (4.13) that

$$\int_\Omega B_{k,G}^n(x,u_n(\tau))dx \leq \; kC. \tag{4.14}$$

Now we turn to proving the almost everywhere convergence of u_n and $b_n(x,u_n)$. Consider a non decreasing function $g_k \in C^2(\mathbb{R})$ such that

$$g_k(s)=\begin{cases} s & if\ |s|\leq \frac{k}{2} \\ k & if\ |s|\geq k \end{cases}$$

Multiplying the approximate equation by $g_k'(u_n)$, we get

$$\frac{\partial B_k^n(x,u_n)}{\partial t}-\operatorname{div}(a(x,t,u_n,\nabla u_n)g_k'(u_n))+a(x,t,u_n,\nabla u_n)g_k''(u_n)\nabla u_n$$

$$+H_n(x,t,u_n,\nabla u_n)g_k'(u_n)=f_n g_k'(u_n)-\operatorname{div}(Fg_k'(u_n))+Fg_k''(u_n)\nabla u_n. \tag{4.15}$$

where

$$B_k^n(x,z)=\int_0^z \frac{\partial b_n(x,s)}{\partial s}g_k'(s)ds.$$

As a consequence of (4.13), we deduce that $g_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(.)}$ $(\Omega))$ and $\frac{\partial B_k^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q)+V^*$. Due to the properties of g_k and (3.2), we conclude that $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $L^1(Q)+V^*$, which implies that $g_k(u_n)$ is compact in $L^1(Q)$.
Due to the choice of g_k, we conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q, which implies that u_n converges almost everywhere to some measurable function v in Q. Thus by using the same argument as in [9], [10], [21], we can show the following lemma.

Lemma 4.4. *Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$. Then,*

$$\begin{aligned} u_n &\rightarrow u \quad a.e.\ in\ Q, \\ b_n(x,u_n) &\rightarrow b(x,u) \quad a.e.\ in\ Q. \end{aligned}$$

We can deduce from (4.13) that

$$T_k(u_n)\rightharpoonup T_k(u) \quad in \quad L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)),$$

which implies, by using (3.3), that for all $k>0$ there exists $\varphi_k\in(L^{p'(.)}(Q))^N$ such that

$$a(x,t,u,T_k(u_n),\nabla T_k(u_n))\rightharpoonup\varphi_k \quad in \quad (L^{p'(.)}(Q))^N.$$

Remark 4.5. .
$b(.,u)$ it belongs to $L^\infty(0,T;L^1(\Omega))$.

Proof. Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$, passing to $\liminf$ in (4.14) as $n \to \infty$, we obtain

$$\frac{1}{k}\int_{\Omega} B_{k,G}(x,u(\tau))dx \leq C, \text{ for a.e. } \tau \text{ in } [0,\tau].$$

Due to the definition of $B_{k,G}(x,s)$ and the fact that $\frac{1}{k}B_{k,G}(x,s)$ converges pointwise to $\int_0^u sgn(s)\frac{\partial b(x,s)}{\partial s}\exp(G(s))ds \geq |b(x,u)|$ as $k \to \infty$, it follows that $b(.,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$. $\square$

Lemma 4.6. *Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$. Then,*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla(u_n))\nabla u_n dxdt = 0. \tag{4.16}$$

Proof. See Appendix.

STEP 3:**Almost everywhere convergence of the gradients** :
This step is devoted to prove the strong convergence of truncation of $T_k(u_n)$ that, we will use the following function of one real variable for $m > k$

$$h_m(s) = \begin{cases} 1 & if \quad |s| \leq m \\ 0 & if \quad |s| > m+1 \\ m+1+|s| & if \quad m \leq |s| \leq m+1. \end{cases}$$

Let $\psi_i \in D(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$
Set $\omega^i_\mu = (T_k(u))_\mu + e^{-\mu t}T_k(\psi_i)$ where $(T_k(u))_\mu$ is the mollification of $T_k(u)$ with respect to time. Note that ω^i_μ is a smooth function having the following properties:

$$\frac{\partial \omega^i_\mu}{\partial t} = \mu(T_k(u) - \omega^i_\mu), \quad \omega^i_\mu(0) = T_k(\psi_i), \quad |\omega^i_\mu| \leq k, \tag{4.17}$$

$$\omega^i_\mu \to T_k(u) \quad \text{in } L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \quad \text{as } \mu \to \infty. \tag{4.18}$$

The very definition of the sequence ω^i_μ makes it possible to establish the following lemma.

Lemma 4.7. *(See [20,6]). For $k \geq 0$, we have*

$$\int_{\{T_k(u_n)-\omega^i_\mu\geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))(T_k(u_n)-\omega^i_\mu)h_m(u_n)dxdt \geq \varepsilon(n,m,\mu,i).$$

Proposition 4.8. *The subsequence of u_n solution of problem $(\mathcal{P}_n)$ satisfies for any $k \geq 0$ following assertion*

$$\lim_{n\to\infty}\int_Q \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))\Big]\cdot\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]dxdt = 0.$$

Proof. See Appendix.
Thanks to the lemma (2.11), we have

$$T_k(u_n) \to T_k(u) \quad \text{strongly in} \quad L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \quad \forall k. \tag{4.19}$$

and

$$\nabla u_n \to \nabla u. \quad \text{a.e. in } Q, \text{which implies that}$$

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u)) \ \ in \ (L^{P'(.)}(Q))^N. \tag{4.20}$$

STEP 4: **Equi-Integrability of the non Linearity Sequence** :
We shall now prove that $H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u)$ *strongly in* $L^1(Q)$. by using Vitali's theorem. Since $H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u)$ *a.e. in* Q, considering now $\varphi = \rho_h(u_n) = \int_0^{u_n} g(s)\chi_{\{s>h\}}ds$ as a test function in (4.5), we obtain

$$\begin{aligned}
&\Big[\int_\Omega B_h^n(x,u_n)dx\Big]_0^T + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n g(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))dxdt \\
&\quad\le \Big(\int_h^\infty g(s)\chi_{\{s>h\}}ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)}\Big] \\
&\qquad + \int_Q F\nabla u_n g(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))dxdt \\
&\qquad + \Big(\int_h^\infty g(s)\chi_{\{s>h\}}ds\Big)\int_Q |F\nabla u_n|\frac{g(u_n)}{\alpha}\exp(G(u_n))\chi_{\{u_n>h\}}dxdt,
\end{aligned}$$

where $B_h^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho_h(s)\exp(G(s))ds$,
which implies, in view of $B_h^n(x,r) \ge 0$, (3.5) and Young's Inequality,

$$\begin{aligned}
&\alpha\int_{\{u_n>h\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
&\qquad\le \Big(\int_h^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}\|\gamma\|_{L^1(Q)} \\
&\qquad\quad + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big] + C'\int_h^\infty g(s)ds \\
&\qquad\quad + \frac{\alpha}{2}\int_{\{u_n>h\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
&\qquad\quad + \Big(\int_h^\infty g(s)ds\Big)\int_Q |F\nabla u_n|\frac{g(u_n)}{\alpha}\exp(G(u_n))dxdt
\end{aligned}$$

$$\text{hence} \quad \frac{\alpha}{2}\int_{\{u_n>h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt$$
$$\le\Big(\int_h^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}$$
$$+\|b(x,u_0\|_{L^1(\Omega)}+C'\Big]$$
$$+\Big(\int_h^\infty g(s)ds\Big)\int_Q|F\nabla u_n|\frac{g(u_n)}{\alpha}\exp(G(u_n))dxdt$$

and since $g\in L^1(\mathbb{R})$, we deduce that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}}|\nabla u_n|^{p(x)}g(u_n)dxdt=0.$$

Similarly, taking $\varphi=\rho_h(u_n)=\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds$ as a test function in (4.6), we conclude that, $\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}}|\nabla u_n|^{p(x)}g(u_n)dxdt=0$. Consequently, $\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}}|\nabla u_n|^{p(x)}g(u_n)dxdt=0$. Which implies, for h large enough and for a subset E of Q,

$$\lim_{measE\to 0}\int_E|\nabla u_n|^{p(x)}g(u_n)dxdt \le \|g\|_\infty\lim_{measE\to 0}\int_E|\nabla T_hu_n|^{p(x)}dxdt$$
$$+\int_{\{|u_n|>h\}}|\nabla u_n|^{p(x)}g(u_n)dxdt$$

so $g(u_n)|\nabla u_n|^{p(x)}$ is equi-integrable. Thus, we have shown that

$$g(u_n)|\nabla u_n|^{p(x)}\to g(u)|\nabla u|^{p(x)} \quad \text{stongly in} \quad L^1(Q).$$

consequently, by using (3.6), we conclude that

$$H_n(x,t,u_n,\nabla u_n)\to H(x,t,u,\nabla u) \quad \textit{strongly in} \quad L^1(Q). \quad \square \tag{4.21}$$

STEP 5: **Passing to the limit**:

a) Proof that u satisfies (3.8). For any fixed $m\ge 0$, we have

$$\int_{\{m\le|u_n|\le m+1\}}a(x,t,u_n,\nabla u_n)\nabla u_n dxdt$$
$$=\int_Q a(x,t,u_n,\nabla u_n)\Big[\nabla T_{m+1}(u_n)-\nabla T_m(u_n)\Big]dxdt$$
$$=\int_Q a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n)$$
$$-\int_Q a(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dxdt$$

According to (4.19) and (4.20), one can passing to the limit as $n \to \infty$ for fixed $m \geq 0$ to obtain

$$\begin{aligned}\lim_{n\to\infty} \int_{\{m\leq |u_n|\leq m+1\}} & a(x,t,u_n,\nabla u_n)\nabla u_n dxdt \\ &= \int_Q a(x,t,T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u) \\ &\quad - \int_Q a(x,t,T_m(u),\nabla T_m(u))\nabla T_m(u)dxdt \\ &= \int_{\{m\leq |u_n|\leq m+1\}} a(x,t,u,\nabla u)\nabla u dxdt \end{aligned} \tag{4.22}$$

Taking the limit as $m \to \infty$ in (4.22) and using the estimate (4.16) shows that u satisfies(3.8). □

b) Proof that u satisfies (3.9)

Let $S \in W^{2,\infty}(\mathbb{R})$ be such that S' has a compact support . Let $M > 0$ such that supp(S')$\subset [-M, M]$. Pointwise multiplication of the approximate problem $(\mathcal{P}_n)$ by $S'(u_n)$ leads to

$$\frac{\partial B^n_S(x,u_n)}{\partial t} - \operatorname{div}\Big[S'(u_n)a(x,t,u_n,\nabla u_n)\Big] + S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n$$

$$+ H_n(x,t,u_n,\nabla u_n)S'(u_n) = f_n S'(u_n) - \operatorname{div}\Big(S'(u_n)F\Big) + S''(u_n)F\nabla u_n \text{ in } D'(Q). \tag{4.23}$$

In what follows we pass to the limit in (4.23) as n tends to ∞.

• Limit of $\frac{\partial B^n_S(x,u_n)}{\partial t}$. Since S is bounded and continuous, $u_n \to u$ a.e. in Q implies that $B^n_S(x,u_n)$ converge to $B_S(x,u)$ a.e. in Q and L^∞ weakly .

$$Then, \quad \frac{\partial B^n_S(x,u_n)}{\partial t} \to \frac{\partial B_S(x,u)}{\partial t} \quad in\ D'(Q).\ as\ n \to \infty.$$

• Limit of $-\operatorname{div}\Big[S'(u_n)a(x,t,u_n,\nabla u_n)\Big]$. Since supp($S'$)$\subset [-M, M]$, we have, for $n \geq M$

$$S'(u_n)a(x,t,u_n,\nabla u_n) = S'(u_n)a(x,t,T_M(u_n),\nabla T_M(u_n))\ a.e.\ in\ Q.$$

The pointwise convergence of u_n to u and (4.20) and the boundedness of S' yied, as $n \to \infty$,

$$S'(u_n)a(x,t,u_n,\nabla u_n) \rightharpoonup S'(u)a(x,t,T_M(u),\nabla T_M(u))\ in\ (L^{p'(\cdot)}(Q))^N as\ n \to \infty \tag{4.24}$$

$S'(u)a(x,t,T_M(u),\nabla T_M(u))$ has been denoted by $S'(u)a(x,t,u,\nabla u)$ in equation (3.9).

• Limit of $S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n$ .Consider the "energy" term $S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n = S''(u_n)a(x,t,T_M(u_n),\nabla T_M(u_n))\nabla T_M(u_n)$ a.e. in Q. The pointwise convergence of $S'(u_n)$ to $S'(u)$ and (4.20) as $n \to \infty$ and the boundedness of S'' yield

$$S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n \rightharpoonup S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M(u) \text{ in } L^1(Q). \quad (4.25)$$

Recall that $S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M((u)) = S''(u)a(x,t,u,\nabla u)\nabla u$ a.e. in Q. • Limit of $S'(u_n)H_n(x,t,u_n,\nabla u_n)$. From supp($S'$)$\subset [-M,M]$ and (4.21), we have

$$S'(u_n)H_n(x,t,u_n,\nabla u_n) \to S'(u)H(x,t,u,\nabla u) \text{ strongly in } L^1(Q) \text{ as } k\ n \to \infty. \quad (4.26)$$

• Limit of $S'(u_n)f_n$. Since $u_n \to u$ a.e. in Q, we have $S'(u_n)f_n \to S'(u)f$ strongly in $L^1(Q)$, *as* $n \to \infty$
• Limit of div$(S'(u_n)F)S'(u_n)$ is bounded and converges to $S'(u)$ a.e. in Q.

then div$(S'(u_n)F) \to$ div$(S'(u)F)$ strongly in $L^{p'^-}(0,T;W^{-1,p'(\cdot)}(\Omega))$ as $n \to \infty$.

• Limit of $S''(u_n)F\nabla u_n$. This term is equal to $F\nabla S'(u_n)$.
Since $\nabla S'(u_n)$ converge to $\nabla S'(u)$ weakly in $(L^{p(\cdot)}(Q))^N$, we obtain
$S''(u_n)F\nabla u_n = F\nabla S'(u_n) \rightharpoonup F\nabla S'(u)$ weakly in $L^1(Q)$ as $n \to \infty$. The term $F\nabla S'(u)$ identifies with $S''(u)F\nabla u$.
As a consequence of the above convergence result, we are in a position to pass to the limit as $n \to \infty$ in equation (4.23) and to conclude that u satisfies (3.9). □

c) Proof that u satisfies (3.10)

S is bounded, and $B_S^n(x,u_n)$ is bounded in $L^\infty(Q)$. Secondly, by (4.23) we have $\frac{\partial B_S^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q)+V^*$.
As a consequence, an Aubin type Lemma (see, e.g, [17] implies that $B_S^n(x,u_n)$ lies in a compact set in $C^0([0,T],L^1(\Omega))$.
It follows that on the hand, $B_S^n(x,u_n)\mid_{t=0}= B_S^n(x,u_0^n)$ converge to $B_S(x,u)\mid_{t=0}$ strongly in $L^1(\Omega)$ implies that :$B_S(x,u)\mid_{t=0}= B_S(x,u_0)$ in Ω.
As a conclusion of Steps 1 to 5, the proof of theorem 4.1 is complete . □

5. APPENDIX

Proof of theorem 4.2

We define the operator $L_n : L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \to L^{p'^-}(0,T;W^{-1,p'(\cdot)}(\Omega))$ by
$\Big\langle L_n u, v\Big\rangle = \int_Q \frac{\partial b_n(x,u)}{\partial t} v dxdt = \int_Q \frac{\partial b_n(x,u)}{\partial u}\frac{\partial u}{\partial t} v dxdt \ \ \forall u,v \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ then,

$$\begin{aligned}
&\Big|\Big\langle L_n u, v\Big\rangle\Big| \leq \Big|\int_0^T\int_\Omega A_n(x)\frac{\partial u}{\partial t} v dxdt\Big| \\
&\leq \Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|A_n\|_{L^\infty}\int_0^T \|\frac{\partial u}{\partial t}\|_{L^{p'(x)}(\Omega)}\|v\|_{L^{p(x)}(\Omega)}dt \\
&\leq C\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|A_n\|_{L^\infty}\int_0^T \|\frac{\partial u}{\partial t}\|_{W^{-1,p'(\cdot)}(\Omega)}\|v\|_{W_0^{1,p(x)}(\Omega)}dt \\
&\leq C\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|A_n\|_{L^\infty}\|\frac{\partial u}{\partial t}\|_{L^{p'^-}(0,T,W^{-1,p'(\cdot)}(\Omega))}\|v\|_{L^{p^-}(0,T,W_0^{1,p(x)}(\Omega))} \\
&\qquad \leq C_1\|v\|_{L^{p^-}(0,T,W_0^{1,p(x)}(\Omega))}. \qquad (5.1)
\end{aligned}$$

We define the operator $G_n : L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \to L^{p^-}(0,T,W^{-1,p'(.)}(\Omega))$

$$by, \quad \Big\langle G_n u, v \Big\rangle = \int_Q H_n(x,t,u,\nabla u) v dx dt \quad \forall u, v \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)).$$

Thanks to the Hölder Inequality, we have that for $u, v \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$

$$\begin{aligned}
&\int_Q H_n(x,t,u,\nabla u) v dx dt \le \Big| \int_0^T \int_\Omega H_n(x,t,u,\nabla u) v dx dt \Big| \\
&\le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big) \int_0^T \Big(\int_\Omega \Big| H_n(x,t,u,\nabla u) \Big|^{p'(x)} dx \Big)^\theta \|v\|_{L^{p(x)}(\Omega)} dt \\
&\le C\Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big) \int_0^T (n^{\theta p'^+} (meas\Omega)^\theta \|v\|_{W_0^{1,p(x)}(\Omega)} dt \\
&\le C_2 \|v\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))}. \qquad (5.2)
\end{aligned}$$

with $\theta = \begin{cases} 1/p'^- & if \quad \|H_n(x,t,u,\nabla u)\|_{L^1(Q)} > 1 \\ 1/p'^+ & if \quad \|H_n(x,t,u,\nabla u)\|_{L^1(Q)} \le 1. \end{cases}$

Lemma 5.1. *Let $B_n : L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \to L^{p'^-}(0,T,W^{-1,p'(.)}(\Omega))$.*
The operator $B_n = A + G_n$ is
a) coercive
b) pseudo-monotone
c) bounded and demi continuous.

Proof. a) For the coercivity, we have for any $u \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$

$$\begin{aligned}
&\Big\langle B_n u, u \Big\rangle = \Big\langle G_n u, u \Big\rangle + \Big\langle Au, u \Big\rangle \\
&\Rightarrow \Big\langle B_n u, u \Big\rangle - \Big\langle G_n u, u \Big\rangle = \Big\langle Au, u \Big\rangle \\
&\quad then, \ \Big\langle B_n u, u \Big\rangle - \Big\langle G_n u, u \Big\rangle = \int_Q a(x,t,u,\nabla u) \nabla u dx dt \\
&= \int_0^T \int_\Omega a(x,t,u,\nabla u) \nabla u dx dt \\
&\ge \int_0^T \alpha\Big(\int_\Omega |\nabla u|^{p(x)} dx\Big) dt \quad \text{(using (3.5))} \\
&\ge \alpha \|\nabla u\|^\delta_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} \ge \beta \|u\|^\delta_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))},
\end{aligned}$$

which is due to Poincaré Inequality with
$\delta = \begin{cases} p_- & if \ \|\nabla u\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} > 1 \\ p_+ & if \ \|\nabla u\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} \le 1 \end{cases}$

$$hence, \ \Big\langle B_n u, u \Big\rangle - \Big\langle G_n u, u \Big\rangle \ge \beta \|u\|^\delta_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))}$$

$$then, \ \Big\langle B_n u, u \Big\rangle \ge \beta \| u\|^\delta_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} - C_2 \|u\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))}$$

then, we have

$$\frac{\Big\langle B_n u, u\Big\rangle}{\|u\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))}} \geq \beta\|u\|^{\delta-1}_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))} - C_2 \to +\infty$$

$$\Rightarrow \frac{\Big\langle B_n u, u\Big\rangle}{\|u\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))}} \to +\infty \quad as\ \|u\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))} \to +\infty$$

then B_n is coercive. □

b)It remains to show that B_n is pseudo–monotone.

Let $(u_k)_k$ a sequence in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ such that

$$\begin{aligned} u_k &\rightharpoonup u \text{ in } L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \\ L_n u_k &\rightharpoonup L_n u \text{ in } L^{p'^-}(0,T;W^{-1,p'(\cdot)}(\Omega)) \\ \lim_{k\to\infty}\sup\Big\langle B_n u_k, u_k - u\Big\rangle &\leq 0 \end{aligned} \tag{5.3}$$

that, we have prove that

$$B_n u_k \rightharpoonup B_n u \quad in \quad L^{p'^-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \quad and\ \langle B_n u_k, u_k\rangle \to \langle B_n u, u\rangle.$$

By the definition of the operator L_n defined in definition (2.1), we obtain that u_k is bounded in $W_0^{1,p(\cdot)}(\Omega)$ and since $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p'(\cdot)}(\Omega)$ then $u_k \to u$ in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, then the growth condition (3.3) $(a(x,t,u_k,\nabla u_k))_k$ is bounded in $(L^{p'(\cdot)}(Q))^N$ therefore, there exists a function $\varphi \in (L^{p'(\cdot)}(Q))^N$ such that

$$a(x,t,u_k,\nabla u_k) \rightharpoonup \varphi\ as\ k \to +\infty. \tag{5.4}$$

Similarly, using condition (3.6) $\Big(H_n(x,t,u_k,\nabla u_k)\Big)_k$ is bounded in $(L^1(Q))$ then, there exists a function $\psi_n \in L^1(Q)$ such that

$$H_n(x,t,u_k,\nabla u_k) \to \psi_n \text{ in } L^1(Q)\ as\ k \to +\infty. \tag{5.5}$$

$$\begin{aligned} \lim_{k\to\infty}\Big\langle B_n u_k, u_k\Big\rangle &= lim_{k\to\infty}\Big[\Big\langle G_n u_k, u_k\Big\rangle + \Big\langle A u_k, u_k\Big\rangle\Big] \\ &= \lim_{k\to\infty}\Big[\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt + \int_Q H(x,t,u_k,\nabla u_k)u_k dxdt\Big] \\ &= \int_Q \varphi\nabla u_k dxdt + \int_Q \psi_n u_k dxdt \end{aligned} \tag{5.6}$$

using (5.3) and, (5.6), we obtain

$$\begin{aligned} \lim_{k\to\infty}\sup\Big\langle B_n u_k, u_k\Big\rangle &= \lim_{k\to\infty}\sup\Big\{\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt \\ &+ \int_Q H(x,t,u_k,\nabla u_k)u_k dxdt\Big\} \\ &\leq \int_Q \varphi\nabla u dxdt + \int_Q \psi_n u dxdt \end{aligned} \tag{5.7}$$

thanks to (5.5), we have

$$\int_Q H_n(x,t,u_k,\nabla u_k)dxdt \to \int_Q \psi_n dxdt. \tag{5.8}$$

therefore,

$$\lim_{k\to\infty} \sup \int_Q a(x,t,u_k,\nabla u_k)\nabla u_k \leq \int_Q \varphi \nabla u dxdt \tag{5.9}$$

on the other hand, using (3.4), we have

$$\int_Q \Big[a(x,t,u_k,\nabla u_k) - a(x,t,u_k,\nabla u)\Big](\nabla u_k - \nabla u)dxdt \geq 0. \tag{5.10}$$

Then,

$$\begin{aligned}\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt \quad &\geq \quad -\int_Q a(x,t,u_k,\nabla u)\nabla u dxdt \\ &+\int_Q a(x,t,u_k,\nabla u_k)\nabla u dxdt \\ &+\int_Q a(x,t,u_k,\nabla u)\nabla u_k dxdt\end{aligned}$$

and by (5.4), we get

$$\lim_{k\to\infty} \inf \int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt \geq \int_Q \varphi \nabla u dxdt.$$

this implies, thanks to (5.9), that

$$\lim_{k\to\infty} \int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt = \int_Q \varphi \nabla u dxdt \tag{5.11}$$

Now by (5.11), we can obtain

$$\lim_{k\to\infty} \int_Q a(x,t,u_k,\nabla u_k) - a(x,t,u_k,\nabla u))(\nabla u_k - \nabla u)dxdt = 0$$

In view of the lemma 2.11, we obtain

$$\begin{array}{rcll} u_k & \to & u \quad in & L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \\ \nabla u_k & \to & \nabla u \quad a.e.\ in & Q. \end{array}$$

Then,

$$\begin{array}{rll} a(x,t,u_k,\nabla u_k) \rightharpoonup a(x,t,u,\nabla u) & in & (L^{p'(.)}(Q))^N \\ H_n(x,t,u_k,\nabla u_k) \rightharpoonup H(x,t,u,\nabla u) & in & L^1(Q), \end{array}$$

we deduce that

$$Au_k \rightharpoonup Au \quad in \quad (L^{p'^-}(Q))^N$$

and

$$G_n u_k \rightharpoonup G_n u \quad in \quad (L^1(Q))$$

which implies

$$B_n u_k \rightharpoonup B_n u \quad in \quad L^{p'^-}(0,T;W_0^{1,p(.)}(\Omega))$$

and

$$\Big\langle B_n u_k, u_k \Big\rangle \to \Big\langle B_n u, u \Big\rangle$$

completing the proof of assertion(b). $\square$

c) Using $H\ddot{o}lder's$ inequality and the growth condition (3.3), we can show that the operator$A$ is bounded and by using (5.2), we conclude that B_n is bounded. For to show that B_n is demicontinuous

Let $u_k \to u$ in $L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$ and prove that

$$\Big\langle B_n u_k, \psi \Big\rangle \to \langle B_n u, \psi\rangle \quad for\ all\ \psi \in\ L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)).$$

Since $a(x,t,u_k,\nabla u_k) \to a(x,t,u,\nabla u)$ as $k \to \infty$ a.e. in Q. Then, by the growth condition (3.3) and lemma 2.12

$$a(x,t,u_k,\nabla u_k) \rightharpoonup a(x,t,u,\nabla u)\ in \quad L^{p'(.)}(Q))^N$$

and for all $\varphi \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$, $\langle Au_k, \varphi\rangle \to \langle Au, \varphi\rangle$ as $k \to \infty$
similarly, $G_n u_k \to G_n u$ as $k \to \infty$ a.e. in Q then, by the (3.6) and lemma 2.12 $G_n u_k \rightharpoonup G_n u$ in $L^{p'(.)}(Q)$ and for all $\phi \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$,
$\Big\langle G_n u_k, \phi \Big\rangle \to \Big\langle G_n u, \phi \Big\rangle$ as $k \to \infty$ which implies B_n is demi continuous. $\square$

Proof of lemma 4.6.
Set $\varphi = T_1(u_n - T_m(u_n))^+ = \alpha_m(u_n)$ in (4.5), this function is admissible since $\varphi \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$ and $\varphi \geq 0$. Then, we have

$$\begin{aligned}
&\int_Q \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))\alpha_m(u_n)dxdt \\
&\qquad + \int_{\{m \le u_n \le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))dxdt \\
&\qquad \le \int_Q |\gamma(x,t)| \exp(G(u_n))\alpha_m(u_n)dxdt + \int_Q |f_n| \exp(G(u_n))\alpha_m(u_n)dxdt \\
&\qquad + \int_Q F\nabla u_n \frac{g(u_n)}{\alpha} \exp(G(u_n))\alpha_m(u_n)dxdt \\
&\qquad + \int_{\{m \le u_n \le m+1\}} F\nabla u_n \exp(G(u_n))dxdt.
\end{aligned}$$

This gives, by setting $B^m_{n,G}(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s} \exp(G(s))\alpha_m(s)ds$ and by Young's Inequality,

$$\begin{aligned}
&\int_\Omega B^m_{n,G}(x,u_n)(T)dx \quad + \quad \int_{\{m \le u_n \le m+1\}} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla u_n dxdt \\
&\le \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\int_{\{|u_n|>m\}} (|\gamma| + |f_n|)dxdt + \int_{\{|u_{0n}|>m\}} |b_n(x,u_{0n})|dx\Big]dxdt \\
&+C_1 \int_{\{u_n \ge m\}} |F|^{p'(x)}dxdt + \frac{\alpha}{2}\int_{\{m \le u_n \le m+1\}} |\nabla u_n|^{p(x)} \exp(G(u_n))dxdt \\
&+C_2 \int_{\{u_n \ge m\}} |F|^{p'(x)}dxdt + \frac{\alpha}{2}\int_{\{|u_n|>m\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt.
\end{aligned}$$

Since $B^m_{n,G}(x,u_n)(T) > 0$ and use (3.5), we obtain

$$\begin{aligned}
\frac{1}{2}\int_{\{m\le u_n\le m+1\}} & a(x,t,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))dxdt \\
&\le \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\int_{\{|u_n|>m\}}(|\gamma)|+|f_n|)dxdt. \\
&\quad+\int_{\{|u_{0n}|>m\}}|b_n(x,u_{0n})|dx\Big]+C_3\int_{\{u_n>m\}}|F|^{p'(x)}dxdt \\
&\quad+C_4\int_{\{u_n>m\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt
\end{aligned} \tag{5.12}$$

Taking $\varphi=\rho_m(u_n)=\int_0^{u_n} g(s)\chi_{\{s>m\}}ds$ as a test function in (4.5), we obtain

$$\begin{aligned}
\Big[\int_\Omega B^m_{m,n}(x,u_n)dx\Big]_0^T &+ \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))g(u_n)\chi_{\{u_n>m\}}dxdt \\
\le \Big(\int_m^\infty g(s)\chi_{\{s>m\}}ds\Big) &\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}\Big] \\
+\int_Q F\nabla u_n g(u_n)&\chi_{\{u_n>m\}}\exp(G(u_n))dxdt \\
+\Big(\int_m^\infty g(s)\chi_{\{s>m\}}ds&\int_Q F\nabla u_n\frac{g(u_n)}{\alpha}\chi_{\{u_n>m\}}\exp(G(u_n))dxdt
\end{aligned}$$

where $B^m_{m,n}(x,r)=\int_0^r\frac{\partial b_n(x,s)}{\partial s}\rho_m(s)\exp(G(s))ds$, which implies, since $B^m_{m,n}(x,r)\ge 0$, by (3.5) and Young's Inequality,

$$\begin{aligned}
\Big(\frac{\alpha-1}{2}\Big)\int_{\{u_n>m\}} & |\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt \\
&\le \Big(\int_m^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} \\
&\quad+\|\gamma\|_{L^1(Q)}+\|b_n(x,u_{0n}\|_{L^1(\Omega)}+C_5\Big]
\end{aligned} \tag{5.13}$$

Using (5.13) and the strong convergence of f_n in $L^1(\Omega)$ and $b_n(x,u_{0n})$ in $L^1(\Omega)$ $\gamma\in L^1(\Omega)$, $g\in L^1(\mathbb{R})$ and $F\in(L^{p'(\cdot)}(Q))^N$, by Lebesgue's theorem, passing to limit in (5.12), we conclude that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le u_n\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt=0. \tag{5.14}$$

On the other hand, taking $\varphi=T_1(u_n-T_m(u_n))^-$ as a test function in (4.6) and reasoning as in the proof (5.14), we deduce that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\le u_n\le -m\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt=0. \tag{5.15}$$

By using (5.14) and (5.15), we have

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le |u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt=0. \quad\Box \tag{5.16}$$

Proof of Proposition 4.2
For $m > k$, let $\varphi = (T_k(u_n) - \omega_\mu^i)^+ h_m(u_n) \in L^{p^-}(0,T; W_0^{1,p(.)}(\Omega)) \cap L^\infty(Q)$ and $\varphi \geq 0$. If we take this function in (4.5), we obtain

$$
\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))(T_k(u_n) - w_\mu^i) h_m(u_n) dxdt \\
&+ \int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,u_n,\nabla u_n) \nabla (T_k(u_n) - w_\mu^i) h_m(u_n) dxdt \\
&- \int_{\{m\leq u_n\leq m+1\}} \exp(G(u_n)) a(x,t,u_n,\nabla u_n) \nabla u_n (T_k(u_n) - w_\mu^i)^+ dxdt \\
&\leq \int_Q (f_n + \gamma) \exp(G(u_n))(T_k(u_n) - w_\mu^i)^+ h_m(u_n) dxdt \qquad (5.17)\\
&+ \int_Q F \nabla u_n \frac{g(u_n)}{\alpha} \exp(G(u_n))(T_k(u_n) - \omega_\mu^i)^+ h_m(u_n) dxdt \\
&+ \int_{\{T_k(u_n)-\omega_\mu^i\geq 0\}} F \exp(G(u_n))(T_k(u_n) - \omega_\mu^i) h_m(u_n) dxdt \\
&- \int_{\{m\leq u_n\leq m+1\}} F \exp(G(u_n))(T_k(u_n) - \omega_\mu^i)^+ dxdt
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\Big| \int_{\{m\leq u_n\leq m+1\}} \exp(G(u_n)) a(x,t,u_n,\nabla u_n) \nabla u_n (T_k(u_n) - w_\mu^i)^+ dxdt \Big| \\
&\quad \leq 2k \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big) \int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n) \nabla u_n dxdt.
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big| \int_{\{m\leq u_n\leq m+1\}} F \nabla u_n \exp(G(u_n))(T_k(u_n) - \omega_\mu^i)^+ dxdt \Big| \\
&\leq 2k \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big) \frac{\|F\|_{L^{p'(.)}(Q))^N}}{\alpha^{\frac{1}{p^-}}} \Big(\int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla(u_n)) \nabla u_n dxdt \Big)^{\frac{1}{p^-}}
\end{aligned}
$$

Tanks to (4.16) the third and fourth integrals on the right hand side tend to zero as n and m tend to infinity and by Lebesgue's theorem and $F \in (L^{p'(.)}(Q))^N$, we deduce that the right hand side converges to zero as n, m and μ tend to infinity . Since

$$
\Big(T_k(u_n) - \omega_\mu^i\Big)^+ h_m(u_n) \rightharpoonup \Big(T_k(u) - \omega_\mu^i\Big)^+ h_m(u) \; in \; L^\infty(Q) \; as \; n \to \infty
$$

and strongly in $L^{p^-}(0,T; W_0^{1,p(.)}(\Omega))$ and $(T_k(u_n) - \omega_\mu^i)^+ h_m(u_n) \rightharpoonup 0$ *in* $L^\infty(Q)$ and strongly in $L^{p^-}(0,T; W_0^{1,p(.)}(\Omega))$ *as* $\mu \to \infty$, it follows that the first and second integrals on the right-hand side of (5.17) converge to zeros as n, m, $\mu \to \infty$, using [3] lemma 4.7 and lemma 2.11 the proof of Proposition 4.2 is complete. □

6. Example

Consider the following special case : $b(x,s) = F(x)K(s)$, where $F \in W^{1,p(.)}(\Omega)$ with $p(x) = \sin|x| + 3$, $p \in C_+(\bar{\Omega})$ and $K \in C^1(\mathbb{R}), K(0) = 0$
b is a Carathéodory function satisfing the following assertions :
$b(x,0) = 0$. Next, for any $k > 0$, there exist $\lambda_k > 0$ and function $A_k \in L^\infty(\Omega)$ $B_k \in L^{p(.)}(\Omega)$ such that

$$\lambda_k = \inf_{|s|\leq k} K'(s) \leq \frac{\partial b(x,s)}{\partial s} \leq A_k(x) \text{ and } \left|D_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\right| \leq B_k(x). \tag{6.1}$$

for almost every $x \in \Omega$ and every s such that $|s| \leq k$, we have

$$Au = -\triangle_{p(x)} = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u). \tag{6.2}$$

we are $(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)(u-v) > 0$ for almost all $x \in \Omega$, $u, v \in \mathbb{R}^{\mathbb{N}}$ and $u \neq v$ then the monotonicity condition is satisfying.
The operator $-div(|\nabla u|^{p(x)-2}\nabla u)$ is a Carathéodory function satisfing the growth condition (3.3) and the coercivity (3.5).

$$H(x,t,u,\nabla u) = \frac{-u}{2+u^4}|\nabla u|^{p(x)} + \gamma(x,t). \tag{6.3}$$

where $\gamma \in L^1(Q)$, $H(x,t,u,\nabla u)$ is a Carathéodory function and

$$\begin{aligned} |H(x,t,u,\nabla u)| &\leq \frac{|u|}{2+u^4}|\nabla u|^{p(x)} + \gamma(x,t) \\ &= g(u)|\nabla u|^{p(x)} + \gamma(x,t), \end{aligned}$$

where $g(u) = \frac{|u|}{2+u^4}|$ is bounded positive continuous function which belongs to $L^1(\mathbb{R})$. Note that $H(x,t,u,\nabla u)$ does not satisfy the sign condition or the coercivity condition. Finally, the hypotheses of Theorem 4.1 are satisfied. Therefore, the problem $(\mathcal{P})$ has at least one renormalized solution.

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Youssef Akdim, Nezha El gorch and Mounir Mekkour
Sidi Mohamed Ben Abdellah University,
Laboratory LSI Poly-Disciplinary Faculty of Taza
P.O. Box 1223, Taza Gare, Morocco.
E-mail address: akdimyoussef@yahoo.fr
E-mail address: nezhaelgorch@gmail.com
E-mail address: mekkour.mounir@yahoo.fr