$g_{*\lambda_\mu}$−closed sets in Generalized Topological Spaces

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ABSTRACT: In this paper we introduce some new classes of generalized closed sets called $^*\lambda_\mu - g$−closed, $^*\lambda_\mu - g_\mu$−closed and $g_{*\lambda_\mu}$−closed sets in generalized topological spaces, which are related to the classes of g_μ−closed sets, $g - \lambda_\mu$−closed sets and $\lambda_\mu - g$−closed sets. We investigate the properties of the newly introduced classes, as well as the connections among the above mentioned classes of generalized closed sets. Also, we give a unified framework for the study of several types of generalized closed sets in a space endowed with two generalized topologies.

Key Words: Generalized topology, λ_μ−closed, $^*\lambda_\mu$−closed, $^*\lambda_\mu - g$−closed, $^*\lambda_\mu - g_\mu$−closed and $g_{*\lambda_\mu}$−closed sets.

Contents

1. Introduction

In 1997, $\acute{A}.Cs\acute{a}sz\acute{a}r$ [2] introduced the concept of generalization of topological space, which is a generalized topological space. A subset μ of $exp(X)$ is called a generalized topology [4] on X if $\emptyset \in \mu$ and μ is closed under arbitrary union. Elements of μ are called μ−open sets. A set X with a GT μ on it is said to be a generalized topological space (briefly GTS) and is denoted by (X, μ). For a subset A of X, we denote by $c_\mu(A)$ the intersection of all closed sets containing A and by $i_\mu(A)$ the union of all μ−open sets contained in A. Then $c_\mu(A)$ is the smallest closed set containing A and $i_\mu(A)$ is the largest μ−open set contained in A. A point $x \in X$ is called a μ−cluster point of A if for every $U \in \mu$ with $x \in U$ we have $A \cap U \neq \emptyset$. It is known from [4] that $c_\mu(A)$ is the set of all μ−cluster points of A. A GTS (X, μ) is called a quasi-topological space [3] if μ is closed under finite intersections. A subset A of X is said to be π−regular [5] (resp. σ−regular) if $A = i_\mu c_\mu(A)$ (resp. $A = c_\mu i_\mu(A)$).

Definition 1.1. [6] *Let (X, μ) be a GTS and $A \subset X$. Then the subsets $\wedge_\mu(A)$ and $\vee_\mu(A)$ are defined as follows:*

$$\wedge_\mu(A) = \begin{cases} \cap\{G : A \subseteq G, G \in \mu\} & \text{if there exists } G \in \mu \text{ such that } A \subseteq G; \\ X & \text{otherwise.} \end{cases}$$

and

$$\vee_\mu(A) = \begin{cases} \cup\{H : H \subseteq A, H^c \in \mu\} & \text{if there exists } H \text{ such that } H^c \in \mu \text{ and } H \subseteq A; \\ \emptyset & \text{otherwise.} \end{cases}$$

Definition 1.2. [1] *A subset A of a GTS (X, μ) is said to be a $(\wedge, \mu)-$closed (briefly $\lambda_\mu-$closed) set if $A = T \cap C$, where T is a $\wedge_\mu-$set and C is a $\mu-$closed set. The complement of a $\lambda_\mu-$closed set is called a $\lambda_\mu-$open set.*

Given a GTS (X, μ), we denote the class of $\mu-$closed sets by $C(X, \mu)$ and the class of $\wedge_\mu-$sets by $\wedge_\mu(X, \mu)$. Moreover, following [1] we denote the class of $\lambda_\mu-$closed sets (resp. $\lambda_\mu-$open sets) by $\wedge_\mu - C(X, \mu)$ (resp. $\wedge_\mu - O(X, \mu)$). It is known that $\mu \subset \wedge_\mu(X, \mu)$ and $\wedge_\mu(X, \mu)$ is a GT closed to arbitrary intersections, by Proposition 2.3 of [6] and Theorem 2.7, Remark 2.8 [6]. Note that $X \in \wedge_\mu(X, \mu) \cap C(X, \mu)$ by Theorem2.7 [6]. By Theorem 2.6 and 2.7 [1], we have

$$\wedge_\mu - C(X, \mu) \supset \wedge_\mu(X, \mu) \cup C(X, \mu)$$

and $\wedge_\mu - C(X, \mu)$ is closed to arbitrary intersections, hence $\wedge_\mu - O(X, \mu)$ is a GT. From $\mu \subset \wedge_\mu - C(X, \mu)$ and $C(X, \mu) \subset \wedge_\mu - C(X, \mu)$ we conclude that $\mu-$closed sets and $\mu-$open sets are simultaneously $\lambda_\mu-$closed and $\wedge_\mu-$open, i.e.

$$\mu \cup C(X, \mu) \subset \wedge_\mu - C(X, \mu) \cap \wedge_\mu - O(X, \mu).$$

Definition 1.3. [1] *Let (X, μ) be a GTS. A point $x \in X$ is called a $(\wedge, \mu)-$cluster point of A if for every $(\wedge, \mu)-$ open set U of X containing x, we have $A \cap U \neq \emptyset$. The set of all $(\wedge, \mu)-$ cluster points of A is called the $(\wedge, \mu)-$closed of A and is denoted by $A^{(\wedge, \mu)}$.*

Lemma 1.4. [1] *Let (X, μ) be a GTS. Then the following properties hold:*
(a). $A \subseteq A^{(\wedge, \mu)}$;
(b). $A^{(\wedge, \mu)} = \cap\{F : A \subseteq F and F is (\wedge, \mu) - closed\}$;
(c). $A \subseteq B \Rightarrow A^{(\wedge, \mu)} \subseteq B^{(\wedge, \mu)}$;
(d). A is $(\wedge, \mu)$- closed iff $A = A^{(\wedge, \mu)}$;
(e). $A^{(\wedge, \mu)}$ is $(\wedge, \mu)-$ closed.

Also, throughout the paper , it should be "$A^{(\wedge, \mu)}$" instead "$c_{\lambda_\mu}(A)$"

Definition 1.5. [10] *A subset A of a GTS (X, μ) is said to be $* \wedge_\mu -$set if $A =^* \wedge_\mu(A)$ where $^* \wedge_\mu(A) = \cap\{U \subset X | A \subset U$ and U is a $\lambda_\mu-$open$\}$. Denote the class of $^* \wedge_\mu -$sets by $^* \wedge_\mu(X, \mu)$.*

We have $\wedge_\mu - O(X, \mu) \subset^* \wedge_\mu(X, \mu)$ and $X \in^* \wedge_\mu(X, \mu)$. Since $^* \wedge_\mu (\cup_{i \in I} A_i) = \cup_{i \in I}^* \wedge_\mu(A_i)$ whenever A_i, $i \in I$, are subsets of X, the class $^* \wedge_\mu(X, \mu)$ is a generalized topology. Since $^* \wedge_\mu (\cap_{i \in I} A_i) \subset \cap_{i \in I}^* \wedge_\mu(A_i)$ whenever A_i, $i \in I$, are subsets of X, the class $^* \wedge_\mu(X, \mu)$ is closed to arbitrary intersections.

Definition 1.6. [10] *A subset A of a GTS (X, μ) is said to be a $^*\lambda_\mu$ −closed set if $A = T \cap C$, where T is a $^* \wedge_\mu$ −set and C is a λ_μ −closed set. The complement of a $^*\lambda_\mu$ −closed set is called a $^*\lambda_\mu$ −open set.*

Denote the class of $^*\lambda_\mu$ −closed sets (resp. $^*\lambda_\mu$ −open sets) by $^* \wedge_\mu -C(X, \mu)$ (resp. $^* \wedge_\mu -O(X, \mu)$).

The class of $^*\lambda_\mu$ −closed sets is closed to arbitrary intersections, since $^* \wedge_\mu (X, \mu)$ and $\wedge_\mu - C(X, \mu)$ have this property. In addition, X is a $^*\lambda_\mu$ −closed set. Therefore, the class of $^*\lambda_\mu$ −open sets is a generalized topology.

Lemma 1.7. *For a GTS (X, μ), $\wedge_\mu (X, \mu) \subset^* \wedge_\mu (X, \mu) \subset^* \wedge_\mu - C(X, \mu)$.*

Proof: a) Since every μ −open set is λ_μ −open, we have $^* \wedge_\mu (A) \subset \wedge_\mu (A)$ for every $A \subset X$. Now assume that A is a $\wedge_\mu$ −set, i.e. $\wedge_\mu (A) \subset A$. Then $^* \wedge_\mu (A) \subset A$, hence A is a $^* \wedge_\mu$-set.
b) Let $B \in^* \wedge_\mu (X, \mu)$. Writing $B = T \cap C$, where $T = B$ is a $^* \wedge_\mu$ −set and $C = X$ is a λ_μ −closed set, we see that $B \in^* \wedge_\mu - C(X, \mu)$. □

Theorem 1.8. [10] *For a GTS (X, μ), λ_μ −closed sets and λ_μ −open sets are $^*\lambda_\mu$ −closed and $^*\lambda_\mu$ −open, i.e.*
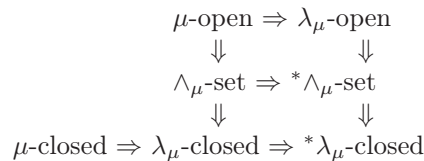
$$\wedge_\mu - C(X, \mu) \cup \wedge_\mu - O(X, \mu) \subset^* \wedge_\mu - C(X, \mu) \cap^* \wedge_\mu - O(X, \mu).$$

Proof. a) Let $A \in \wedge_\mu - C(X, \mu)$. Writing $A = T \cap C$, where $T = X$ is a $^* \wedge_\mu$ −set and $C = A$ is a λ_μ −closed set, we see that $A \in^* \wedge_\mu - C(X, \mu)$.
b) Since $\wedge_\mu - O(X, \mu) \subset^* \wedge_\mu (X, \mu) \subset^* \wedge_\mu - C(X, \mu)$, all the λ_μ −open sets are $^*\lambda_\mu$ −closed.
c) By a) and b), λ_μ −closed sets and λ_μ −open sets are $^*\lambda_\mu$ −closed, hence they are $^*\lambda_\mu$ −open.

The above disscusion can be summarized in the following diagram:

$$\mu\text{-open} \Rightarrow \lambda_\mu\text{-open}$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$\wedge_\mu\text{-set} \Rightarrow {}^*\wedge_\mu\text{-set}$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$\mu\text{-closed} \Rightarrow \lambda_\mu\text{-closed} \Rightarrow {}^*\lambda_\mu\text{-closed}$$

From the above diagram, we have the following observation, which are used in the subsequent chapters.

Observation 1.9. *For a GTS (X, μ), the following hold:*
i). Since X is μ −closed and Φ is a $\wedge_\mu$ −set, X and Φ are λ_μ −closed.
ii). μ −closed sets and μ −open sets are λ_μ −closed and λ_μ −open.
iii). Every λ_μ −open set is a $^* \wedge_\mu -$ set.
iv). λ_μ −closed sets and λ_μ −open sets are $^*\lambda_\mu$ −closed and $^*\lambda_\mu$ −open.

Definition 1.10. *A subset A of a GTS (X, μ) is called g_μ −closed [11] (resp. $g - \lambda_\mu$ −closed [9], $\lambda_\mu - g$ −closed [9]) if $c_\mu(A) \subset U$ (resp. $c_{\lambda_\mu}(A) \subset U$, $c_\mu(A) \subset U$) whenever $A \subset U$ and U is μ-open (resp. U is μ-open, U is λ_μ-open) in (X, μ).*

For $A \subset X$ we denote by $c_{*\lambda_\mu}(A)$ the intersection of all $^*\lambda_\mu-$closed subsets of X containing A,we have

$$c_{*\lambda_\mu}(A) \subset c_{\lambda_\mu}(A) \subset c_\mu(A)$$

for every $A \subset X$.

The purpose of this present paper is to define some new class of generalized closed sets called $^*\lambda_\mu-g-$closed, $^*\lambda_\mu-g_\mu-$closed and $g_{*\lambda_\mu}-$closed sets and also we obtain some basic properties of these closed sets in generalized topological spaces. Moreover the relations between these classes of sets are established.

2. $^*\lambda_\mu - g-$closed sets

Definition 2.1. *Let (X, μ) be a GTS. A subset A of X is called $^*\lambda_\mu - g-closed$ set if $c_\mu(A) \subset U$ whenever $A \subset U$ and U is $^*\lambda_\mu-open$ in X. The complements of a $^*\lambda_\mu - g-closed$ sets are called $^*\lambda_\mu - g-open$ sets.*

Theorem 2.2. *Every $\mu-closed$ set is a $^*\lambda_\mu - g-closed$ set.*

Proof: Let A be μ-closed set and U be any $^*\lambda_\mu-$open set containing A. Since A is μ-closed, $c_\mu(A) = A$. Therefore $c_\mu(A) \subset U$ and hence A is $^*\lambda_\mu - g-$closed.

Example 2.3 shows that the converse of the above theorem is not true.

Example 2.3. *Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a, b\}, X\}$. Then $\{a, c\}$ is $^*\lambda_\mu - g-closed$ but not$\mu-closed$.*

Theorem 2.4 shows that every $\lambda_\mu^* - g-$closed set is a $g_\mu-$closed set (a $g - \lambda_\mu-$closed set, a $\lambda_\mu - g-$closed set) and Example 2.5 shows that no converse is **true.**

Theorem 2.4. *Let(X, μ) be a GTS. Then the following hold:*
a). Every $^*\lambda_\mu - g-closed$ set is a $g_\mu-closed$ set.
b). Every $^*\lambda_\mu - g-closed$ set is a $g - \lambda_\mu-closed$ set.
c). Every $^*\lambda_\mu - g-closed$ set is a $\lambda_\mu - g-closed$ set.

Proof: a). Let A be a $^*\lambda_\mu - g-$closed set and U be an $\mu-$open set containing A in(X, μ). Since every μ- open set is $^*\lambda_\mu-$open and A is $^*\lambda_\mu - g-$closed, $c_\mu(A) \subset U$. Therefore A is $g_\mu-$closed.
b). Let A be a $^*\lambda_\mu - g-$closed set and U be an $\mu-$open set containing A in(X, μ). From the above part, $c_\mu(A) \subset U$. Since $c_{\lambda_\mu}(A) \subset c_\mu(A)$,$c_{\lambda_\mu}(A) \subset U$ and hence A is $g - \lambda_\mu-$closed.
c). Let A be a $^*\lambda_\mu - g-$closed set and U be an $\lambda_\mu-$open set containing A in(X, μ). Since every $\lambda_\mu-$open set is $^*\lambda_\mu-$open and A is $^*\lambda_\mu - g-$closed, $c_\mu(A) \subset U$. Therefore A is $\lambda_\mu - g-$closed.

Example 2.5. *Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a, b, c\}\{b, c, d\}, X\}$. If $A = \{a, b, d\}$ then A is both g_μ- closed and $g - \lambda_\mu-closed$ but not $^*\lambda_\mu - g-closed$. Further if $B = \{a, c, d\}$ is $\lambda_\mu - g-closed$ but not $^*\lambda_\mu - g-closed$.*

Theorem 2.6 gives a characterization of $^*\lambda_\mu - g-$closed sets.

Theorem 2.6. *Let (X,μ) be a GTS. Then a subset A of X is $^*\lambda_\mu - g-$closed set if and only if $F \subset c_\mu(A)\backslash A$ and F is $^*\lambda_\mu-$closed imply that F is empty.*

Proof: Let F be a $^*\lambda_\mu-$closed subset of $c_\mu(A)\backslash A$. Then $A \subset X\backslash F$ and $X\backslash F$ is $^*\lambda_\mu-$open. Since A is $^*\lambda_\mu - g-$closed, we have $c_\mu(A) \subset X\backslash F$. Consequently $F \subset X\backslash c_\mu(A)$. Hence F is empty.
Conversely, Suppose the implication holds and $A \subset U$, where U is $^*\lambda_\mu-$open. If $c_\mu(A) \not\subset U$, then $c_\mu(A) \cap (X - U)$ is a non-empty $^*\lambda_\mu-$closed subset of $c_\mu(A)\backslash A$. Therefore A is $^*\lambda_\mu - g-$closed.

Theorem 2.7. *If A is $^*\lambda_\mu - g-$closed set in a GTS (X,μ), then $c_\mu(A)\backslash A$ contains no non-empty $\lambda_\mu-$closed ($\lambda_\mu-$open / $\mu-$open / $\mu-$closed) subset of X.*

Proof: Suppose $c_\mu(A)\backslash A$ contains on non empty $\lambda_\mu-$closed ($\lambda_\mu-$open /$\mu-$open /$\mu-$closed) subset of X. Since every $\lambda_\mu-$closed ($\lambda_\mu-$open /$\mu-$open /$\mu-$closed) is $^*\lambda_\mu-$closed , we have F is a non-empty $^*\lambda_\mu-$closed set contained in $c_\mu(A)\backslash A$, which is a contradiction to Theorem 2.6. Hence the proof.
Example 2.8 shows that the converse of the above theorem is not true.

Example 2.8. *Let $X = \{a,b,c,d\}$ and $\mu = \{\emptyset, \{a,b,c\}\{b,c,d\}, X\}$. If $A = \{a,b,d\}$ then $c_\mu(A)\backslash A = \{c\}$, which is neither contain a nonempty $\lambda_\mu-$closed set nor contain a nonempty $\lambda_\mu-$open set, but A is not a $^*\lambda_\mu - g-$closed set.*
If $B = \{a,c,d\}$ then $c_\mu(B)\backslash B = \{b\}$, which is neither contain a nonempty $\mu-$closed set nor contain a nonempty $\mu-$open set, but B is not a $^*\lambda_\mu - g-$closed set.

Lemma 2.9. [7] *For a GTS (X,μ) and $S,T \subset X$, the following properties hold:*
(a).$i_\mu(s \cap T) \subset i_\mu(S) \cap i_\mu(T)$.
(b).$c_\mu(s) \cup c_\mu(T) \subset c_\mu(S) \cup T)$.

Remark 2.10. [7] *$i_\mu(s \cap T) \supset i_\mu(S) \cap i_\mu(T)$ is not true in general for subset S and T of a GTS (X,μ).*

Theorem 2.11. [5] *Let (X,μ) be a quasi-topological space. Then $c_\mu(A \cup B) = c_\mu(A) \cup c_\mu(B)$ for every A and B of X.*

Theorem 2.12. *Let (X,μ) be a quasi-topological space. Then $A \cup B$ is a $^*\lambda_\mu - g-$closed set whenever A and B are $^*\lambda_\mu - g-$closed sets.*

Proof: Let U be a $^*\lambda_\mu-$closed set such that $A \cup B \subset U$. Then $A \subset U$ and $B \subset U$. Since A and B are $^*\lambda_\mu - g-$closed, $c_\mu(A) \subset U$ and $c_\mu(B) \subset U$. Hence $c_\mu(A \cup B) = c_\mu(A) \cup c_\mu(B) \subset U$ and so that proof follows.

Example 2.13. *Let $X = \{a,b,c\}$ and $\mu = \{\emptyset, \{a,b\}\{b,c\}, X\}$. Then μ is a GT but not a quasi-topology. If $A = \{a\}$ and $B = \{c\}$, then A and B are $^*\lambda_\mu - g-$closed sets but their union is not a $^*\lambda_\mu - g-$closed set.*

Example 2.14. *Let $X = \{a,b,c\}$ and $\mu = \{\emptyset, \{a\}\}$. If $A = \{a,b\}$ and $B = \{a,c\}$, then A and B are $^*\lambda_\mu - g-$closed sets but $A \cap B = \{a\}$ is not a $^*\lambda_\mu - g-$closed set.*

Theorem 2.15. *Let (X, μ) be a GTS. If A is $^*\lambda_\mu - g-closed$ and B is $\mu-closed$, then $A \cap B$ is a $^*\lambda_\mu - g-closed$ set.*

Proof: Suppose $A \cap B \subset U$ where U is $^*\lambda_\mu-$open. Then $A \subset U \cup (X \backslash B)$. Since A is $^*\lambda_\mu - g-$ closed, $c_\mu(A) \subset U \cup (X \backslash B)$ and so $c_\mu(A) \cap B \subset U$. Hence $c_\mu(A \cap B) \subset U$, which implies that $A \cap B$ is a $^*\lambda_\mu - g-$closed set.

Theorem 2.16. *Let (X, μ) be a GTS. If A is $^*\lambda_\mu-$open and $^*\lambda_\mu - g-closed$, then A is $\mu-closed$.*

Proof: Since A is $^*\lambda_\mu-$open and $^*\lambda_\mu - g-$closed, $c_\mu(A) \subset A$ and hence A is $\mu-$closed.

3. $^*\lambda_\mu - g_\mu-$ Closed sets

Definition 3.1. *Let (X, μ) be a GTS. A subset A of X is called $^*\lambda_\mu - g_\mu-closed$ set if $c_{\lambda_\mu}(A) \subset U$ whenever $A \subset U$ and U is $^*\lambda_\mu-open$ set in X. The complements of a $^*\lambda_\mu - g_\mu-closed$ sets are called $^*\lambda_\mu - g_\mu-open$ sets.*

Theorem 3.2. *For a GTS (X, μ), Every $\lambda_\mu-closed$ set is $^*\lambda_\mu - g_\mu-closed$.*

Proof: Let A be $\lambda_\mu-$closed set and U be any $^*\lambda_\mu - g-$open set containing A. Since A is $\lambda_\mu-$closed, $c_{\lambda_\mu}(A) = A$. Therefore $c_{\lambda_\mu}(A) \subset U$and hence A is $^*\lambda_\mu - g_\mu-$closed.

Corollary 3.3. *For a GTS (X, μ), the following hold:*
a).Every $\mu-closed$ set is $^*\lambda_\mu - g_\mu-closed$.
b).Every $\mu-open$ set is $^*\lambda_\mu - g_\mu-closed$.

Example 3.4 shows that the converse of the above theorem is not true.

Example 3.4. *Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a\}\}$. If $A = \{c\}$ then A is neither $\mu-$ closed nor $\mu-open$ but it is $^*\lambda_\mu - g_\mu-closed$. If $B = \{a, b\}$, then B is $^*\lambda_\mu - g_\mu-closed$ but not $^*\lambda_\mu-closed$.*

Theorem 3.5. *Let(X, μ) be a GTS. If $A \subset X$ and A is a $^*\lambda_\mu - g_\mu-closed$ set, then A is a $g - \lambda_\mu-closed$ set.*

Proof: Let U be a $\mu-$open set containing A in(X, μ). Since every μ- open set is $^*\lambda_\mu-$open and A is $^*\lambda_\mu - g_\mu-$closed, $c_{\lambda_\mu}(A) \subset U$. Therefore A is $g - \lambda_\mu-$closed.
    Example3.6 shows that the converse of Theorem 3.5 is not true.

Example 3.6. *Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a, b\}\{b, c\}, X\}$. If $A = \{a, c\}$ then A is $g - \lambda_\mu-closed$ but not $^*\lambda_\mu - g_\mu-closed$.*

Theorem 3.7 shows that the relation between $^*\lambda_\mu - g-$closed set and $^*\lambda_\mu - g_\mu-$closed set.

Theorem 3.7. *In a GTS (X, μ), every $^*\lambda_\mu - g-closed$ set is $^*\lambda_\mu - g_\mu-closed$.*

Proof: Let A be a $^*\lambda_\mu - g-$closed set and U be an $\lambda_\mu-$open set containing A in (X, μ). Then $c_\mu(A) \subset U$. Since $c_{\lambda_\mu}(A) \subset c_\mu(A)$, we have $c_{\lambda_\mu}(A) \subset U$. Therefore A is $^*\lambda_\mu - g_\mu-$closed.

Example 3.8 shows that the converse of the above theorem is not true.

Example 3.8. *Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a, b\}, X\}$. If $A = \{a\}$, then A is $^*\lambda_\mu - g_\mu-$closed set but it is not $^*\lambda_\mu - g-$ closed.*

Remark 3.9. *The concepts of $g_\mu-$closed set and $^*\lambda_\mu - g_\mu-$closed set are independent.*

Example 3.10. *Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Then $\{b, d\}$ is $g_\mu-$closed but not $^*\lambda_\mu - g_\mu-$closed and $\{c\}$ is $^*\lambda_\mu - g_\mu-$closed but not $g_\mu-$closed.*

Theorem 3.11 gives a characterization of $^*\lambda_\mu - g_\mu-$closed sets.

Theorem 3.11. *Let (X, μ) be a GTS. Then a subset A of X is $^*\lambda_\mu - g_\mu-$ closed set if and only if $F \subset c_{\lambda_\mu}(A) \backslash A$ and F is $^*\lambda_\mu-$closed imply that F is empty.*

Proof: The proof is similar to that of Theorem 2.6 so that it is omitted.

Theorem 3.12. *If A is $^*\lambda_\mu - g_\mu-$ closed set in a GTS (X, μ), then $c_{\lambda_\mu}(A) \backslash A$ contains no non empty $\lambda_\mu-$ closed ($\lambda_\mu-$open / $\mu-$open / $\mu-$closed) subset of X.*

Proof: The proof is similar to that of Theorem 2.7 so that it is omitted.

Example 3.13 shows that the converse of above Theorem is not true.

Example 3.13. *Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a, b, c\}\{b, c, d\}, X\}$. If $A = \{a, d\}$, $c_{\lambda_\mu}(A) \backslash A = \{b, c\}$, which does not contain both a nonempty $\mu-$open set and $\lambda_\mu-$open set but A is not $^*\lambda_\mu - g_\mu-$closed set. If $B = \{c, d\}$, $c_{\lambda_\mu}(B) \backslash B = \{b\}$, which does not contain both a nonempty $\mu-$closed set and $\lambda_\mu-$closed set but B is not a $^*\lambda_\mu - g_\mu-$closed set.*

Theorem 3.14. *Let (X, μ) be a GTS and A and B be subsets of X. If $A \subset B \subset c_{\lambda_\mu}(A)$ and A is $^*\lambda_\mu - g_\mu-$closed set then B is $^*\lambda_\mu - g_\mu-$closed.*

Proof: If F is $^*\lambda_\mu-$closed such that $F \subset c_{\lambda_\mu}(B) \backslash B$, then $F \subset c_{\lambda_\mu}(A) \backslash A$. Since A is $^*\lambda_\mu - g_\mu-$closed, by Theorem 3.10, $F = \emptyset$ and so B is $^*\lambda_\mu - g_\mu-$closed.

Theorem 3.15. *Let A be a $^*\lambda_\mu - g_\mu-$closed set in a quasi- topological space (X, μ). Then the following hold:*
a).If A is a $\pi-$regular set, then $i_\pi(A)$ and $c_\sigma(A)$ are $^*\lambda_\mu - g_\mu-$closed sets.
b).If A is a $\sigma-$regular set, then $c_\pi(A)$ and $i_\sigma(A)$ are $^*\lambda_\mu - g_\mu-$closed sets.

Proof: a). Since A is $\pi-$regular set, $c_\sigma(A) = A \cup i_\mu c_\mu(A) = A$ and $i_\pi(A) = A \cap i_\mu c_\mu(A) = A$. Thus $i_\pi(A)$ and $c_\sigma(A)$ are $^*\lambda_\mu - g_\mu-$closed sets.
b). Since A is a $\sigma-$regular set, $c_\pi(A) = A$ and $i_\sigma(A) = A$. Thus $c_\pi(A)$ and $i_\sigma(A)$ are $^*\lambda_\mu - g_\mu-$closed sets.

Remark 3.16. *The union (resp. intersection) of two $^*\lambda_\mu - g_\mu-closed$ sets need not be $^*\lambda_\mu - g_\mu-closed$ set.*

Example 3.17. *Let $X = \{a, b, c, d, e\}$ and $\mu = \{\emptyset, \{a\}, \{a, b, c\}, X\}$. Then $\{a\}$ and $\{d, e\}$ are $^*\lambda_\mu - g_\mu-closed$ sets but their union is not a $^*\lambda_\mu - g_\mu-closed$ set. Further $\{a, b, d\}$ and $\{a, c, d, e\}$ are $^*\lambda_\mu - g_\mu-closed$ sets but their intersection is not a $^*\lambda_\mu - g_\mu-closed$ set.*

4. $g_{*\lambda_\mu}-$ Closed sets

Definition 4.1. *Let (X, μ) be a GTS. A subset A of X is called $g_{*\lambda_\mu}-closed$ set if $c_{*\lambda_\mu}(A) \subset U$ whenever $A \subset U$ and U is $^*\lambda_\mu-open$ set in X. The complements of a $g_{*\lambda_\mu}-closed$ sets are called $g_{*\lambda_\mu}-open$ sets.*

Theorem 4.2. *For a GTS (X, μ), Every $^*\lambda_\mu-closed$ set is $g_{*\lambda_\mu}-closed$.*

Proof: Let A be $^*\lambda_\mu-$closed set and U be any $^*\lambda_\mu-$open set containing A.Since A is $^*\lambda_\mu-$closed, $c_{*\lambda_\mu}(A) = A$. Therefore $c_{*\lambda_\mu}(A) \subset U$ and hence A is $g_{*\lambda_\mu}-$closed.

Corollary 4.3. *For a GTS (X, μ), the following hold:*
a).Every $\lambda_\mu-closed$ set is $g_{*\lambda_\mu}-closed$.
b).Every $\lambda_\mu-open$ set is $g_{*\lambda_\mu}-closed$.
c).Every $\mu-closed$ set is $g_{*\lambda_\mu}-closed$.
d).Every $\mu-$ open set is $g_{*\lambda_\mu}-closed$.

Example 4.4 shows that the converse of Theorem 4.2 and the converses of a)-d) from corollary 4.3 are not true.

Example 4.4. *Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a\}\}$. If $A = \{c\}$ then A is neither $\mu-$closed nor $\mu-$open but it is $g_{*\lambda_\mu}-closed$. If $B = \{a, b\}$, then B is neither $\lambda_\mu-closed$ but nor $\lambda_\mu-open$ but it is $g_{*\lambda_\mu}-$ closed. If $C = \{a, c\}$ then C is $^*\lambda_\mu-colsed$ but is not $g_{*\lambda_\mu}-closed$.*

Remark 4.5. *The concepts of $g - \lambda_\mu-closed$ set and $g_{*\lambda_\mu}-closed$ set are independent.*

Example 4.6. *Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a, b, c\}\{b, c, d\}, X\}$. Then $\{a, d\}$ is $g_{*\lambda_\mu}-closed$ but $g - \lambda_\mu-closed$ and $\{c\}$ is $g - \lambda_\mu-closed$ but not $g_{*\lambda_\mu}-closed$.*

Theorem 4.7 shows that the relation between $g_{*\lambda_\mu}-$closed set and generalized closed sets defined in section 3.

Theorem 4.7. *For a GTS (X, μ), Every $^*\lambda_\mu - g_\mu-closed$ set is $g_{*\lambda_\mu}-closed$.*

Proof: Let A be a $^*\lambda_\mu - g_\mu-$closed sets and U be an $^*\lambda_\mu-$open set containing A in(X, μ). Then $c_{\lambda_\mu}(A) \subset U$. Since $c_{*\lambda_\mu}(A) \subset c_{\lambda_\mu}(A)$,we have $c_{*\lambda_\mu}(A) \subset U$. Therefore A is $g_{*\lambda_\mu}-$closed.
Example 4.8 shows that the converse of Theorem 4.7 is not true.

Example 4.8. *Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. If $A = \{a, c\}$ then A is $g_{*\lambda_\mu}-closed$ but not $^*\lambda_\mu - g_\mu-closed$.*

5. A Unified framework of generalized closed sets in GTS'

Given an ordered pair of topologies (τ_1, τ_2) on X, a subset $A \subset X$ is said to be (i,j)-g-closed if $\tau_j - cl(A) \subset U$ whenever $A \subset U$ and $U \in \tau_i$, where $i, j \in \{1, 2\}$ with $i \neq j$ [6]. This definition could be easily generalized: Given a pair of generalized topologies (μ_1, μ_2) on X, a subset $A \subset X$ is said to be $g(\mu_i, \mu_j)$−closed if $c_{\mu_j}(A) \subset U$ whenever $A \subset U$ and $U \in \mu_i$, where $i, j \in \{1, 2\}$.
If $i = j$, the class of g(μ_i, μ_i)-closed sets is the class of g_{μ_i}−closed sets.

Observation 5.1. *For a space (X, μ_1, μ_2), the following are hold:*
a). If $\mu_1 = \mu$ and μ_2 is the class of λ_μ−open sets, then A is $g(\mu_1, \mu_2)$−closed if and only if A is $g - \lambda_\mu$−closed.
b). If μ_1 is the class of λ_μ−open sets and $\mu_2 = \mu$, then A is $g(\mu_1, \mu_2)$−closed if and only if A is $\lambda_\mu - g$−closed.
c). If μ_1 is the class of $^*\lambda_\mu$−open sets and $\mu_2 = \mu$, then A is $g(\mu_1, \mu_2)$−closed if and only if A is $^*\lambda_\mu - g$−closed.
d). If μ_1 is the class of $^*\lambda_\mu$−open sets and μ_2 is the class of λ_μ−open sets, then A is $g(\mu_1, \mu_2)$−closed if and only if A is $^*\lambda_\mu - g_\mu$−closed.
e). If $\mu_1 = \mu_2$ represents the class of $^*\lambda_\mu$−open sets, then A is $g(\mu_1, \mu_2)$−closed if and only if A is $g*\lambda_\mu$−closed.

Theorem 5.2. *If $\mu_1 \subset \mu_2$, then:*
a). A is $g(\mu_2, \mu_1)$−closed $\Rightarrow$ A is g_{μ_1}−closed and g_{μ_2}−closed.
b). A is g_{μ_1}−closed $\Rightarrow$ A is $g(\mu_1, \mu_2)$−closed.
c). A is μ_1−closed $\Rightarrow$ A is $g(\mu_2, \mu_1)$−closed.

Proof: Since $\mu_1 \subset \mu_2$, we have $c_{\mu_2}(A) \subset c_{\mu_1}(A)$ for every $A \subset X$.
a) Let A be $g(\mu_2, \mu_1)$−closed. Assume that $A \subset U$ and $U \in \mu_1$. We prove that $c_{\mu_1}(A) \subset U$. Since $\mu_1 \subset \mu_2$, we have $U \in \mu_2$. But A is $g(\mu_2, \mu_1)$−closed, therefore $c_{\mu_1}(A) \subset U$.
Assume that $A \subset V$ and $V \in \mu_2$. We prove that $c_{\mu_2}(A) \subset V$. Since A is $g(\mu_2, \mu_1)$−closed, we have $c_{\mu_1}(A) \subset V$. But $c_{\mu_2}(A) \subset c_{\mu_1}(A)$, hence $c_{\mu_2}(A) \subset V$.
b) Let A be g_{μ_1}−closed. Assume that $A \subset U$ and $U \in \mu_1$. We prove that $c_{\mu_2}(A) \subset U$. By our assumption, we have $c_{\mu_1}(A) \subset U$, but $c_{\mu_2}(A) \subset c_{\mu_1}(A)$, hence the claim follows.
c) Let A be μ_1−closed. If $A \subset W$ and $W \in \mu_2$, then $c_{\mu_1}(A) = A \subset W$.

Theorem 5.3. *Assume that $\mu_1' \subset \mu_1''$ and $\mu_2'' \subset \mu_2'$. If A is $g(\mu_1'', \mu_2'')$−closed, then A is $g(\mu_1', \mu_2')$−closed.*

Proof: Let A be $g(\mu_1'', \mu_2'')$−closed. Consider $U \in \mu_1'$ such that $A \subset U$. We prove that $c_{\mu_2'}(A) \subset U$. Since $U \in \mu_1''$ and A is $g(\mu_1'', \mu_2'')$−closed, we have $c_{\mu_2''}(A) \subset U$. But $\mu_2'' \subset \mu_2'$ implies $c_{\mu_2'}(A) \subset c_{\mu_2''}(A)$, therefore the claim holds true.

Remark 5.4. *Assume that μ_1 and μ_2 are generalized topologiesbon X.*
(i). Let $\mu_1 = \mu$ and μ_2 be the class of $^*\lambda_\mu$−open sets. From remark 5.2 a), we obtain Theorem 2.4 (a) and Theorem 4.7.

(ii). Let $\mu_1 = \mu$ and μ_2 be the class of $\lambda_\mu-$open sets. From remark 5.2 b), we recover the fact every $g_\mu-$closed set $g - \lambda_\mu-$closed. So Theorem 2.4 b), follows from this and Theorem 2.4.

(iii). Let μ_1' be the class of $\lambda_\mu-$open sets and μ_1'' be the class of $^*\lambda_\mu-$open sets and take $\mu_2' = \mu_2'' = \mu$. From Theorem 5.3, we obtain Theorem 2.4 c.

(iv) Let μ_1 be the class of $^*\lambda_\mu-$open sets and μ_2 be the class of $\lambda_\mu-$open sets. Since $\mu_2 \subset \mu_1$, by Theorem 5.2 we obtain that every $^*\lambda_\mu - g-$closed set is $g_{\lambda_\mu}-$closed (and $g_{*\lambda_\mu}-$closed). By Theorem 5.2 c) we get Theorem 3.2 a).

(v) Let $\mu_1' = \mu$ and μ_1'' be the class of λ_μ^*-open sets, and let $\mu_2' = \mu_2''$ be the class of $\lambda_\mu-$open sets.From Theorem 5.3 we obtain Theorem 3.4.

(vi) Let $\mu_1' = \mu_1''$ be the class of λ_μ^*-open sets and let μ_2' be the class of $\lambda_\mu-$open sets, while $\mu_2'' = \mu$. From Theorem 5.3 we get Theorem 3.6.

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