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# A Decomposition of Pairwise Continuity via Ideals

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ABSTRACT: In this paper, we introduce and study the notions of $(i, j)$-regular-$\mathcal{I}$-closed sets, $(i, j)$-$A_{\mathcal{I}}$-sets, $(i, j)$-$\mathcal{I}$-locally closed sets, $p$-$A_{\mathcal{I}}$-continuous functions and $p$-$\mathcal{I}$-$LC$-continuous functions in ideal bitopological spaces and investigate some of their properties. Also, a new decomposition of pairwise continuity is obtained using these sets.

Key Words: $(j, i)$-regular closed sets, $(i, j)$-$A$-sets, $(i, j)$-locally closed sets, $(i, j)$-regular-$\mathcal{I}$-closed sets, $(i, j)$-$A_{\mathcal{I}}$-sets, $(i, j)$-$\mathcal{I}$-locally closed sets, $p$-$A_{\mathcal{I}}$-continuous functions and $p$-$\mathcal{I}$-$LC$-continuous functions.

## Contents



## 1. Introduction and Preliminaries

In 1963, J. C. Kelly [9] introduced the notion of bitopological spaces. Such spaces are equipped with two arbitrary topologies. B. Dvalishvili [3] introduced the concept of $(i, j)$-regular closed sets in bitopological spaces. In [8], M. Jelic introduced the concepts of $(i, j)$-$A$-sets, $(i, j)$-locally closed sets, $p$-$A$-continuity and $p$-$LC$-continuity in bitopological spaces. Throughout this paper, $\tau_j$-cl(A) and $\tau_i$-int(A) denote the closure of A with respect to $\tau_j$ and the interior of A with respect to $\tau_i$ and the spaces $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ are bitopological spaces on which no separation axioms are assumed unless explicitly stated.

An ideal topological space is a topological space $(X, \tau)$ with an ideal $\mathcal{I}$ on X and is denoted by $(X, \tau, \mathcal{I})$. The subject of ideals in topological spaces has been introduced and studied by Kuratowski [11] and Vaidyanathaswamy [17].

Let $(X, \tau_1, \tau_2)$ be a bitopological space and let $\mathcal{I}$ be an *ideal* of subsets of $X$. An *ideal bitopological* space is a bitopological space $(X, \tau_1, \tau_2)$ with an ideal $\mathcal{I}$ on $X$ and is denoted by $(X, \tau_1, \tau_2, \mathcal{I})$. For a subset $A$ of $X$ and $j = 1, 2$, $A^*_{\tau_j}(\mathcal{I}) = \{x \in X : A \cap U \notin \mathcal{I}$ for every $U \in \tau_j(X, x)\}$ is called the *local function* [11] of $A$ with respect to $\mathcal{I}$ and $\tau_j$. We simply write $A^*_j$ instead of $A^*_{\tau_j}(\mathcal{I})$ in case there is no chance for confusion. For every ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$, there exists

2000 *Mathematics Subject Classification:* 54A05 a topology $\tau_j^*(\mathcal{I})$, finer than $\tau_j$. Additionally, $\tau_j$-$cl^*(A)$=$A \cup A_j^*$ defines a *Kuratowski closure operator* [18] for $\tau_j^*(\mathcal{I})$. Also, $\tau_j$-$cl^*(A) \subseteq \tau_j$-$cl(A)$ for any subset $A$ of $X$. The hypothesis $X = X_j^*$ is equivalent to the hypothesis $\tau_j \cap \mathcal{I} = \emptyset$. In an ideal topological spaces $(X, \tau, \mathcal{I})$, a space is called Hayashi-Samuels space if $\tau \cap \mathcal{I} = \emptyset$. In an ideal bitopological spaces $(X, \tau_1, \tau_2, \mathcal{I})$, we call a space is a Hayashi-Samuels space if $\tau_j \cap \mathcal{I} = \emptyset$, j = 1 or 2. For every ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$, there exists a topology $\tau_j^*(\mathcal{I})$, finer than $\tau_j$, generated by $\beta(\mathcal{I}, \tau_j) = \{U - I : U \in \tau_j$ and $I \in \mathcal{I}\}$, but in general $\beta(\mathcal{I}, \tau_j)$ is not always a topology. If $\mathcal{I} = \{\emptyset\}$, then $A_j^* = \tau_j$-$cl(A)$. Hence in this case $\tau_j$-$cl^*(A) = \tau_j$-$cl(A)$ and $\tau_j^* = \tau_j$. If $\mathcal{I} = \mathcal{P}(X)$, then $A_j^* = \emptyset$ for every $A \subseteq X$. The family of all nowhere dense subsets of a bitopological space $(X, \tau_1, \tau_2)$ is defined by $ij$-$\mathcal{N} = \{A \subseteq X : \tau_i$-$int(\tau_j$-$cl(A)) = \emptyset\}$, where $i, j = 1, 2$ and $i \neq j$. Recently, M. Rajamani et al. [14] introduced the notions of $(i,j)-B_{\mathcal{I}}$-sets, $(i,j)-C_{\mathcal{I}}$-sets, $(i,j)-S_{\mathcal{I}}$-sets, $(i,j)$-$\alpha$-$\mathcal{I}$-open sets, $(i,j)$-*semi*-$\mathcal{I}$-open sets, $(i,j)$-*pre*-$\mathcal{I}$-open sets and $(i,j)$-$\beta$-$\mathcal{I}$-open sets and obtained decompositions of pairwise continuity. In this paper, we introduce the notions of $(i,j)$-regular-$\mathcal{I}$-closed sets, $(i,j)$-$A_{\mathcal{I}}$-sets, $(i,j)$-$\mathcal{I}$-locally closed sets, $p$-$A_{\mathcal{I}}$-continuous functions and $p$-$\mathcal{I}$-$LC$-continuous functions to obtain another decomposition of pairwise continuity in ideal bitopological spaces.

**Definition 1.1.** *A subset $A$ of a space $(X, \tau_1, \tau_2)$ is said to be*
*1.* $(i,j)$-regular closed [3] *if* $A = \tau_i$-$cl(\tau_j$-$int(A))$,
*2.* *$(i,j)$-semi-open [12] if* $A \subseteq \tau_j$-$cl(\tau_i$-$int(A))$,
*3.* *$(i,j)$-pre-open [8] if* $A \subseteq \tau_i$-$int(\tau_j$-$cl(A))$,
*4.* $(i,j)$-$\alpha$-open [15] *if* $A \subseteq \tau_i$-$int(\tau_j$-$cl(\tau_i$-$int(A)))$,
*5.* *$(i,j)$-$\alpha^*$-set [16] if* $\tau_i$-$int(A) = \tau_i$-$int(\tau_j$-$cl(\tau_i$-$int(A)))$,
*6.* $(i,j)$-$A$-set [8] *if $A = U \cap V$, where $U$ is $\tau_i$-open and $V$ is $(j,i)$-regular closed,*
*7.* $(i,j)$-locally closed set (briefly $(i,j)$-LC-set) [8] *if $A = U \cap V$, where $U$ is $\tau_i$-open and $V$ is $\tau_j$-closed,*
*8.* $(i,j)$-$C$-set [16] *if $A = U \cap V$, where $U$ is $\tau_i$-open and $V$ is an $(i,j)$-$\alpha^*$-set, where $i \neq j$, $i$, $j$ = 1, 2.*

**Definition 1.2.** *A subset $A$ of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be*
*1.* $*$-dense-in-itself [5] *if* $A \subseteq A^*$,
*2.* $\tau^*$-closed [6] *if* $A^* \subseteq A$,
*3.* $*$-perfect [5] *if* $A = A^*$,
*4.* *$\alpha$-$\mathcal{I}$-open [4] if* $A \subseteq int(cl^*(int(A)))$,
*5.* *semi-$\mathcal{I}$-open [4] if* $A \subseteq cl^*(int(A))$,
*6.* *pre-$\mathcal{I}$-open [2] if* $A \subseteq int(cl^*(A))$,
*7.* *$\mathcal{I}$-open [7] if* $A \subseteq int(A^*)$,
*8.* *$\alpha^*$-$\mathcal{I}$-set [4] if* $int(A) = int(cl^*(int(A)))$,
*9.* *regular-$\mathcal{I}$-closed* [10] *if* $A = (int(A))^*$,
*10.* $\mathcal{I}$-locally closed [2] *if $A = U \cap V$, where $U \in \tau$ and $V$ is $*$-perfect,*
*11.* *$C_{\mathcal{I}}$-set [4] if $A = U \cap V$, where $U \in \tau$ and $V$ is an $\alpha^*$-$\mathcal{I}$-set,*
*12.* *$A_{\mathcal{I}}$-set* [10] *if $A = U \cap V$, where $U \in \tau$ and $V$ is a regular-$\mathcal{I}$-closed set.*

**Definition 1.3.** *[14] A subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be*

*1.* $(i,j)$*-pre-*$\mathfrak{I}$*-open if* $A \subseteq \tau_i\text{-}int(\tau_j\text{-}cl^*(A))$,
*2.* $(i,j)$*-*$\mathfrak{I}$*-open if* $A \subseteq \tau_i\text{-}int(A_j^*)$,
*3.* $(i,j)$*-semi-*$\mathfrak{I}$*-open if* $A \subseteq \tau_j\text{-}cl^*(\tau_i\text{-}int(A))$,
*4.* $(i,j)$*-*$\alpha$*-*$\mathfrak{I}$*-open if* $A \subseteq \tau_i\text{-}int(\tau_j\text{-}cl^*(\tau_i\text{-}int(A)))$,
*5.* $(i,j)$*-*$\alpha^*$*-*$\mathfrak{I}$*-set if* $\tau_i\text{-}int(\tau_j\text{-}cl^*(\tau_i\text{-}int(A))) = \tau_i\text{-}int(A)$,
*6.* $(i,j)$*-*$C_{\mathfrak{I}}$*-set if* $A = U \cap V$*, where* $U \in \tau_i$ *and* $V$ *is an* $(i,j)$*-*$\alpha^*$*-*$\mathfrak{I}$*-set, where* $i \neq j$*,* $i,\ j = 1,\ 2$*.*

**Lemma 1.4.** *[6] Let* $(X,\tau)$ *be a topological space with ideals* $\mathfrak{I}$ *and* $\mathfrak{J}$ *and* $A$*,* $B$ *subsets of* $X$*. Then the following properties hold:*
*1. If* $A \subseteq B$*, then* $A^* \subseteq B^*$*,*
*2. If* $\mathfrak{I} \subseteq \mathfrak{J} \Rightarrow A^*(\mathfrak{J}) \subseteq B^*(\mathfrak{I})$*,*
*3.* $A^* = cl(A^*) \subseteq cl(A)$*,*
*4.* $(A^*)^* \subseteq A^*$*,*
*5.* $(A \cup B)^* = A^* \cup B^*$*.*

**Lemma 1.5.** *[14] Let* $(X,\tau_1,\tau_2,\mathfrak{I})$ *be an ideal bitopological space and let* $\mathfrak{I} = ij\text{-}\mathfrak{N}$ *be the family of all nowhere dense subsets of* $(X,\tau_1,\tau_2)$*. Then* $A_j^*(\mathfrak{I}) = \tau_j\text{-}cl(\tau_i\text{-}int(\tau_j\text{-}cl(A)))$*.*

**Lemma 1.6.** *[14] Let* $(X,\tau_1,\tau_2,\mathfrak{I})$ *be an ideal bitopological space and* $\mathfrak{I} = \{\emptyset\}$ *or* $\mathfrak{I} = ij\text{-}\mathfrak{N}$*. Then a subset* $A$ *of* $X$ *is* $(i,j)$*-pre-*$\mathfrak{I}$*-open if and only if* $A$ *is* $(i,j)$*-pre open.*

## 2. $(i,j)$-*Regular*-$\mathfrak{I}$-closed sets

**Definition 2.1.** *A subset* $A$ *of an ideal bitopological space* $(X,\tau_1,\tau_2,\mathfrak{I})$ *is said to be* $(i,j)$*-regular-*$\mathfrak{I}$*-closed if* $A = (\tau_i\text{-}int(A))_j^*$*.*

**Proposition 2.2.** *For a subset of an ideal bitopological space* $(X,\tau_1,\tau_2,\mathfrak{I})$*, the following hold:*
*1. Every* $(i,j)$*-regular-*$\mathfrak{I}$*-closed set is* $(i,j)$*-*$\alpha^*$*-*$\mathfrak{I}$*-set and* $(i,j)$*-semi-*$\mathfrak{I}$*-open.*
*2. Every* $(i,j)$*-regular-*$\mathfrak{I}$*-closed set is* $\tau_j$*-*$*$*-perfect.*

**Proof:** **(1)** Let $A$ be an $(i,j)$-regular-$\mathfrak{I}$-closed set. Then, we have $\tau_j\text{-}cl^*(\tau_i\text{-}int(A))) = \tau_i\text{-}int(A) \cup (\tau_i\text{-}int(A))_j^* = \tau_i\text{-}int(A) \cup A = A$. Thus $\tau_i\text{-}int(\tau_j\text{-}cl^*(\tau_i\text{-}int(A))) = \tau_i\text{-}int(A)$ and $A \subseteq \tau_j\text{-}cl^*(\tau_i\text{-}int(A)))$. Therefore, A is $(i,j)$-$\alpha^*$-$\mathfrak{I}$-set and $(i,j)$-semi-$\mathfrak{I}$-open.
**2.** Let $A$ be an $(i,j)$-regular-$\mathfrak{I}$-closed set. Then, $A = (\tau_i\text{-}int(A))_j^*$. Since $\tau_i\text{-}int(A) \subseteq A$, $(\tau_i\text{-}int(A))_j^* \subseteq A_j^*$ by (1) of Lemma 1.4. Then we have $A = (\tau_i\text{-}int(A))_j^* \subseteq A_j^*$. On the other hand, by (5) of Lemma 1.4, $A_j^* = ((\tau_i\text{-}int(A))_j^*)_j^* \subseteq (\tau_i\text{-}int(A))_j^* = A$. Therefore, we obtain $A = A_j^*$. Thus, A is $\tau_j$-$*$-perfect. □

**Remark 2.3.** *The converses of Proposition 2.2 need not be true as the following examples show.*

**Example 2.4.** *Let* $X = \mathbb{R}$*,* $\tau_1 = \tau_2 =$ *usual topology on* $\mathbb{R}$ *and* $\mathfrak{I}$ *be the ideal of finite subsets of* $\mathbb{R}$*. Let* $A = \mathbb{Q}$*. Then* $A$ *is* $(1,2)$*-*$\alpha^*$*-*$\mathfrak{I}$*-set but it is not* $(1,2)$*-regular-*$\mathfrak{I}$*-closed.*

**Example 2.5.** *Let $X = \mathbb{R}$, $\tau_1 = \tau_2 =$ usual topology on $\mathbb{R}$ and $\mathcal{I}$ be the ideal of finite subsets of $(0,1]$. Let $A = (0,1]$. Then $A$ is $(1,2)$-semi-$\mathcal{I}$-open but it is not $(1,2)$-regular-$\mathcal{I}$-closed.*

**Example 2.6.** *Let $X = \mathbb{N}$, $\tau_1 = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, ....\}$, $\tau_2 = \mathcal{P}(\mathbb{N})$ and $\mathcal{I} = \mathcal{P}(2\mathbb{N}-1)$, where $2\mathbb{N}-1$ denotes the set of all odd natural numbers. Let $A = \{2,4,6,8,...\}$. Then $A$ is $\tau_2$-$*$-perfect but it is not $(1,2)$-regular-$\mathcal{I}$-closed.*

**Corollary 2.7.** *Every $(i,j)$-regular-$\mathcal{I}$-closed set is $\tau_j^*$-closed and $\tau_j$-$*$-dense-in-itself.*

**Proof:** The proof is obvious from (2) of Proposition 2.2. □

**Proposition 2.8.** *In an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$, every $(i,j)$-regular-$\mathcal{I}$-closed set is $(j,i)$-regular closed.*

**Proof:** Let $A$ be any $(i,j)$-regular-$\mathcal{I}$-closed set. Then, we have $A = (\tau_i\text{-}int(A))_j^*$. Thus, $\tau_j\text{-}cl(A) = \tau_j\text{-}cl((\tau_i\text{-}int(A))_j^*) = (\tau_i\text{-}int(A))_j^* = A$ by (3) of Lemma 1.4. Also from (3) of Lemma 1.4, we have $(\tau_i\text{-}int(A))_j^* \subseteq \tau_j\text{-}cl(\tau_i\text{-}int(A))$ and hence $A = (\tau_i\text{-}int(A))_j^* \subseteq \tau_j\text{-}cl(\tau_i\text{-}int(A)) \subseteq \tau_j\text{-}cl(A) = A$. Thus we have $A = \tau_j\text{-}cl((\tau_i\text{-}int(A))$ and hence $A$ is a $(j,i)$-regular closed set. □

**Remark 2.9.** *The converse of Proposition 2.8 need not be true as the following example shows.*

**Example 2.10.** *Let $X = \mathbb{R}$, $\tau_1 = \tau_2 =$ usual topology on $\mathbb{R}$ and $\mathcal{I} = \mathcal{P}(\mathbb{R})$. Let $A = [0,1]$. Then $A$ is $(2,1)$-regular closed which is not $(1,2)$-regular-$\mathcal{I}$-closed.*

**Proposition 2.11.** *Every $\tau_j^*$-closed set is an $(i,j)$-$\alpha^*$-$\mathcal{I}$-set in an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$.*

**Proof:** Let $A$ be $\tau_j^*$-closed. Then we have $\tau_i\text{-}int(\tau_j\text{-}cl^*(\tau_i\text{-}int(A))) \subseteq \tau_i\text{-}int(\tau_j\text{-}cl^*(A)) = \tau_i\text{-}int(A)$, because A is $\tau_j^*$-closed. Also, $\tau_i\text{-}int(A) \subseteq \tau_j\text{-}cl^*(\tau_i\text{-}int(A))$. Clearly, $\tau_i\text{-}int(A) \subseteq \tau_i\text{-}int(\tau_j\text{-}cl^*(\tau_i\text{-}int(A)))$. This shows that $A$ is $(i,j)$-$\alpha^*$-$\mathcal{I}$-open. □

**Example 2.12.** *The converse of the above proposition need not be true. In Example 2.4, the set $A = \mathbb{Q}$ is $(1,2)$-$\alpha^*$-$\mathcal{I}$-set but it is not $\tau_2^*$-closed.*

**Proposition 2.13.** *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space and $\mathcal{I} = \{\emptyset\}$ or $\mathcal{I} = ij$-$\mathcal{N}$, where $ij$-$\mathcal{N}$ is the ideal of all nowhere dense sets in $(X, \tau_1, \tau_2)$. Then a subset $A$ of $X$ is $(i,j)$-regular-$\mathcal{I}$-closed if and only if $A$ is $(j,i)$-regular closed.*

**Proof:** By Proposition 2.8, we need to show only sufficiency in both cases.

If $\mathcal{I} = \{\emptyset\}$, then $A_j^*(\{\emptyset\}) = \tau_j\text{-}cl(A)$. If A is $(j,i)$-regular closed, we have $(\tau_i\text{-}int(A))_j^* = \tau_j\text{-}cl(\tau_i\text{-}int(A)) = A$. Thus $A$ is $(i,j)$-regular-$\mathcal{I}$-closed.

If $\mathcal{I} = ij - \mathcal{N}$, then $A_j^*(ij - \mathcal{N}) = \tau_j\text{-}cl(\tau_i\text{-}int(\tau_j\text{-}cl(A)))$, for any subset $A$ of $X$. If A is $(j,i)$-regular closed, we obtain $(\tau_i\text{-}int(A))_j^* = \tau_j\text{-}cl(\tau_i\text{-}int(\tau_j\text{-}cl(\tau_i\text{-}int(A))))$ $= \tau_j\text{-}cl(\tau_i\text{-}int(A)) = A$. This shows that $A$ is $(i,j)$-regular-$\mathcal{I}$-closed. □

**Remark 2.14.** *Following examples show that the notion of $(i,j)$-regular-$\mathcal{I}$-closedness is independent with the notions of $\tau_i$-openness and $(i,j)$-$\alpha$-$\mathcal{I}$-openness.*

**Example 2.15.** *In Example 2.5, the set $A=[0,1]$ is $(1,2)$-regular-$\mathcal{I}$-closed which is not $\tau_1$-open and $(1,2)$-$\alpha$-$\mathcal{I}$-open.*

**Example 2.16.** *Let $X=\mathbb{R}$, $\tau_1=\{\emptyset,\mathbb{Q},X\}$, $\tau_2=$ usual topology on $\mathbb{R}$ and $\mathcal{I}=\mathcal{P}(\mathbb{Q})$. Let $A=\mathbb{Q}$. Then the set $A$ is $\tau_1$-open and $(1,2)$-$\alpha$-$\mathcal{I}$-open but it is not $(1,2)$-regular-$\mathcal{I}$-closed.*

**Remark 2.17.** *From the above definitions and results, we have the following diagram. None of them is reversible.*

$$
\begin{tikzcd}
\tau_j^*\text{-closed} \arrow[r] & (i,j)\text{-}\alpha^*\text{-}\mathcal{I}\text{-open} & (i,j)\text{-}\alpha\text{-}\mathcal{I}\text{-open} \arrow[d] \\
\tau_j\text{-}*\text{-perfect} \arrow[u] \arrow[d] & (i,j)\text{-regular-}\mathcal{I}\text{-closed} \arrow[l] \arrow[u] \arrow[r] \arrow[d] & (i,j)\text{-semi-}\mathcal{I}\text{-open} \arrow[d] \\
\tau_j\text{-}*\text{-dense-in-itself} & (j,i)\text{-regular closed} \arrow[r] & (i,j)\text{-semi-open}
\end{tikzcd}
$$

**Remark 2.18.** *We can say that $(j,i)$-regular closed and $\tau_j$-$*$-dense-in-itself are independent. In Example 2.10, the set $A=[1,2]$ is $(2,1)$-regular closed but not $\tau_2$-$*$-dense-in-itself. Also, in Example 2.4 the set $A=\mathbb{Q}$ is $\tau_2$-$*$-dense-in-itself but not $(2,1)$-regular closed.*

## 3. $(i,j)$-$A_{\mathcal{I}}$-sets and $(i,j)$-$\mathcal{I}$-locally closed sets

**Definition 3.1.** *A subset $A$ of an ideal bitopological space $(X,\tau_1,\tau_2,\mathcal{I})$ is called an*
*1. $(i,j)$-$A_{\mathcal{I}}$-set if $A=U\cap V$, where $U\in\tau_i$ and $V$ is an $(i,j)$-regular-$\mathcal{I}$-closed set,*
*2. $(i,j)$-$\mathcal{I}$-locally-closed set (briefly $(i,j)$-$\mathcal{I}$-LC set) if $A=U\cap V$, where $U\in\tau_i$ and $V$ is $\tau_j$-$*$-perfect.*

**Proposition 3.2.** *Let $(X,\tau_1,\tau_2,\mathcal{I})$ be an ideal bitopological space and $A$ a subset of $X$. Then the following hold:*
*1. If $A$ is a $\tau_i$-open set and $(X,\tau_1,\tau_2,\mathcal{I})$ is a Hayashi-Samuels space, then $A$ is an $(i,j)$-$A_{\mathcal{I}}$-set.*
*2. If $A$ is an $(i,j)$-regular-$\mathcal{I}$-closed set, then $A$ is an $(i,j)$-$A_{\mathcal{I}}$-set.*

**Proof:** Since $X\in\tau_i$ and $X$ is an $(i,j)$-regular-$\mathcal{I}$-closed set, the proof is obvious. □

**Remark 3.3.** *The converses of Proposition 3.2 need not be true as the following example shows.*

**Example 3.4.** *Let $X=\mathbb{R}$, $\tau_1=\{\emptyset,\mathbb{Q},X\}$, $\tau_2=$ usual topology on $\mathbb{R}$ and $\mathcal{I}$ be the ideal of finite subsets of $\mathbb{R}$. Let $A=\mathbb{Q}$. Then $A$ is a $(1,2)$-$A_{\mathcal{I}}$-set but it is not $(1,2)$-regular-$\mathcal{I}$-closed. In Example 2.5, the set $A=[0,1]$ is a $(1,2)$-$A_{\mathcal{I}}$-set but it is not $\tau_1$-open.*

**Proposition 3.5.** *Let $(X, \tau_1, \tau_2, \mathfrak{I})$ be an ideal bitopological space and $A$ a subset of $X$. Then the following hold:*
*1. If $A$ is an $(i,j)$-$A_{\mathfrak{I}}$-set then $A$ is an $(i,j)$-$C_{\mathfrak{I}}$-set and an $(i,j)$-$\mathfrak{I}$-locally-closed set.*
*2. If $A$ is an $(i,j)$-$A_{\mathfrak{I}}$-set then $A$ is an $(i,j)$-$A$-set.*

**Proof:** This is an immediate consequence of Proposition 2.2 and 2.8. □

**Remark 3.6.** *The converses of Proposition 3.5 need not be true. In Example 2.10, the set $A = [1,2]$ is a $(1,2)$-$A$-set but it is not a $(1,2)$-$A_{\mathfrak{I}}$-set. In Example 2.16, the set $A = \mathbb{Q} \cup \{\sqrt{2}\}$ is a $(1,2)$-$C_{\mathfrak{I}}$-set but it is not a $(1,2)$-$A_{\mathfrak{I}}$-set. In Example 2.6, the set $A = \{2,4,6,8,...\}$ is $(1,2)$-$\mathfrak{I}$-locally-closed but it is not a $(1,2)$-$A_{\mathfrak{I}}$-set.*

**Proposition 3.7.** *For a subset $A$ of a Hayashi-Samuels space $(X, \tau_1, \tau_2, \mathfrak{I})$, the following are equivalent:*
*1. $A$ is a $\tau_i$-open set.*
*2. $A$ is an $(i,j)$-$\alpha$-$\mathfrak{I}$-open set and an $(i,j)$-$A_{\mathfrak{I}}$-set.*
*3. $A$ is an $(i,j)$-pre-$\mathfrak{I}$-open set and an $(i,j)$-$A_{\mathfrak{I}}$-set.*

**Proof:** **1 ⇒ 2**. Let $A$ be a $\tau_i$-open set. Hence $A$ is an $(i,j)$-$\alpha$-$\mathfrak{I}$-open set. On the other hand, $A$=$A \cap X$, where $A \in \tau_i$ and $X$ is an $(i,j)$-regular-$\mathfrak{I}$-closed set. Hence $A$ is an $(i,j)$-$A_{\mathfrak{I}}$-set.
**2 ⇒ 3**. This is obvious since every $(i,j)$-$\alpha$-$\mathfrak{I}$-open set is $(i,j)$-pre-$\mathfrak{I}$-open.
**3 ⇒ 1**. Let $A$ be an $(i,j)$-pre-$\mathfrak{I}$-open set and an $(i,j)$-$A_{\mathfrak{I}}$-set. Then $A$=$U \cap V$, where $U \in \tau_i$ and V is an $(i,j)$-regular-$\mathfrak{I}$-closed set. Now, $A$=$U \cap A \subseteq U \cap \tau_i$-$int(\tau_j$-$cl^*(A)) = U \cap \tau_i$-$int(\tau_j$-$cl^*(U \cap V)) \subseteq U \cap \tau_i$-$int(\tau_j$-$cl^*(U) \cap \tau_j$-$cl^*(V))$ $= U \cap \tau_i$-$int(\tau_j$-$cl^*(U) \cap V) = U \cap \tau_i$-$int(\tau_j$-$cl^*(U)) \cap \tau_i$-$int(V) = U \cap \tau_i$-$int(V)$ $=\tau_i$-$int(U \cap V)$. That is, $A \subseteq \tau_i$-$int(U \cap V)$ and also $A = U \cap V \supseteq U \cap \tau_i$-$int(V) \supseteq \tau_i$-$int(U \cap V)$. Therefore $A = \tau_i$-$int(U \cap V) = \tau_i$-$int(A)$. Hence A is $\tau_i$-open. □

## 4. Decomposition of Pairwise continuity

**Definition 4.1.** *[13] A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be p-continuous if the induced mappings $f : (X, \tau_k) \to (Y, \sigma_k)$,$(k = 1, 2)$ are continuous.*

**Definition 4.2.** *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be p-$\alpha$-continuous [15] (resp. p-pre-continuous [8], p-A-continuous [8] if for every $V \in \sigma_i$, $f^{-1}(V)$ is $(i,j)$-$\alpha$-open (resp.$(i,j)$-pre-open, $(i,j)$-$A$-set) of $(X, \tau_1, \tau_2)$.*

**Definition 4.3.** *[14] A function $f : (X, \tau_1, \tau_2, \mathfrak{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be p-$\alpha$-$\mathfrak{I}$-continuous (resp. p-pre-$\mathfrak{I}$-continuous, p-$C_{\mathfrak{I}}$-continuous) if for every $V \in \sigma_i$, $f^{-1}(V)$ is $(i,j)$-$\alpha$-$\mathfrak{I}$-open (resp. $(i,j)$-pre-$\mathfrak{I}$-open, $(i,j)$-$C_{\mathfrak{I}}$-set) of $(X, \tau_1, \tau_2, \mathfrak{I})$.*

**Definition 4.4.** *A function $f : (X, \tau_1, \tau_2, \mathfrak{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be p-$A_{\mathfrak{I}}$-continuous (resp. p-$\mathfrak{I}$-LC-continuous) if for every $V \in \sigma_i$, $f^{-1}(V)$ is an $(i,j)$-$A_{\mathfrak{I}}$-set (resp. $(i,j)$-$\mathfrak{I}$-LC-set) of $(X, \tau_1, \tau_2, \mathfrak{I})$.*

**Proposition 4.5.** *1. Every p-$A_{\mathfrak{I}}$-continuous function is p-$C_{\mathfrak{I}}$-continuous.*
*2. Every p-$A_{\mathfrak{I}}$-continuous function is p-$\mathfrak{I}$-$LC$-continuous.*
*3. Every p-$A_{\mathfrak{I}}$-continuous function is p-$A$-continuous.*

**Proof:** The proof follows from Proposition 3.5. □

**Remark 4.6.** *The converses of Proposition 4.5 need not be true as seen from the following examples show.*

**Example 4.7.** *Let $X=\{a,b,c\}$ with topologies $\tau_1=\{\emptyset,\{a\},\{a,b\},X\}$ and $\tau_2 = \{\emptyset,\{c\},\{a,b\},X\}$ and an ideal $\mathfrak{I}=\{\emptyset,\{a\},\{b\},\{a,b\}\}$ and let $Y=\{p,q,r\}$ with topologies $\sigma_1=\{\emptyset,\{r\},Y\}$ and $\sigma_2 = \{\emptyset,\{p,q\},Y\}$. Let $f : (X,\tau_1,\tau_2,\mathfrak{I}) \to (Y,\sigma_1,\sigma_2)$ be a function defined as f(a)=p and f(b)=q and f(c)=r. Then f is p-$C_{\mathfrak{I}}$-continuous but not p-$A_{\mathfrak{I}}$-continuous.*

**Example 4.8.** *Let $X=\{a,b,c\}$ with topologies $\tau_1=\{\emptyset,\{c\},\{a,b\},X\}$ and $\tau_2 = \{\emptyset,\{a\},\{b\},\{a,b\},\{b,c\},X\}$ and an ideal $\mathfrak{I}=\{\emptyset,\{a\},\{c\},\{a,c\}\}$ and let $Y=\{p,q,r\}$ with topologies $\sigma_1=\{\emptyset,\{q\},\{q,r\},Y\}$ and $\sigma_2 = \{\emptyset,\{p\},Y\}$. Let $f$: $(X,\tau_1,\tau_2,\mathfrak{I}) \to (Y,\sigma_1,\sigma_2)$ be a function defined as $f(a) = p, f(b) = q and f(c) = r$. Then f is p-$\mathfrak{I}$-$LC$-continuous but not p-$A_{\mathfrak{I}}$-continuous.*

**Example 4.9.** *Let $X=\{a,b,c,d\}$ with topologies $\tau_1=\{\emptyset,\{d\},\{a,c\},\{a,c,d\},X\}$ and $\tau_2 = \{\emptyset,\{a\},\{d\},\{a,d\},X\}$ and an ideal $\mathfrak{I}=\{\emptyset,\{a\},\{d\},\{a,d\}\}$ and let $Y=\{p,q,r\}$ with topologies $\sigma_1=\{\emptyset,\{q,r\},Y\}$ and $\sigma_2=\{\emptyset,\{p\},Y\}$. Let $f : (X,\tau_1,\tau_2,\mathfrak{I}) \to (Y,\sigma_1,\sigma_2)$ be a function defined as f(a) = r, f(b) = f(c) = q and f(d) = p. Then f is p-A-continuous but not p-$A_{\mathfrak{I}}$-continuous.*

**Theorem 4.10.** *Let $(X,\tau_1,\tau_2,\mathfrak{I})$ be a Hayashi-Samuels space. For a function $f : (X,\tau_1,\tau_2,\mathfrak{I}) \to (Y,\sigma_1,\sigma_2)$, the following are equivalent:*
*1. $f$ is p-continuous.*
*2. $f$ is p-$\alpha$-$\mathfrak{I}$-continuous and p-$A_{\mathfrak{I}}$-continuous.*
*3. $f$ is p-pre-$\mathfrak{I}$-continuous and p-$A_{\mathfrak{I}}$-continuous.*

**Proof:** The proof is obvious from Proposition 3.7. □

**Corollary 4.11.** *Let $(X,\tau_1,\tau_2,\mathfrak{I})$ be an ideal bitopological space and $\mathfrak{I} = \{\emptyset\}$ or $ij$-$\mathfrak{N}$. For a function $f : (X,\tau_1,\tau_2,\mathfrak{I}) \to (Y,\sigma_1,\sigma_2)$, the following are equivalent:*
*1. $f$ is p-continuous.*
*2. $f$ is p-$\alpha$-continuous and p-A-continuous.*
*3. $f$ is p-pre-continuous and p-A-continuous.*

**Proof: 1** Let $I = \{\emptyset\}$. Then we have $A^*_j=\tau_j$-$cl(A)$ and hence $\tau_j$-$cl^*(A)=\tau_j$-$cl(A)$ for any subset $A$ of $X$. Therefore, we obtain $A$ is $(i,j)$-$\alpha$-$\mathfrak{I}$-open if and only if it is $(i,j)$-$\alpha$-open. By Proposition 2.13, $A$ is an $(i,j)$-$A_{\mathfrak{I}}$-set if and only if it is an $(i,j)$-A-set and $A$ is $(i,j)$-*pre*-$\mathfrak{I}$-open if and only if it is $(i,j)$-*pre*-open. The proof of the corollary follows immediately from Lemma 1.6 and Theorem 4.10.

**2** Let $\mathcal{I} = ij\text{-}\mathcal{N}$. Then we have $A_j^* = \tau_j\text{-}cl(\tau_i\text{-}int(\tau_j\text{-}cl(A)))$ and $\tau_j\text{-}cl^*(A) = A \cup A_j^* = A \cup \tau_j\text{-}cl(\tau_i\text{-}int(\tau_j\text{-}cl(A)))$ for any subset A of X. Therefore, $\tau_i\text{-}int(\tau_j\text{-}cl^*(\tau_i\text{-}int(A))) = \tau_i\text{-}int[\tau_i\text{-}int(A) \cup \tau_j\text{-}cl(\tau_i\text{-}int(\tau_j\text{-}cl(\tau_i\text{-}int(A))))] = \tau_i\text{-}int[\tau_i\text{-}int(A) \cup \tau_j\text{-}cl(\tau_i\text{-}int(A))] = \tau_i\text{-}int(\tau_j\text{-}cl(\tau_i\text{-}int(A)))$. We obtain A is $(i,j)$-$\alpha$-$\mathcal{I}$-open if and only if it is $(i,j)$-$\alpha$-open. By Proposition 2.13, A is an $(i,j)$-$A_{\mathcal{I}}$-set if and only if it is an $(i,j)$-A-set and $A$ is $(i,j)$-*pre*-$\mathcal{I}$-open if and only if it is $(i,j)$-*pre*-open. The proof of the corollary follows immediately from Lemma 1.6 and Theorem 4.10. □

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