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Ideally slowly oscillating sequences

B. Hazarika

abstract: An ideal I is a family of subsets of positive integers $\mathbb{N}$ which is closed under taking finite unions and subsets of its elements. In this paper, we introduce the notion of ideally slowly oscillating sequences, which is lying between ideal convergent and ideal quasi-Cauchy sequences, and study on ideally slowly oscillating continuous functions, and ideally slowly oscillating compactness.

Key Words: ideal convergence; compact; continuity; quasi-Cauchy sequence; slowly oscillating sequences.

Contents

1. Introduction

The idea of statistical convergence first appeared, under the name of almost convergence, in the first edition of celebrated monograph [37] of Zygmund. Later, this idea was introduced by Fast [22] and Steinhaus [34]. Actually, this concept is based on the natural density of subsets of $\mathbb{N}$ of positive integers. A subset E of $\mathbb{N}$ is said to have natural or asymptotic density $\delta(E)$, if

$$\delta(E)=\lim_{n\to\infty}\frac{|E(n)|}{n} \text{ exists},$$

where $E(n)=\{k\leq n: k\in E\}$ and $|E|$ denotes the cardinality of the set E.

Any number sequence $\mathbf{x}=(x_k)$ is said to be *statistically convergent* to the number L if for each $\varepsilon>0$, $\delta(\{k\in\mathbb{N}:|x_k-L|\geq\varepsilon\})=0$, i.e.

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n:|x_k-L|\geq\varepsilon\}|=0.$$

Kostyrko et al. [29] introduced the notion of ideal convergence which is a generalization of statistical convergence (see [23]) based on the structure of the

2000 *Mathematics Subject Classification:* 40A30; 40A05; 26A15 admissible ideal I of subsets of natural numbers $\mathbb{N}$.

A family of sets $I \subset P(\mathbb{N})$ (the power sets of $\mathbb{N}$) is said to be an *ideal* on $\mathbb{N}$ if and only if

(i) $\phi \in I$

(ii) for each $A, B \in I$, we have $A \cup B \in I$

(ii) for each $A \in I$ and each $B \subset A$, we have $B \in I$.

A non-empty family of sets $F \subset P(\mathbb{N})$ is said to be a *filter* on $\mathbb{N}$ if and only if

(i) $\phi \notin F$

(ii) for each $A, B \in F$, we have $A \cap B \in F$

(iii) each $A \in F$ and each $B \supset A$, we have $B \in F$.

An ideal I is called *non-trivial* ideal if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly $I \subset P(\mathbb{N})$ is a non-trivial ideal if and only if $F = F(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on $\mathbb{N}$. A non-trivial ideal $I \subset P(\mathbb{N})$ is called *admissible* if and only if $\{\{n\} : n \in \mathbb{N}\} \subset I$.

Recall that a sequence $\mathbf{x} = (x_n)$ of points in $\mathbb{R}$ is said to be I-convergent to the number ℓ if for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\} \in I$. In this case we write $I\text{-}\lim x_n = \ell$. A sequence $\mathbf{x} = (x_n)$ of points in $\mathbb{R}$ is said to be I-quasi-Cauchy if $I - \lim_n (x_{n+1} - x_n) = 0$. We see that I-convergence of a sequence (x_n) implies I-quasi-Cauchyness of (x_n). We note that the definition of a quasi-Cauchy sequence is a special case of an ideal quasi-Cauchy sequences where I is taken as the finite subsets of the set of positive integers. Cakalli and Hazarika [3] introduced the concept of ideal quasi Cauchy sequences and proved some results related to ideal ward continuity and ideal ward compactness. For more details on ideal convergence we refer to [25,26,27,28,32,33].

A real valued function is continuous on the set of real numbers if and only if it preserves Cauchy sequences. Using the idea of continuity of a real function and the idea of compactness in terms of sequences, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: forward continuity [6], slowly oscillating continuity [9,14,15,21], statistical ward continuity [7], δ-ward continuity [11], ideal ward continuty [3,12], N_θ-ward continuity [4,5] and λ-statistical ward continuity [13]. The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to zero. Nevertheless, sequences which satisfy this weaker property are interesting in their own right. A sequence (x_n) of points in $\mathbb{R}$ is called quasi-Cauchy if (Δx_n) is a null sequence where $\Delta x_n = x_{n+1} - x_n$. In [2] Burton and Coleman named these sequences as "quasi-Cauchy" and in [8] Çakallı used the term "ward convergent to 0" sequences. In terms of quasi-Cauchy we restate the definitions of ward compactness and ward continuity as follows: a function f is ward continuous if it preserves quasi-Cauchy sequences, i.e. $(f(x_n))$ is quasi-Cauchy whenever (x_n) is, and a subset E of $\mathbb{R}$ is ward compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence $\mathbf{x}$.

2. Preliminaries and Notations

Throughout this paper, $\mathbb{N}$, and $\mathbb{R}$ will denote the set of all positive integers, and the set of all real numbers, respectively. We will use boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... for sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, ... of terms in $\mathbb{R}$.

It is known that a sequence (x_n) of points in $\mathbb{R}$, the set of real numbers, is slowly oscillating if

$$\lim_{\lambda \to 1^+} \overline{\lim}_n \max_{n+1 \le k \le [\lambda n]} |x_k - x_n| = 0$$

where $[\lambda n]$ denotes the integer part of λn. This is equivalent to the following if $(x_m - x_n) \to 0$ whenever $1 \le \frac{m}{n} \to 1$ as $m, n \to \infty$. Using $\varepsilon > 0$ and δ this is also equivalent to the case when for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that $|x_m - x_n| < \varepsilon$ if $n \ge N(\varepsilon)$ and $n \le m \le (1+\delta)n$ (see [9]). For more details on slowly oscillating sequences we refer to [1,16,17,18,19,24,30,31,36].

A function defined on a subset E of $\mathbb{R}$ is called slowly oscillating continuous if it preserves slowly oscillating sequences, i.e. $(f(x_n))$ is slowly oscillating whenever (x_n) is.

Throughout this paper we assume I is a non-trivial admissible ideal in $\mathbb{N}$, also, $I(\mathbb{R})$ and ΔI will denote the set of all I-convergent sequences and the set of all I-quasi-Cauchy sequences of points in $\mathbb{R}$, respectively.

Connor and Grosse-Erdman [20] gave sequential definitions of continuity for real functions calling G-continuity (see [10]) instead of A-continuity and their results covers the earlier works related to A-continuity where a method of sequential convergence, or briefly a method, is a linear function G defined on a linear subspace of s, space of all sequences, denoted by c_G, into $\mathbb{R}$. A sequence $\mathbf{x} = (x_n)$ is said to be G-convergent to ℓ if $\mathbf{x} \in c_G$ and $G(\mathbf{x}) = \ell$. In particular, lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the linear space c and st-lim denotes the statistical limit function $st\text{-}\lim \mathbf{x} = st\text{-}\lim_n x_n$ on the linear space $st(\mathbb{R})$. Also I-lim denotes the I-limit function $I\text{-}\lim \mathbf{x} = I\text{-}\lim_n x_n$ on the linear space $I(\mathbb{R})$.

A method G is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. A method is called subsequential if whenever $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = \ell$, then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = \ell$.

3. Ideally slowly oscillating sequences

In this section we introduce the concepts of ideally slowly oscillating continuity and ideally slowly oscillating compactness and establish some intresting results related to these notions.

Tripathy and Hazarika [35], we say that $\mathbf{x} = (x_k)$ is an ideally Cauchy sequence if for any $\varepsilon > 0$ there exists $m = m(\varepsilon) \in \mathbb{N}$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \varepsilon\} \in I$. Any sequence is ideally convergent if and only if it is ideally Cauchy.

Definition 3.1. *A sequence $\mathbf{x} = (x_n)$ of real or complex numbers is said to be ideally slowly oscillating if for any given $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ and the positive integer $N = N(\varepsilon)$, the set*

$$\{N(\varepsilon) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : |x_k - x_n| \geq \varepsilon\} \in I.$$

It is clear that an ideally convergent sequence is ideally slowly oscillating, but the converse need not to be true. The sequence $(x_n) = \left(\sum_{j=1}^{n} \frac{1}{j}\right)$ is ideally slowly oscillating but not ideally convergent.

Example 3.2. *Define the sequence (x_n) by*

$$x_n = \begin{cases} (-2)^n & , \text{ if } n = i^2, n = i^2+1, i \in \mathbb{N}; \\ 0 & , \text{ otherwise} \end{cases}$$

Then the sequence (x_n) is ideally slowly oscillating, but not slowly oscillating because

$$|x_{i^2+1} - x_{i^2}| = 3.2^{i^2} \nrightarrow 0 \text{ as } i \to \infty,$$

whenever $1 < \frac{(i^2+1)}{i^2} \to 1$ as $i \to \infty$.

Cakalli [9] introduced the notion of slowly oscillating continuity. The following definition is an ideal version of slowly oscillating continuity.

Definition 3.3. *A function f is called ideally slowly oscillating continuous if it transforms ideally slowly oscillating sequences to ideally slowly oscillating sequences, that is, $(f(x_n))$ is ideally slowly oscillating whenever (x_n) is ideally slowly oscillating.*

Theorem 3.4. *If f is ideally slowly oscillating continuous on a subset E of $\mathbb{R}$ then it is ideally continuous on E.*

Proof: Suppose that f is ideally slowly oscillating continuous on E and let (x_n) be any ideally convergent sequence in E with $I - \lim x_n = x_0$. Then the sequence

$$(y_n) = (x_1, x_0, x_2, x_0, ..., x_{n-1}, x_0, x_n, x_0, ...)$$

is also ideally convergent to x_0 and hence (y_n) is ideally slowly oscillating. Since f is ideally slowly oscillating continuous, then the sequence

$$(f(y_n)) = (f(x_1), f(x_0), f(x_2), f(x_0), ..., f(x_{n-1}), f(x_0), f(x_n), f(x_0), ...)$$

is also ideally slowly oscillating. Hence $(f(y_n))$ is an ideally quasi-Cauchy sequence, so $I - \lim[f(x_n) - f(x_0)] = 0$. It follows that $I - \lim f(x_n) = f(x_0)$. This completes the proof of theorem. □

In general the converse is not true. For instance, $f : [0, \infty) \to \mathbb{R}, f(x) = e^x$ is an ideally continuous function. On the other hand $(x_n) = (\ln n)$ is an ideally slowly oscillating sequence while $(f(x_n)) = (n)$ is not. So, f is not ideally slowly oscillating continuous on $[0, \infty)$.

Theorem 3.5. *Sum of two ideally slowly oscillating continuous functions is ideally slowly oscillating continuous.*

Proof: Let f and g be ideally slowly oscillating continuous functions on a subset E of $\mathbb{R}$. To prove that $f + g$ is ideally slowly oscillating continuous on E. Let $\varepsilon > 0$ and $\mathbf{x} = (x_n)$ is any ideally slowly oscillating sequence in E. Then $(f(x_n))$ and $(g(x_n))$ are ideally slowly oscillating sequences. Since $(f(x_n))$ and $(g(x_n))$ are ideally slowly oscillating sequences, there exist a positive integer $n_1 = n_1(\varepsilon)$ and $\delta > 0$ such that

$$\{n_1(\varepsilon) \leq n \leq k \leq (1 + \delta)n \text{ and } k \in \mathbb{N} : |f(x_k) - f(x_n)| \geq \frac{\varepsilon}{2}\} \in I$$

and

$$\{n_1(\varepsilon) \leq n \leq k \leq (1 + \delta)n \text{ and } k \in \mathbb{N} : |g(x_k) - g(x_n)| \geq \frac{\varepsilon}{2}\} \in I.$$

Since I is an admissible ideal, therefore we have

$$\begin{aligned}
&\{n_1(\varepsilon) \leq n \leq k \leq (1 + \delta)n \text{ and } k \in \mathbb{N} : |(f + g)(x_k) - (f + g)(x_n)| \geq \varepsilon\} \\
&\subseteq \{n_1(\varepsilon) \leq n \leq k \leq (1 + \delta)n \text{ and } k \in \mathbb{N} : |f(x_k) - f(x_n)| \geq \frac{\varepsilon}{2}\} \\
&\cup \{n_1(\varepsilon) \leq n \leq k \leq (1 + \delta)n \text{ and } k \in \mathbb{N} : |g(x_k) - g(x_n)| \geq \frac{\varepsilon}{2}\}
\end{aligned}$$

i.e.

$$\{n_1(\varepsilon) \leq n \leq k \leq (1 + \delta)n \text{ and } k \in \mathbb{N} : |(f + g)(x_k) - (f + g)(x_n)| \geq \varepsilon\} \in I.$$

This compltes the proof of the theorem. □

We know that a function is ideally continuous at a point x_0 if it continuous at x_0 in the ordinary sense. Hence we have the following corollary.

Corollary 3.6. *If f is ideally slowly oscillating continuous, then it is continuous in the ordinary sense.*

Theorem 3.7. *If f is a uniformly continuous on a subset E of $\mathbb{R}$, then it is ideally slowly oscillating continuous.*

Proof: Let f be uniformly continuous function and $\mathbf{x} = (x_n)$ be any ideally slowly oscillating sequence in E. Since f is uniformly continuous on E, for given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $x, y \in E$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$. Since (x_n) is ideally slowly oscillating, for the same $\delta > 0$ there exist $\eta > 0$ and the positive integer $N = N(\delta)$ we have

$$\{N(\delta) \leq n < k \leq (1 + \eta)n \text{ and } k \in \mathbb{N} : |x_k - x_n| < \delta\} \in F.$$

Hence we have

$$\{N(\delta) \leq n < k \leq (1 + \eta)n \text{ and } k \in \mathbb{N} : |f(x_k) - f(x_n)| < \varepsilon\} \in F.$$

This completes the proof of theorem. □

Definition 3.8. *A sequence (x_n) of real numbers is called ideally Cesáro slowly oscillating if (t_n) is ideally slowly oscillating, where $t_n = \frac{1}{n}\sum_{k=1}^{n} x_k$, is the Cesáro means of the sequence (x_n). Also a function f is called ideally Cesáro slowly oscillating continuous if it preserves ideally Cesáro slowly oscillating sequences.*

By using the similar argument used in proof of Theorem 3.7, we immediately have the following result.

Theorem 3.9. *If f is a uniformly continuous on a subset E of $\mathbb{R}$ and (x_n) is an ideally slowly oscillating sequence in E, then $(f(x_n))$ is ideally Cesáro slowly oscillating.*

Definition 3.10. *A sequence of functions (f_n) defined on a subset E of $\mathbb{R}$ is said to be uniformly ideally convergent to a function f if for each $\varepsilon > 0$, the set*

$$\{x \in E, n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon\} \in I.$$

Note that ordinary uniform convergence implies uniform ideal convergence.

Theorem 3.11. *If (f_n) is a sequence of ideally slowly oscillating continuous functions defined on a subset E of $\mathbb{R}$ and (f_n) is uniformly ideally convergent to a function f on E, then f is ideally slowly oscillating continuous on E.*

Proof: Let $\varepsilon > 0$ and (x_n) be any ideally slowly oscillating sequence of points in E. By uniform ideal convergence of (f_n) for each $\varepsilon > 0$, we have

$$\{x \in E \text{ and } n \in \mathbb{N} : |f_n(x) - f(x)| \geq \frac{\varepsilon}{3}\} \in I.$$

Also since f_n is ideally slowly oscillating continuous, there exist a positive integer $n_1 = N(\varepsilon)$ and $\delta > 0$ such that

$$\{N(\varepsilon) \leq n \leq k \leq (1 + \delta)n \text{ and } k \in \mathbb{N} : |f_{n_1}(x_k) - f_{n_1}(x_n)| \geq \frac{\varepsilon}{3}\} \in I.$$

Since I is an admissible ideal, we have

$$\{N(\varepsilon) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : |f(x_k) - f(x_n)| \geq \varepsilon\}$$

$$\subseteq \{N(\varepsilon) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : |f(x_k) - f_{n_1}(x_k)| \geq \frac{\varepsilon}{3}\}$$

$$\cup\{N(\varepsilon) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : |f_{n_1}(x_k) - f_{n_1}(x_n)| \geq \frac{\varepsilon}{3}\}$$

$$\cup\{N(\varepsilon) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : |f_{n_1}(x_n) - f(x_n)| \geq \frac{\varepsilon}{3}\}$$

i.e.

$$\{N(\varepsilon) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : |f(x_k) - f(x_n)| \geq \frac{\varepsilon}{3}\} \in I.$$

Thus $(f(x_n))$ is an ideally slowly oscillating sequence and this completes the proof of theorem. □

Corollary 3.12. *If (f_n) is a sequence of ideally slowly oscillating continuous functions defined on a subset E of $\mathbb{R}$ and (f_n) is uniformly convergent to a function f on E, then f is ideally slowly oscillating continuous on E.*

Using the same techniques as in the Theorem 3.11, the following result can be obtained easily.

Theorem 3.13. *If (f_n) is a sequence of ideally Cesáro slowly oscillating continuous functions defined on a subset E of $\mathbb{R}$ and (f_n) is uniformly ideally convergent to a function f on E, then f is ideally Cesáro slowly oscillating continuous on E.*

Theorem 3.14. *The set of all ideally slowly oscillating continuous functions defined on a subset E of $\mathbb{R}$ is a closed subset of all continuous functions on E, that is $\overline{isoc(E)} = isoc(E)$, where $isoc(E)$ is the set of all ideally slowly oscillating continuous functions defined on E and $\overline{isoc(E)}$ denotes the set of all cluster points of $isoc(E)$.*

Proof: Let f be any element of $\overline{isoc(E)}$. Then there exists a sequence of points in $isoc(E)$ such that $\lim f_k = f$. Now let (x_n) be any ideally slowly oscillating sequence in E. Since (f_k) converges to f, there exist a positive integer $n_1 = n_1(\varepsilon)$ such that for all $x \in E$ and for all $n \geq n_1$, $|f(x) - f_n(x)| < \frac{\varepsilon}{3}$. By definition of ideal, for all $x \in E$ we have

$$\{n \in \mathbb{N} : |f(x) - f_n(x)| \geq \frac{\varepsilon}{3}\} \in I.$$

Also since f_n is ideally slowly oscillating continuous on E, for $\varepsilon > 0$ there exists a positive integer $n_1 = n_1(\varepsilon)$ and $\delta > 0$, we have

$$\{n_1(\varepsilon) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : |f_{n_1}(x_k) - f_{n_1}(x_n)| \geq \frac{\varepsilon}{3}\} \in I.$$

Also we have

$$\{n_1(\varepsilon) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : |f(x_k) - f(x_n)| \geq \varepsilon\}$$

$$\subseteq \{n_1(\varepsilon) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : |f(x_k) - f_{n_1}(x_k)| \geq \frac{\varepsilon}{3}\}$$

$$\cup \{n_1(\varepsilon) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : |f_{n_1}(x_k) - f_{n_1}(x_n)| \geq \frac{\varepsilon}{3}\}$$

$$\cup \{n_1(\varepsilon) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : |f_{n_1}(x_n) - f(x_n)| \geq \frac{\varepsilon}{3}\}.$$

Since I is an admissible ideal, it implies that

$$\{n_1(\varepsilon) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : |f(x_k) - f(x_n)| \geq \varepsilon\} \in I.$$

Thus f is ideally slowly oscillating continuous function on E and this completes the proof of theorem. □

Corollary 3.15. *The set of all ideally slowly oscillating continuous functions defined on a subset E of $\mathbb{R}$ is a complete subspace of the space of all continuous functions on E.*

Next we define the ideal version of the concept of slowly oscillating compactness.

Definition 3.16. *A subset E of $\mathbb{R}$ is called ideally slowly oscillating compact if any sequence of points in E has an ideally slowly oscillating subsequence.*

Theorem 3.17. *An ideally slowly oscillating continuous image of an ideally slowly oscillating compact subset of $\mathbb{R}$ is ideally slowly oscillating compact.*

Proof: Suppose that f is an ideally slowly oscillating continuous function on a subset E of $\mathbb{R}$ and E is an ideally slowly oscillating compact subset of $\mathbb{R}$. Let (y_n) be a sequence of points in $f(E)$. Then we can write $y_n = f(x_n)$ where $x_n \in E$ for each $n \in \mathbb{N}$. Since E is ideally slowly oscillating compact, there is an ideally slowly oscillating subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of (x_n). Then, ideally slowly oscillating continuity of f implies that $f(z_k)$ is an ideally slowly oscillating subsequence of $f(x_n)$. Hence $f(E)$ is ideally slowly oscillating compact. □

Corollary 3.18. *An ideally slowly oscillating continuous image of any compact subset of $\mathbb{R}$ is ideally slowly oscillating compact.*

Proof: The proof follows for the preceding theorem. □

Theorem 3.19. *Let E be an ideally slowly oscillating compact subset of $\mathbb{R}$ and let $f : E \to \mathbb{R}$ be an ideally slowly oscillating continuous function. Then f is uniformly continuous on E.*

Proof: Suppose that f is not uniformly continuous on E, there exist $\varepsilon > 0$ and the sequences (x_n) and (y_n) of points in E such that $|x_n - y_n| < \frac{1}{n}$ but

$$|f(x_n) - f(y_n)| \geq \varepsilon \text{ for all } n \in \mathbb{N}. \tag{3.1}$$

Since E is ideally slowly oscillating compact, there is an ideally slowly oscillating subsequence (x_{n_k}) of (x_n). It is clear from the inequality

$$|y_{n_k} - y_{n_m}| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - x_{n_m}| + |x_{n_m} - y_{n_m}|$$

that the corresponding subsequence (y_{n_k}) of (y_n) is also ideally slowly oscillating. Then observe that the sequence

$$(x_{n_1}, y_{n_1}, x_{n_2}, y_{n_2}, ..., x_{n_k}, y_{n_k}, ...)$$

is ideally slowly oscillating. Since f is ideally slowly oscillating continuous, the sequence

$$(f(x_{n_1}), f(y_{n_1}), f(x_{n_2}), f(y_{n_2}), ..., f(x_{n_k}), f(y_{n_k}), ...)$$

must be ideally slowly oscillating. But this is contradicts the result (3.1). This contradiction completes the proof of the theorem. □

Corollary 3.20. *A real valued function defined on a bounded subset of* $\mathbb{R}$ *is uniformly continuous if and only if it is ideally slowly oscillating continuous.*

Proof: The proof of the result follows from the fact that totally boundedness coincides with ideally slowly oscillating compactness and boundedness coincides with totally boundedness in $\mathbb{R}$. □

4. Conclusions

In this paper, the concept of ideally slowly oscillating continuity of a real function and the concept of ideally slowly oscillating compactness of a subset of $\mathbb{R}$ are introduced and investigated. In this investigation we have obtained theorems related to continuity, compactness, ideally continuity, uniform continuity and ideally slowly oscillating continuity. Finally, we note that the results of this paper can be obtained by defining the ideas of ideal quasi-slowly oscillating and ideal Δ-quasi-slowly oscillating sequences.

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B. Hazarika
Department of Mathematics,
Rajiv Gandhi University, Rono Hills,
Doimukh-791112, Arunachal Pradesh, India
E-mail address: `bh_rgu@yahoo.co.in; bipan.hazarika@rgu.ac.in`